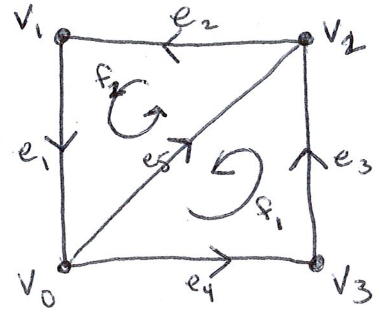
Ex.

K



$C_0(K) \cong \mathbb{Z}^4$ gen. by $\{v_0, v_1, v_2, v_3\}$

$C_1(K) \cong \mathbb{Z}^5$ gen. by $\{\overset{\langle v_1, v_0\rangle}{\overset{=}{e_1}}, e_2, e_3, e_4, e_5\}$

$C_2(K) \cong \mathbb{Z}^2$ gen by $\{f_1, f_2\}$

$C_p(K) \cong 0$ if $p \geq 3$.

$\partial_0(\text{any 0-chain}) = 0$

$\partial_1 : C_1(K) \to C_0(K)$

$$\begin{aligned}
&\partial_1(n_1e_1 + n_2e_2 + n_3e_3 + n_4e_4 + n_5e_5) \\
&= n_1\partial_1(e_1) + n_2\partial_1(e_2) + n_3\partial_1(e_3) + n_4\partial_1(e_4) + n_5\partial_1(e_5) \\
&= n_1(v_0 - v_1) + n_2(v_1 - v_2) + n_3(v_2 - v_3) + n_4(v_3 - v_0) + n_5(v_2 - v_0) \\
&= (n_1 - n_4 - n_5)v_0 + (-n_1 + n_2)v_1 + (-n_2 + n_3 + n_5)v_2 + (-n_3 + n_4)v_4
\end{aligned}$$

Image ∂_1: $y \in C_0(K) \Rightarrow y = c_1v_0 + c_2v_1 + c_3v_2 + c_4v_3$

$$\Rightarrow \left.\begin{aligned} c_1 &= n_1 - n_4 - n_5 \\ c_2 &= -n_1 + n_2 \\ c_3 &= -n_2 + n_3 + n_5 \\ c_4 &= -n_3 + n_4 \end{aligned}\right\} \Rightarrow c_1 + c_2 + c_3 = n_3 - n_4 = -c_4$$

$\Rightarrow \boxed{B_0(K) \text{ gen. by } \{c_1, c_2, c_3 \text{ or } \{v_0, v_1, v_2\}}$

Ker ∂_1:

$$\left.\begin{aligned} n_1 - n_4 - n_5 &= 0 \\ -n_1 + n_2 &= 0 \\ -n_2 + n_3 + n_5 &= 0 \\ -n_3 + n_4 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} n_1 &= n_2 \\ n_3 &= n_4 \\ n_1 - n_3 &= n_5 \end{aligned}$$

$\Rightarrow \boxed{Z_1(K) \text{ gen by } \{(1,1,0,0,1), (0,0,1,1,}$

or $\{e_1 + e_2 + e_5, e_3 + e_4 - e_5$

$$\partial_2 : C_2(K) \to C_1(K)$$

$$\partial_2(af_1 + bf_2) = a\,\partial_2(f_1) + b\,\partial_2(f_2)$$

$$= a(e_1 + e_5 + e_2) + b(e_4 + e_3 - e_5)$$

$\Rightarrow \mathrm{Im}\,\partial_2 = B_1(K)$ gen by $\{e_1 + e_5 + e_2,\ e_4 + e_3 - e_5\}$

$\ker \partial_2 = 0 = Z_2(K)$ since $af_1 + bf_2$ is a cycle only when $a = b = 0$.

Then

$$H_0(K) = \frac{Z_0(K)}{B_0(K)} = \frac{\langle v_0\rangle \times \langle v_1\rangle \times \langle v_2\rangle \times \langle v_3\rangle}{\langle v_0\rangle \times \langle v_1\rangle \times \langle v_2\rangle} \cong \langle v_3\rangle \cong \boxed{\mathbb{Z}.}$$

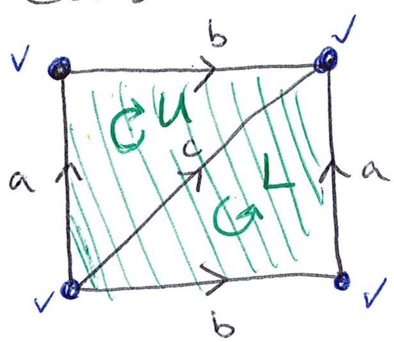
$$H_1(K) = \frac{Z_1(K)}{B_1(K)} = \frac{\langle e_1 + e_2 + e_5\rangle \times \langle e_3 + e_4 - e_5\rangle}{\langle \text{ same } \rangle \times \langle \text{thing} \rangle} \cong \boxed{0}$$

$$H_2(K) = \frac{Z_2(K)}{B_2(K)} \cong \boxed{0} \cong H_p(K),\ p \geq 1$$

<u>Ex</u> H_p(torus)?

Consider the torus written as

(This is not a simplicial complex.)



$C_0 = \langle v \rangle$

$C_1 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$

$C_2 = \langle U \rangle \times \langle L \rangle$

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$\partial_0(v) = 0$

$\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$

$\partial_2(U) = \partial_2(L) = a + b - c$

$$H_0(T) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v \rangle}{\boxed{0}}$$

$\partial_1(n_1 a + n_2 b + n_3 c) = 0$

$\Rightarrow \operatorname{Im} \partial_1 = 0$

$\ker \partial_1 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$

$\cong \langle v \rangle \cong \boxed{\mathbb{Z}}$ gen. by vertex

$$H_1(T) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\boxed{\langle a \rangle \times \langle b \rangle \times \langle c \rangle}}{\boxed{\langle a+b-c \rangle}}$$

$\hookrightarrow a+b-c = 0$

$\Rightarrow c = a+b$

$\partial_2(n_1 U + n_2 L)$

$= n_1(a+b-c)$

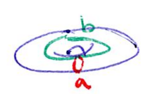
$+ n_2(a+b-c$

$= n(a+b-c)$

$\Rightarrow \operatorname{Im} \partial_2 = \langle a+b-c \rangle$

$\cong \langle a \rangle \times \langle b \rangle$

$\cong \boxed{\mathbb{Z} \times \mathbb{Z}}$ gen. by usual gen.



$$H_2(T) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\boxed{\langle U - L \rangle}}{0} \cong \boxed{\mathbb{Z}}$$

gen by "entire surface"

$n_1 U + n_2 L \in \ker \partial_2 \Longleftrightarrow n_1 = -n_2$

$\Rightarrow \ker \partial_2 = \langle U - L \rangle$

$H_p(T) \cong 0$

for $p \geq 3$.