Hour Exam 1 Solutions

Math 222

March 2015

The following are outlines of my solutions to our first hour exam. While I have given some explanations of the thinking behind these solutions, I've concentrated on outlining my approaches, not explaining them in detail. But please do feel free to ask me questions about any this!

Question 1

Evaluate

$$\int \frac{-x - \frac{13}{2}}{(x+3)(x-\frac{1}{2})} dx$$

Use partial fractions to decompose the integrand into simpler fractions and then integrate. Start by finding A and B such that

$$\frac{-x - \frac{13}{2}}{(x+3)(x-\frac{1}{2})} = \frac{A}{x+3} + \frac{B}{x-\frac{1}{2}}$$

You can use the Heaviside technique to find A and B, i.e., multiply both sides of the above by $x+3$ and then set $x = -3$ to find A, and multiply both sides by $x - \frac{1}{2}$ and set $x = \frac{1}{2}$ to find B:

$$\frac{-x - \frac{13}{2}}{x - \frac{1}{2}} = A + \frac{B(x+3)}{x - \frac{1}{2}}$$

$$\frac{3 - \frac{13}{2}}{-3 - \frac{1}{2}} = \frac{-\frac{7}{2}}{-\frac{7}{2}} = 1 = A$$

and

$$\frac{-x - \frac{13}{2}}{x+3} = \frac{A(x-\frac{1}{2})}{x+3} + B$$

$$\frac{-\frac{1}{2} - \frac{13}{2}}{\frac{1}{2} + 3} = \frac{-\frac{14}{2}}{\frac{7}{2}} = -2 = B$$

The original integral can now be evaluated as

$$\begin{aligned}\int \frac{-x-\frac{13}{2}}{(x+3)(x-\frac{1}{2})}dx &= \int \frac{1}{x+3} - \frac{2}{x-\frac{1}{2}}dx \\ &= \ln|x+3| - 2\ln|x-\frac{1}{2}| + C\end{aligned}$$

Question 2

How fast is your angle of sight, θ, to a bird changing?

Since the bird is flying at a constant height of 100 feet, and its horizontal distance from you after t minutes is $200t$ feet, $\tan\theta = \frac{100}{200t} = \frac{1}{2t}$, or $\theta = \tan^{-1}(\frac{1}{2t})$.

The rate at which the angle changes is its derivative:

$$\begin{aligned}\frac{d\theta}{dt} &= \left(\frac{1}{1+\left(\frac{1}{2t}\right)^2}\right)\left(\frac{-1}{2t^2}\right) \\ &= \frac{-1}{2t^2+\frac{1}{2}}\end{aligned}$$

Question 3

Evaluate

$$\int x^2 e^{1-x} dx$$

Use a series of integrations by parts. For the first, choose $u = x^2$ and $dv = e^{1-x}$, so that $du = 2xdx$ and $v = -e^{1-x}$. Then

$$\int x^2 e^{1-x} dx = -x^2 e^{1-x} + 2\int xe^{1-x} dx$$

Use integration by parts again, with $u = x$ and $dv = e^{1-x}$ so that $du = dx$ and $v = -e^{1-x}$, to get

$$\int xe^{1-x} dx = -xe^{1-x} + \int e^{1-x} dx = -xe^{1-x} - e^{1-x} + C$$

Putting the above together,

$$\begin{aligned}\int x^2 e^{1-x} dx &= -x^2 e^{1-x} - 2xe^{1-x} - 2e^{1-x} + C \\ &= -e^{1-x}(x^2+2x+2) + C\end{aligned}$$

Question 4

Find a value of x such that $e^{2x} = 9$.

Take logarithms of both sides, and use the fact that $9 = 3^2$ to simplify:

$$2x = \ln 9 = \ln(3^2) = 2\ln 3$$

so that $x = \ln 3$.

Question 5

Calculate

$$\lim_{x\to 0} \frac{1-e^x}{\sin x}$$

When $x = 0$, the fraction $\frac{1-e^x}{\sin x}$ is the indeterminate form $\frac{0}{0}$, so use L'Hôpital's rule:

$$\lim_{x\to 0} \frac{1-e^x}{\sin x} = \lim_{x\to 0} \frac{-e^x}{\cos x} = \frac{-1}{1} = -1$$

Question 6

Evaluate

$$\int_0^{\frac{1}{3}} \sqrt{9-81x^2}dx$$

Use the substitution $x = \frac{1}{3}\sin\theta$, meaning that $dx = \frac{1}{3}\cos\theta$ and $\theta = \sin^{-1}(3x)$. Then

$$\begin{aligned}
\int_0^{\frac{1}{3}} \sqrt{9-81x^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{9-81(\frac{\sin^2\theta}{9})}\frac{1}{3}\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{9-9(\sin^2\theta)}\frac{1}{3}\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4}
\end{aligned}$$