ProofSpace Problem Set

Functions

Injections, Surjections, and Bijections

Discussed Problems

1 (Sundstrom) For each of the following functions, prove or disprove: (i) the function is an injection, (ii) the function is a surjection, and (iii) the function is a bijection.

a) $f : (\mathbb{R} - \{4\}) \to (\mathbb{R} - \{3\})$ given by $f(x) = \frac{3x}{x-4}$

b) $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $g(a, b) = 2a + b$

c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $h(a, b) = 3a + 6b$

d) $j : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $j(m, n) = (2m, m + n)$

e) $k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $k(m, n) = (2m, m + n)$

2 Let A be any arbitrary set. Find a bijection from A to A, and prove your function is bijective.

3 Let $[0, 1]$ and $[2, 3]$ be intervals in $\mathbb{R}$. Find a bijection from [0,1] to [2,3] and prove that it is a bijection.

4 (Sundstrom) Let $s : \mathbb{N} \to \mathbb{N}$ be a function such that, for each $n \in \mathbb{N}$, $s(n)$ is the sum of the distinct natural number divisors of n. This is the **sum of the divisors function**. Is s an injection? Is s a surjection? Justify your conclusions.

5 Let $f : S \to T$ be a function, and let A and B be subsets of S. Let $C \subseteq T$.

a) Recall that $f[A \cap B] \subseteq f[A] \cap f[B]$. Prove that if f is an injection, $f[A] \cap f[B] \subseteq f[A \cap B]$.

b) Recall that $f[f^{-1}[C]] \subseteq C$. Prove that if f is a surjection, $C \subseteq f[f^{-1}[C]]$.

Evaluated Problems

1 For each of the following functions, prove or disprove: (a) the function is an injection, (b) the function is a surjection, and (c) the function is a bijection.

a) Let $PN^* = \{S \in \mathcal{P}(\mathbb{N}) \mid S \text{ is finite}\}$. That is, PN^* is the set of all finite subsets of $\mathbb{N}$. Let $f : PN^* \to (\mathbb{N} \cup \{0\})$ be the function defined by letting $f(S)$ be the number of elements in S. For example, $f(\{2, 13, 7, 159\}) = 4$.

b) $der : \mathbb{Z}^4 \to \mathbb{Z}^3$ given by $der(a, b, c, d) = (3a, 2b, c)$.

2 Let $[0, 1]$ and $[2, 4]$ be intervals in $\mathbb{R}$. Find a bijection from [0,1] to [2,4] and prove that it is a bijection.

3 (Sundstrom) Let $d : \mathbb{N} \to \mathbb{N}$, where $d(n)$ is the number of the natural number divisors of n. This is the **number of the divisors function**, which was introduced in the previous problem set. Is d an injection? Is d a surjection? Prove your claims. (Hint: It is highly recommended that you look at the results of Section 4.1 Discussion Problem 1.)

4 Let $f : S \to T$ be a function, and let A be a subset of S. Prove that if f is injective, then $f^{-1}[f[A]] = A$. Compare this to the result of Section 4.1 Evaluated Problem 4(c).

5 Consider the following proof.

> **Proposition:** Let $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$. Let $f : \mathbb{R} \to \mathbb{R}^*$ be given by $f(x) = e^x$. Then, f is a surjection.
>
> **Proof:** We will show that he given function f is surjective.
>
> 1) Let $y \in \mathbb{R}^*$.
> 2) Then, $ln(y) \in \mathbb{R}$.
> 3) Let $x = ln(y)$.
> 4) Then, $f(x) = e^{ln(y)} = y$.
> 5) Thus, for all $y \in \mathbb{R}^*$, there exists an $x \in \mathbb{R}$ such that $y = f(x)$.
> 6) Therefore, f is surjective. **Q.E.D.**

Identify any content-related errors in the proof. If there are none, say so.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 6.3: 4, 5, 6, 8, 13, 17, 19

Advanced Problems

1 In this problem, we will introduce alternate definitions of injection and surjection.

a) Recall Discussed Problem 5(a) above. We have proved that the injective condition is sufficient. Now prove that it is necessary. That is, prove f is an injection if and only if $f[A] \cap f[B] \subseteq f[A \cap B]$.

b) Recall Discussed Problem 5(b) above. We have proved that the surjective condition is sufficient. Now prove that it is necessary. That is, prove f is a surjection if and only if $C \subseteq f[f^{-1}[C]]$.

2 Recall the definitions from (Functions/Introduction/Advanced Problems/1). Suppose $f : A \to B$ and $g : A \to B$. Let $k \in B$. Define the following:

i) $f + g : A \to B$ given by $(f + g)(x) = f(x) + g(x)$.

ii) $f \cdot g : A \to B$ given by $(f \cdot g)(x) = (f(x))(g(x))$.

iii) $kf : A \to B$ given by $(kf)(x) = k(f(x))$.

Now, let $f : \mathbb{R} \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$. Let $k \in \mathbb{R}$. Prove or disprove the following.

a) If f and g are bijections, $f + g$ is a bijection.

b) If f and g are bijections, $f \cdot g$ is a bijection.

c) If $k \in \mathbb{R}$, then kf is a bijection.

d) If $k \in \mathbb{R}$ and $k \neq 0$, then kf is a bijection.