Algebraic Topology - Homework 2

Problem 1. Show that the Klein bottle can be cut into two Mobius strips that intersect along their boundaries.

Problem 2. Show that a projective plane contains a Mobius strip.

Problem 3. The boundary of a disk D is a circle ∂D. The boundary of a Mobius strip M is a circle ∂M. Let $\sim$ be the equivalence relation defined by the identifying each point on the circle ∂D bijectively to a point on the circle ∂M. Let X be the quotient space $(D \cup M)/\sim$. What is this quotient space homeomorphic to? Show this using cut-and-paste techniques.

Problem 4. Show that the connected sum of two tori can be expressed as a quotient space of an octagonal disk with sides identified according to the word $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Then considering the eight vertices of the octagon, how many points do they represent on the surface.

Problem 5. Find a representation of the connected sum of g tori as a quotient space of a polygonal disk with sides identified in pairs. Do the same thing for a connected sum of n projective planes. In each case give the corresponding word that describes this identification.

Problem 6.

(a) Show that the square disks with edges pasted together according to the words $aabb$ and $aba^{-1}b$ result in homeomorphic surfaces. Hint: Cut along one of the diagonals and glue the two triangular disks together along one of the original edges.

(b) Vague question: What does this tell you about some spaces that you know?

Problem 7. Let T, P, and K denote the torus, projective plane, and Klein bottle, respectively

(a) Write $T\#P$ and $K\#P$ as polygonal disks with pairs of edges identified.

(b) Use cut-and-paste techniques to show that the two surfaces are homeomorphic.

(c) Bid farewell to the possibility of a cancellation law for the operation of connected sum of surfaces.

Problem 8. Determine if the Mobius strip, projective plane, Klein bottle, and dunce cap are manifolds, manifolds with boundary, or neither. Explain your reasoning and use the polygonal representations to assist your explanation by identifying appropriate neighborhoods around various points.

Problem 9. Which of the surfaces below are homeomorphic?