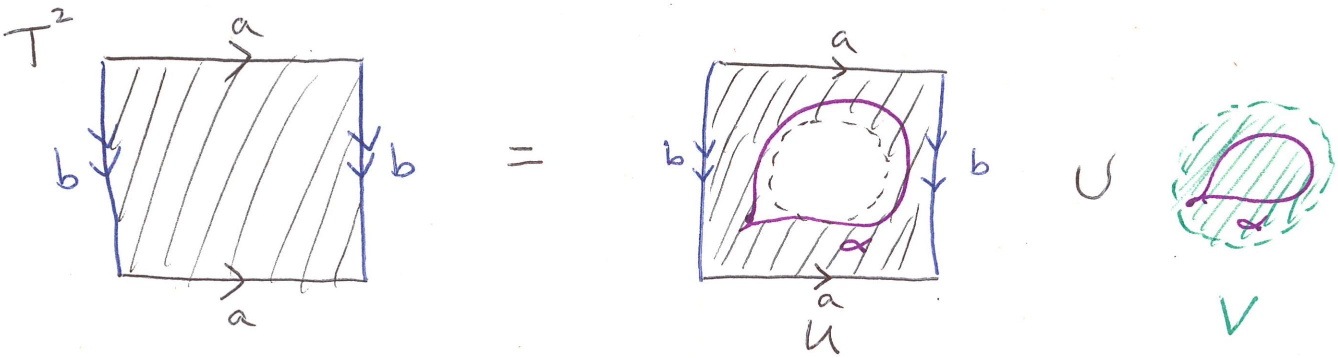
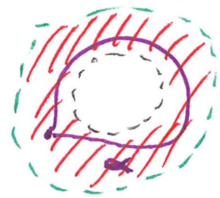
Ex. Use Seifert-van Kampen to find $\pi_1(T^2)$.



and $U \cap V =$ 

$\pi_1(U) \cong \langle a \rangle * \langle b \rangle$ since $U \simeq S^1 \vee S^1$

$\pi_1(V) \cong \{\varepsilon\}$ since $V \simeq D^2$

$\pi_1(U \cap V) \cong \mathbb{Z} \cong \langle \alpha \rangle$ since $U \cap V \simeq S^1$

What is α in $\pi_1(U)$?

$$i_1(\alpha) = aba^{-1}b^{-1}$$

What is α in $\pi_1(V)$?

$$i_2(\alpha) = \varepsilon$$

$$\Rightarrow i_1(\alpha)\, i_2(\alpha)^{-1} = aba^{-1}b^{-1}\varepsilon^{-1} = aba^{-1}b^{-1} \in \ker \Phi$$

$$\Rightarrow \pi_1(X) \cong \frac{\pi_1(U) * \pi_1(V)}{\ker \Phi}$$

$$\cong \frac{\big(\langle a \rangle * \langle b \rangle\big) * \{\varepsilon\}}{\langle i_1(\alpha) \rangle}$$

$$\cong \frac{\langle a \rangle * \langle b \rangle}{\langle aba^{-1}b^{-1} \rangle} \cong \langle a \rangle \times \langle b \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

Make $aba^{-1}b^{-1} = \varepsilon$
$\Rightarrow ab = ba$
$\Rightarrow$ make $\mathbb{Z} * \mathbb{Z}$ abelian
$\Rightarrow \mathbb{Z} \times \mathbb{Z}$

Group Presentations

A free group F_S on a set of generators $S = \{x_1, x_2, \cdots, x_n\}$ can be defined using equivalence relations:

$\boxed{\text{two words } u, v \in F_S \text{ are } \underline{\text{related}} \text{ if } u \text{ can be obtained from } v \text{ by a finite seq. of insertions and deletions of words of form } xx^{-1} \text{ or } x^{-1}x \text{ for } x \in S}$

The free group is free of any other relations.

Universal Mapping Thm

Every group is a homomorphic image of a free group.

$$\phi : F \twoheadrightarrow G$$

where F is free gp on the set of generators of G (which may be G itself.)

Since $\ker\phi$ is a normal subgp of F, we get $G \cong F/_{\ker\phi}$

Univ. Quotient Thm

Every gp is isomorphic to a quotient of a free group.

Let G be a group. Let S be a set of gen. of G (possibly infinite)
Let F_S be the free gp on S.
Let R be a subset of F_S, and let N be the smallest normal subgp of F_S that contains R.

Def. A <u>presentation</u> of G is $\langle S \mid R \rangle$ if $G \cong F_S/N$. S is the set of <u>generators</u> of G, and R is the set of <u>relations</u> in G.

Ex ⓐ $\pi_1(S^1) = \langle x \rangle = \langle x \mid \ \rangle = \langle x \mid 1 \rangle$
has generator x and no relations.

means $aba^{-1}b^{-1} = 1$

$$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z} \cong \langle a \rangle \times \langle b \rangle = \langle a, b \mid aba^{-1}b^{-1} \rangle$$
$$= \langle a, b \mid ab = ba \rangle$$

Recall that
$$\pi_1(T^2) \cong \frac{\langle a \rangle * \langle b \rangle}{\langle aba^{-1}b^{-1} \rangle} \cong \frac{\langle a, b \rangle}{\langle aba^{-1}b^{-1} \rangle}$$

<u>Common questions about a group G:</u>

(1) Is G finitely generated? Is S a finite set?
$G = \langle x_1, x_2, \dots, x_n \mid \text{relations} \rangle$

(2) Is G finitely presentable? Is S and R finite?
$G = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_p \rangle$

or $\langle x, y \mid x^n = 1, y^2 = y^3 \rangle$

Ex ⓐ $\mathbb{Z}_n \cong \langle x \mid x^n \rangle$ or $\langle x \mid x^n = 1 \rangle$

or $\langle x \mid x^3 = x^{n+3} \rangle$

ⓑ $\mathbb{Z}_3 \times \mathbb{Z}_4 = \langle x, y \mid x^3, y^4, xyx^{-1}y^{-1} \rangle$

$= \langle x, y \mid x^3 = y^4 = 1, \; xy = yx \rangle$

ⓒ $D_3 = \langle \rho, r \mid r^2 = \rho^3 = 1, \; (\rho r)^2 = 1 \rangle$

ρ = rotation
r = reflection