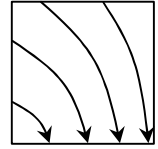


Overview

Recall that for 2D irrotational, steady, incompressible flow, the momentum equation reduces to $\nabla^2 \psi = 0$. Let's imagine a region of fluid where we want to solve this equation computationally. A small portion of some such flow is shown here. We'll discretize this region into small chunks; for each small chunk, we'll assume that ψ is constant. For the flow field shown, a fragment of this grid might look as shown in the second picture. The numbers indicate values for ψ . Of course, in every direction, the flow field might extend further than the amount shown here.



How can we check that this region satisfies Laplace's equation ($\nabla^2 \psi = 0$)? We don't really have a function for ψ , so it is difficult to take a derivative. Instead, we'll have to

approximate the derivatives: $\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$.

Each derivative can be approximated by: $\frac{\partial^2 \psi}{\partial x^2} \Big|_{i,j} \approx \frac{\psi_{i-1,j} - 2\psi_{i,j} + \psi_{i+1,j}}{\Delta x^2}$.

So, the Laplacian becomes: $\nabla^2 \psi_{i,j} \approx \frac{\psi_{i-1,j} - 2\psi_{i,j} + \psi_{i+1,j}}{\Delta x^2} + \frac{\psi_{i,j-1} - 2\psi_{i,j} + \psi_{i,j+1}}{\Delta y^2} = 0$.

Here, the i and j subscripts refer to grid numbers. For example, $\psi_{8,5} = 3.5$.

Fortunately, for our grid, $\Delta x = \Delta y$. Solving for $\psi_{i,j}$: $\psi_{i,j} = \frac{\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1}}{4}$

ψ:	6	3.4	3.8	4.3
	5	3.0	3.5	4.1
	4	2.8	3.1	4.0
	x:	7	8	9

In other words, if the value of ψ in each cell is equal to the average of the 4 directly neighboring cells, then $\nabla^2 \psi = 0$. So as a check, we should see if 3.5 is the average of 3.8, 3.0, 4.1, and 3.4 (it is).

Boundary Conditions

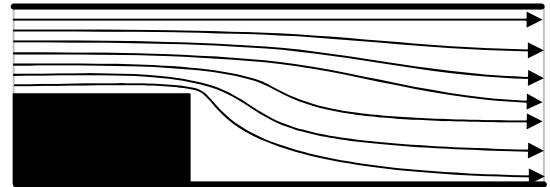
In practice, we are often more interested in going the other way: we want to start out with some boundary conditions for ψ , and determine the value of ψ at every point inside the region of interest. Setting the boundary conditions is the hard part. Let's remind ourselves of what ψ means: it tells us the components of velocity: $\frac{\partial \psi}{\partial y} = u$, and $-\frac{\partial \psi}{\partial x} = v$.

So, if we set $\frac{\partial \psi}{\partial x} = 0$ along some boundary, we are saying that the flow is all in the x direction. Furthermore, we know

that the flow between any two streamlines ψ_1 and ψ_2 is $Q = w(\psi_2 - \psi_1)$, where w is the thickness of the flow (into the page). We can use this to set some boundaries of the flow as exactly equal to some value. Remember that since our flow is frictionless, our result will violate the no-slip condition.

Example

Consider a flow of water in a channel that suddenly expands. The height of the entry channel might be 1m, and the exit channel is 2m tall. Horizontally, the total region shown might be 3m long, with an obstruction of 1m length. Q is given as $100\text{m}^3/\text{s}$, and the channel is $w = 10\text{m}$ wide into the page. We expect the results to look somewhat as shown.



Our region of flow seems to have 6 total boundaries. We are always free to pick any arbitrary of ψ as a starting point, since only differences in ψ have any physical meaning. For convenience, we'll choose $\psi = 0$ every along the bottom of the flow (3 surfaces). These 3 surfaces must all have the same ψ because fluid that starts out adjacent to this surface can't leave it later on (without crossing another streamline, which is forbidden). After this value is picked, the very top surface must therefore be $\psi = Q/w = 10 \text{ m}^2/\text{s}$.

We have a few options for how we set the boundary conditions on the remaining surfaces. We might choose to pick a range of values between $\psi = 0$ and $\psi = 10\text{m}^2/\text{s}$, and distribute them evenly across each of these two boundaries. However, it is probably more appropriate to declare that these flows are horizontal, as shown on the figure.

