ProofSpace Problem Set

Functions

Introduction

Discussed Problems

1 (Sundstrom) Let d be the function that associates with each natural number the number of its natural number divisors. That is, $d : \mathbb{N} \to \mathbb{N}$ where $d(n)$ is the number of natural number divisors of n. For example, $d(6) = 4$ since $1, 2, 3$, and 6 are the natural divisors of 6.

a) Calculate $d(k)$ for each natural number k from 1 through 12.

b) Does there exist a natural number n such that $d(n) = 1$? What is $d^{-1}[\{1\}]$?

c) Does there exist a natural number n such that $d(n) = 2$? What is $d^{-1}[\{2\}]$?

d) Is the following statement true or false? Explain your reasoning.

$$\text{For all } m, n \in \mathbb{N}, \text{ if } m \neq n, \text{ then } d(m) \neq d(n).$$

e) Calculate $d(2^k)$ for all natural numbers k from 1 through 6.

f) Make a conjecture for a formula for $d(2^n)$ for any natural n. Explain your reasoning.

g) Is the following statement true or false? Explain your reasoning.

$$\text{For any } m \in \mathbb{N}, \text{ there exists an } n \in \mathbb{N} \text{ such that } d(n) = m.$$

2 (Quadratic Formula) Let P_2 be the set of all polynomials of the form $ax^2 + bx + c$ where a, b, and c are real numbers.

a) (Review) What are the roots of $x^2 + 2x + 1$ (that is, for which x does $x^2 + 2x + 1 = 0$?)

b) What are the roots of $x^2 + 3x + 2$?

c) What are the roots of $x^2 - 2$?

d) Suppose *roots* is a function that inputs an element of P_2 and outputs the set of its roots. What is an appropriate codomain Y we could use to define this function $roots : P_2 \to Y$?

3 (Multivariable Functions) Let $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{R}$ be given by $f(x, y) = x^y$.

a) What is $f(2, 2)$? What is $f(1, 2)$? $f(2, -1)$?

b) What is $f^{-1}[\{1\}]$?

c) What is the range of f?

4 Let $f : S \to T$. Let A and B be subsets of S, and let C be a subset of T. Prove or disprove the following. For false biconditional statements, is one of the conditionals true?

a) $A \subseteq B \iff f[A] \subseteq f[B]$.

b) $f[A \cup B] = f[A] \cup f[B]$.

c) $f[f^{-1}[C]] = C$.

Evaluated Problems

1 (Sundstrom) Let S be the function that associates with each natural number the set of its natural number divisors. For example, $S(6) = \{1, 2, 3, 6\}$ and $S(10) = \{1, 2, 5, 10\}$.

a) What is the domain of S? Determine an appropriate codomain of S.

b) Determine $S(n)$ for three prime and four composite values of n.

c) Does there exist a natural number n such that $S(n)$ has only one element? Explain your reasoning.

d) Does there exist a natural number n such that $S(n)$ has exactly two elements? Explain your reasoning.

e) Is the following statement true or false? Explain your reasoning.

$$\text{For all } m, n \in \mathbb{N}, \text{ if } m \neq n, \text{ then } S(m) \neq S(n).$$

f) Is the following statement true or false? Explain your reasoning.

$$\text{For any } T \subseteq \mathbb{N}, \text{ there exists an } n \in \mathbb{N} \text{ such that } S(n) = T.$$

2 (Derivatives) Let P_3 be the set of all polynomials of the form $ax^3 + bx^2 + cx + d$ where $a, b, c, d \in \mathbb{R}$.

a) (Review) What is the derivative of $4x^3 + 2x^2 + 3x + 8$?

b) Think of derivative as a function $d : P_3 \to P_3$. What is $d(3x^3 + 9x^2 + 2x + 4)$?

c) What is $d^{-1}[\{2x^2 + 2x + 2\}]$?

d) What is $d^{-1}[\{3x^3 + 4\}]$?

e) What is the range of d?

f) Does it make sense to think of derivatives of elements of P_3 as a function $der : \mathbb{R}^4 \to \mathbb{R}^4$ given by $der(a, b, c, d) = (0, 3a, 2b, c)$. Why or why not?

