Algebraic Topology - Homework 1

Problem 1. Show that the interval $[0, 1)$ is homeomorphic to $[0, \infty)$ by defining an algebraic function from one to the other.

Problem 2. Let a and b be real numbers. Show that any open interval, including (a, b), $(-\infty, b)$, and (a, ∞), is homeomorphic to $\mathbb{R}$.

Problem 3. Show that the open disk $\{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 < 1\}$ is homeomorphic to $\mathbb{R}^2$. Hint: Use your homeomorphism in Problem 1.

Problem 4. Show that the closed disk with the center point removed $\{(x, y) \in \mathbb{R}^2 |\ 0 < x^2 + y^2 \leq 1\}$ is homeomorphic to the closed disk with a smaller disk removed (half-open annulus) $\{(x, y) \in \mathbb{R}^2 |\ 1 < x^2+y^2 \leq 4\}$. Does this mean the removed point is homeomorphic to the removed disk? Explain.

Problem 5. Let A, B, C, and D be sets. Suppose $f : A \to C$ and $g : B \to D$ are homeomorphisms. Use f and g to define a homeomorphism between $A \times B$ and $C \times D$, and prove that it is a bijection (i.e. you do NOT need to show continuity.)

Problem 6. Give a geometric description of a homeomorphism from the sphere S^2 to the unit cube $C = \{(x, y, z) \in \mathbb{R}^3 |\ max\{|x|, |y|, |z|\} = 1\}$.

Problem 7. Give a geometric description of a homeomorphism from the cylinder $S^1 \times [0, 1]$ to the annulus $A = \{(x, y) \in \mathbb{R}^2 |\ 1 \leq x^2 + y^2 \leq 4\}$.

Problem 8. Give a geometric description of a homeomorphism from the sphere with the north pole removed to the plane $\mathbb{R}^2$.

Problem 9. Let X be the unit square region $\{(x, y) \in \mathbb{R}^2 |\ |x| \leq 1 \text{ and } |y| \leq 1\}$. We can assign an equivalence relation to X so that two points (x_1, y_1) and (x_2, y_2) in X are equivalent if:

- they are the same point, or
- $x_1 = x_2$ and $|y_1| = |y_2| = 1$

These equivalence classes can be considered as single points in a new space called Y.

(a) Draw a picture of X showing some typical equivalence classes.

(b) What simple shape does Y appear to be homeomorphic to? Explain.

Problem 10. Define an equivalence relation on the closed interval $[0, 1]$ so there is a simple homeomorphism between the set of equivalence classes and the circle S^1.

Problem 11. Let D be the closed disk $\{(x, y) \in \mathbb{R}^2 |\ x^2+y^2 \leq 1\}$. We can assign an equivalence relation to D so that two points (x_1, y_1) and (x_2, y_2) in D are equivalent if:

- they are the same point, or
- $x_1^2 + y_1^2 = 1, x_2^2 + y_2^2 = 1, x_1 = x_2$ and $y_1 = -y_2$

These equivalence classes can be considered as single points in a new space called Y.

(a) Draw a picture of D showing some typical equivalence classes.

(b) What simple shape does Y appear to be homeomorphic to? Explain.