ProofSpace Problem Set

Relations

Equivalence Relations

Discussed Problems

1 (Sundstrom) Consider the following relation R defined on $\mathbb{Z}$.

$$xRy \text{ if and only if } 2 \mid (x+y).$$

Prove or disprove:

a) R is reflexive.
b) R is symmetric.
c) R is transitive.
d) R is an equivalence relation.

2 Consider the following relation $\sim$ defined on $\mathbb{R}$.

$$a \sim b \text{ if and only if } a+b \text{ is not a rational number.}$$

Prove or disprove:

a) $\sim$ is reflexive.
b) $\sim$ is symmetric.
c) $\sim$ is transitive.
d) $\sim$ is an equivalence relation.

3 Let U be a set. Consider the following relation J defined on $\mathcal{P}(U)$.

$$(A, B) \in J \text{ if and only if there exists a bijection } f : A \to B.$$

Prove or disprove:

a) J is reflexive.
b) J is symmetric.
c) J is transitive.
d) J is an equivalence relation.

4 Consider the following relation $\diamond$ defined on $\mathbb{R}$.

$a \diamond b$ if and only if $a - b$ is a rational number.

Prove that $\diamond$ is symmetric. (We proved that this relation is reflexive and transitive in the videos.)

5 Prove that there are exactly two relations on the set $\{7\}$.

Evaluated Problems

1 (Sundstrom) Consider the following relation R defined on $\mathbb{Z}$.

xRy if and only if $3 \mid (x + y)$.

Prove or disprove:

a) R is reflexive.
b) R is symmetric.
c) R is transitive.
d) R is an equivalence relation.

2 Consider the following relation $\sim$ defined on $\mathbb{R} \times \mathbb{R}$.

$(x, y) \sim (a, b)$ if and only if $(x - a)$ and $(y - b)$ are integers.

Prove or disprove:

a) $\sim$ is reflexive.
b) $\sim$ is symmetric.
c) $\sim$ is transitive.
d) $\sim$ is an equivalence relation.

3 (Sundstrom) Let A be any set, and let R be a relation on A. Call R *circular* if for all $x, y, z \in A$, xRy and yRz implies zRx. Prove that a relation is an equivalence relation if and only if it is circular and reflexive.

4 Explain why there is only one relation on $\emptyset$. Then prove or disprove:

a) The relation is reflexive.
b) The relation is symmetric.
c) The relation is transitive.
d) The relation is an equivalence relation.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 7.2: 4, 6, 7, 11, 13, 15, 16

Advanced Problems

1 In this problem, we will work with geometric reasoning to develop what are called *product spaces.*

a) Let $I = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Describe I geometrically.

b) Describe $I \times I$ geometrically.

c) Describe $I \times I \times I$ geometrically.

d) Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$. Describe S geometrically.

e) Describe $I \times S$ geometrically.

f) Describe $I \times S \times I$ geometrically.

g) Describe $S \times S$ geometrically.

2 In this problem, we will develop some examples of other kinds of relations. Let R be a relation on a set A.

a) We say R is **anti-symmetric** if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$. Give an example of an anti-symmetric relation on $\mathbb{Z}$.

b) A relation R is called a **partial order** on A if it is reflexive, anti-symmetric, and transitive. Give an example of a partial order on $\mathbb{R}$.

c) A relation R on a set A is called **total** if for all $a, b \in A$, $(a, b) \in R$ or $(b, a) \in R$. Give an example of a total relation on $\mathbb{Z}$.

d) Prove that any relation that is total is also reflexive.

e) A relation R is called a **total order** if it is a partial order and is total. Give an example of a total order on $\mathbb{R}$.

Now, find sets A and relations R on A such that:

f) R is anti-symmetric but not a partial order.

g) R is a partial order but is not total.

h) R is a total relation but not a total order.