ProofSpace Problem Set

Proof Techniques

Conditionals with Disjunctions
(Proof by Cases or Elimination)

Discussed Problems

1 Suppose $x \in \mathbb{R}$. Using a proof by elimination, prove that if $x = x^2$, then $x = 0$ or $x = 1$.

2 Consider the following theorem and the start of its proof:

Theorem: There are no rational numbers x such that $x^3 + x + 1 = 0$.

Proof: Suppose for the sake of contradiction that there is a rational number p such that $p^3 + p + 1 = 0$. Since p is rational, there exist integers a and b where $b \neq 0$ such that $p = \frac{a}{b}$ and a and b have no common factors. Thus, $(\frac{a}{b})^3 + \frac{a}{b} + 1 = 0$. Multiplying both sides of the equation by b^3, we get that $a^3 + ab^2 + b^3 = 0$.

We shall complete the proof.

a) What happens in the case where a is odd and b is even?

b) What happens in the case where a is even and b is odd?

c) What happens in the case where a is odd and b is odd?

d) Are there any other cases to consider? Why or why not?

3 Using a proof by cases, prove that if a is an integer, then $2 \mid a^2 + 3a$.

4 Using a proof by cases, prove that if a is an integer, then $3 \mid a^3 + 23a$.

Evaluated Problems

1 Let x be a real number. Using a proof by elimination, prove that if $x^2 - 5x + 6 \geq 0$, then $x \leq 2$ or $x \geq 3$.

2 Prove the following statement: If n is an odd integer, then $8 \mid n^2 - 1$.

3 Prove the following statement: There do not exist positive integers x and y such that $x^2 - y^2 = 1$. HINT: Recall that if a and b are integers such that $ab = 1$, then $a = b = 1$ or $a = b = -1$.

4 Let a and b be integers. Prove the following statement: If $ab \not\equiv 0 (\text{mod } 6)$, then $a \not\equiv 0 (\text{mod } 6)$ and $b \not\equiv 0 (\text{mod } 6)$. (We examined the converse of this in the previous assignment.)

5 Let a and b be integers. If $3 \mid a^2 + b^2$, then $3 \mid a$.

a) Why does it make sense to use an indirect proof technique to prove this statement?

b) What are the cases you would use to prove this?

c) The algebra involved is similar among the cases. Submit the proof for two of the cases.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 3.4: 3, 5, 10, 12, 13

Advanced Problems

A number p is **irreducible** if $a \mid p \Rightarrow (a = \pm 1 \vee a = \pm p)$.

A number p is **prime** if $p \mid ab \Rightarrow (p \mid a \vee p \mid b)$.

1 In this problem, we will consider how to prove two definitions are equivalent.

a) Suppose you wanted to prove $(P \Rightarrow Q) \iff (R \Rightarrow M)$. What would you assume?

b) Prove that if an integer is prime, it is irreducible. HINT: Recall that if $a \mid b$ and $b \mid a$, then $b = \pm a$.

There do exist settings wherein prime and irreducible are not equivalent definitions. We leave such examples for a future course in number theory.

2 Suppose you wanted to write a computer program to find out if some positive integer p greater than two was prime. One way to do this would be is as follows:

i) Let $r := 2$.

ii) Check if $r \mid p$.

iii) (a) If $r \mid p$, STOP. Conclude that p is not prime.
  (b) If $r \nmid p$, then set $r := r + 1$.

iv) (a) If $r < p$, return to step ii).
  (b) If $r = p$, STOP. Conclude that p is prime.

Interpreting this algorithm, we essentially test if any number greater than 1 but less than p is a factor of p. Notice that this means checking $p - 2$ numbers. Is there a way to determine whether or not p is prime by checking fewer numbers?