**Algebraic Topology - Homework 4**

*Problem* 1. Suppose $\alpha$ is any loop based at a point $x_0$ in a space $X$. Draw several stages of a homotopy to illustrate how $\alpha \cdot \alpha^{-1}$ is homotopic to the constant loop $\varepsilon$.

*Problem* 2. Let $G$ and $H$ be groups. Suppose $h : G \to H$ is a homomorphism. Let $e$ denote the identity of $G$.

(a) Show that $h(e)$ is the identity element of $H$.

(b) For any $x \in G$, show that $h(x^{-1}) = (h(x))^{-1}$.

(c) Show $h$ is injective if and only if $\ker(h) = \{e\}$.

*Problem* 3. Let $x_0$ and $x_1$ be points in a topological space $X$, and let $\gamma$ be a path from $x_1$ to $x_0$. Show that the isomorphism $h : \pi_1(X, x_0) \to \pi_1(X, x_1)$, as defined in class with $h(\langle\alpha\rangle) = \langle\gamma\alpha\gamma^{-1}\rangle$, is in fact an isomorphism. To do this, just show that $h$ is a homomorphism and then find an inverse of $h$ to conclude that $h$ is bijective.

*Problem* 4. Let $X$ and $Y$ be topological spaces, and let $x_0$ be a point in a $X$. Let $f : X \to Y$ be a continuous function, and let $\alpha$ and $\beta$ be loops in $X$ based at $x_0$.

(a) Show that $f \circ \alpha$ is a loop in $Y$.

(b) Suppose $\alpha$ and $\beta$ are homotopic in $X$. Use a homotopy between $\alpha$ and $\beta$ to construct a homotopy between $f \circ \alpha$ and $f \circ \beta$ in $Y$. Conclude that $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$, defined by $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$, is well-defined.

(c) Show that $f_*$ is a homomorphism.

*Problem* 5. Suppose that for any topological space $X$ we have an associated group $H(X)$. Suppose that for any continuous function $f : X \to Y$ we are able to get an induced homomorphism $f_* : H(X) \to H(Y)$ that satisfies the two properties: (1) $(id_X)_* = id_{H(X)}$ and (2) if $f : X \to Y$ and $g : Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$.

Consider the circle $S^1$ as the boundary of the disk $D^2$. Assume that $H(D^2) \cong \{0\}$ and $H(S^1) \cong \mathbb{Z}$.

(a) Suppose there is a continuous function $r : D^2 \to S^1$ with $r(x) = x$ for all $x \in S^1$. Describe the homomorphism $r_*$. (Since all other points on $D^2$ are continuously mapped to somewhere in $S^1$, this map $r$ retracts the entire disk onto its boundary. It is called a **retract** of $D^2$ onto $S^1$.)

(b) Consider the inclusion function $i : S^1 \to D^2$ with $i(x) = x$ for all $x \in S^1$. Describe the homomorphism $i_*$.

(c) What is $r \circ i$? What is $(r \circ i)_*$?

(d) Have you found a contradiction? If so, conclude that, if such a group $H$ could be defined, then no continuous function can retract a disk onto its boundary by leaving the boundary points fixed. If not, consider your life's worth and start the problem over again.

(e) Does such a group $H$ exist?

(f) How does the fact that there does not exit a retract of $D^2$ to $S^1$ that leaves $S^1$ fixed relate to deformation retractions?

*Problem* 6. Let $B^n$ be the $n$-ball.

(a) Determine $\pi_1(B^n)$ for $n \geq 2$. Hint: See proof for $\pi_1(B^2)$ since $B^2$ is a disk.

(b) Determine $\pi_1(S^n)$ for $n \geq 2$. Hint: See proof for $\pi_1(S^2)$.

*Problem* 7. Determine the fundamental group of the annulus, Mobius strip, and the solid torus.