3 (Multivariable Functions) Let $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by $f(x, y) = (x + y, xy)$.

a) What is $f(2, 2)$? What is $f(1, 2)$? $f(2, -1)$?

b) What is $f^{-1}[\{(z, 1) \mid z \in \mathbb{Z}\}]$? Why?

c) Suppose $(z, 1)$ is in the range of f. What can you conclude about z?

d) What is $f^{-1}[\{(z, 0) \mid z \in \mathbb{Z}\}]$ Why?

4 Let $f : S \to T$ be a function. Suppose A and B be subsets of S, and C and D be subsets of T.

a) Prove that $C \subseteq D \implies f^{-1}[C] \subseteq f^{-1}[D]$.
Provide a counterexample to show that the converse is false.

b) Prove that$f[A] - f[B] \subseteq f[A - B]$.
Provide a counterexample to show that $f[A - B] \nsubseteq f[A] - f[B]$.

c) Prove that $A \subseteq f^{-1}[f[A]]$.
Provide a counterexample to show that $f^{-1}[f[A]] \nsubseteq A$.
Notice that this means, in general, $f^{-1}[f[A]] \neq A$.

d) Suppose $S = T$ and that S is the universal set. Show that $f[A^c] \neq f[A]^c$ by showing that neither set is a subset of the other. (Both can be shown with a single counterexample, although this is not required.)

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 6.1: 1, 2, 3, 5
Sec. 6.2: 1, 2, 3, 5, 7, 8, 9, 10

Advanced Problems

1 Suppose $f : A \to B$ and $g : A \to B$. Let $k \in B$. Define the following:

i) $f + g : A \to B$ given by $(f + g)(x) = f(x) + g(x)$.

ii) $f \cdot g : A \to B$ given by $(f \cdot g)(x) = (f(x))(g(x))$.

iii) $kf : A \to B$ given by $(kf)(x) = k(f(x))$.

Now, let $f : \mathbb{R} \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$. Let $k \in \mathbb{R}$. Give an example of each of the following, where neither f nor g is a constant function, or prove that one cannot exist:

a) Functions f and g such that $(f + g)(x) = 0$.

b) A function f and real numbers k and q such that $kf(x) = q$.

c) Let $q \in \mathbb{R}$. Functions f and g such that $(f \cdot g)(x) = q$.

2 Let A be a nonempty set, and let $\mathbb{R}^{\dagger} = \{x \in \mathbb{R} \mid x \geq 0\}$. Suppose $x, y, z \in A$. A *metric on* A is a function $d : A \times A \to \mathbb{R}^{\dagger}$ with the following properties:

i) $d(x, y) = 0$ if and only if $x = y$.

ii) $d(x, y) = d(y, x)$.

iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

We might think of a metric as a "distance" between two points.

a) Let A be nonempty. Define the **discrete metric** d on A as follows: $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. Verify that d is a metric.

b) Recall that $|x| : \mathbb{R} \to \mathbb{R}$ is defined to be $|x| = \sqrt{x^2}$. The function $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$. Verify that $|x|$ fills properties i) and ii) above.

c) The **Euclidean metric** on $\mathbb{R}^2$ is given by $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Verify that the Euclidean metric fills properties i) and ii) above.

d) Suppose that the codomain of a metric d was defined to be $\mathbb{R}$ instead of $\mathbb{R}^{\dagger}$. Use the properties of a metric to prove that the range of d would still be $\mathbb{R}^{\dagger}$.

A set B equipped with a metric d is called a **metric space.** Metric spaces are especially important in topology and real analysis, since the notion of "convergence" is well-defined in any metric space.