Algebraic Topology - Homework 5

Problem 1. What is the inverse of ab in the free group $\langle a, b\rangle$? Describe the inverse of an arbitrary element in $\langle a, b\rangle$.

Problem 2. In the free group $\langle a, b\rangle$, let $c = ab$. Show that any element in $\langle a, b\rangle$ can be written in terms of a and c.

Problem 3.

(a) Describe the elements in the free group $\langle a\rangle$ with one generator. What familiar group is $\langle a\rangle$ isomorphic to?

(b) Describe the elements in the free group $\langle a, b, c\rangle$ with three generators.

(c) Draw a picture of $S^1 \vee S^1 \vee S^1$ and determine $\pi_1(S^1 \vee S^1 \vee S^1)$.

(d) Generalize this to compute $\pi_1(\bigvee_{i=1}^{n} S^1)$.

Problem 4. Draw a picture of $S^1 \vee S^2$ and compute $\pi_1(S^1 \vee S^2)$. Explain your reasoning.

Problem 5. Let X be the space obtained by removing the center point from D^2.

(a) Deformation retract X as much as possible. What do you get?

(b) Compute $\pi_1(X)$. Draw generators of the fundamental group as loops in X.

Problem 6.

(a) What is the lowest dimensional space that is a deformation retract of a disk with two holes?

(b) Draw a picture indicating this deformation retract.

(c) Compute the fundamental group of a disk with two holes. Explain your reasoning.

(d) Draw loops in the disk with two holes that represent ab, ba, bab^{-1}, and one of your favorite words in the free group on a and b.

Problem 7. Let X be the space obtained by removing from B^3 the line segment from the north pole to the south pole.

(a) Deformation retract X as much as possible. What do you get?

(b) Compute $\pi_1(X)$. Draw generators of the fundamental group as loops in X.

Problem 8. Let X be the space obtained by removing from $\mathbb{R}^3$ the unit circle in the xy-plane. Let Y be the space obtained by removing from $\mathbb{R}^3$ the unit circle in the xy-plane and the z-axis.

(a) Deformation retract X as much as possible. What do you get?

(b) Compute $\pi_1(X)$. Draw generators of the fundamental group as loops in X.

(c) Deformation retract Y as much as possible. What do you get?

(d) Compute $\pi_1(Y)$. Draw generators of the fundamental group as loops in Y.

Problem 9. Let X be the space obtained by removing from $\mathbb{R}^3$ two non-overlapping circles in the xy-plane.

(a) Deformation retract X as much as possible. What do you get?

(b) Compute $\pi_1(X)$. Draw generators of the fundamental group as loops in X.

Problem 10. Let X be the space obtained by removing a single point from the torus T^2.

(a) Deformation retract X as much as possible. What do you get?

(b) Compute $\pi_1(X)$. Draw generators of the fundamental group as loops in X.

Problem 11. Attempt to draw and describe in words each of the following spaces. Also determine their fundamental groups.

(a) $(S^1 \vee S^1) \times S^1$

(b) $S^2 \times S^1$

Problem 12. Without looking it up anywhere, what do you thing the fundamental group of a genus 2 surface is, i.e. $\pi_1(T^2 \# T^2)$? As we have not yet learned the theorem we will need to do this, just provide a logical guess. I don't care if your answer is right or wrong. Draw a picture indicating any generators of the fundamental group.