Algebraic Topology - Homework 6

Problem 1. Let X be the space obtained by removing two points from the torus T^2.

(a) Deformation retract X as much as possible. What do you get?

(b) Compute $\pi_1(X)$. Draw generators of the fundamental group as loops in X.

Problem 2. Let G be a group. Show that the commutator subgroup of G is a normal subgroup.

Problem 3. Let $\phi : G \to H$ be a homomorphism. Show that $\ker \phi$ is a normal subgroup of G.

Problem 4.

(a) Determine the commutator subgroup of $\mathbb{Z}$.

(b) What is the commutator subgroup of any abelian group?

(c) Determine the commutator subgroup of the symmetric group S_3.

Problem 5. Let P^2 be the projective plane. Use the Seifert-van Kampen Theorem to determine $\pi_1(P^2)$.

Problem 6. A single group can have many different group presentations. Even using the same generators, the relations can be expressed in different ways, and it is not always easy to see that two different presentations describe the same group. In the area of abstract algebra, the **word problem** for a group is the problem of deciding whether two different words in the generators represent the same element or whether two different presentations represent the same group.

(a) Let x, y, z be elements in a group such that $x^2 = y^2 = 1$ and $yz = zxy$. Show that $xy = yx$.

(b) Show that $\langle a, b \mid a^5 = 1, b^2 = 1, ba = a^2 b\rangle$ is isomorphic to $\mathbb{Z}_2$.

Problem 7. Let K^2 be the Klein bottle.

(a) Draw K^2 as a square with sides identified in the usual way, and use the Seifert-van Kampen Theorem to determine $\pi_1(K^2)$.

(b) Recall that $K^2 = P^2 \# P^2$. From this point of view, draw K^2 as a square with sides identified, and use the Seifert-van Kampen Theorem to determine $\pi_1(K^2)$.

(c) Is the group with presentation $\langle x, y \mid xyx^{-1}y \rangle$ isomorphic to the group with presentation $\langle a, b \mid a^2b^2 \rangle$? Explain.

(d) Can you prove your answer to (c) without using topology?

Problem 8. Use the Seifert-van Kampen Theorem to determine the fundamental group of the connected sum of n projective planes.

Problem 9. Let X_g be an orientable surface of genus g.

(a) Use the Seifert-van Kampen Theorem to determine $\pi_1(X_g)$.

(b) Is $\pi_1(X_g)$ an abelian group?

Problem 10. Consider two copies of the torus $S^1 \times S^1$. Let X be the space obtained by identifying a circle $S^1 \times \{x_0\}$ on one torus with the corresponding circle $S^1 \times \{x_0\}$ on the other torus.

(a) Draw a picture of X.

(b) Compute $\pi_1(X)$ using the Seifert-van Kampen Theorem.

(c) Compare your result with Homework 5, #11.