Algebraic Topology - Homework 9

Problem 1. Let A, B, and C be groups. Let α, ϕ, and β be homomorphisms so that the following sequence is exact.

$$A \xrightarrow{\alpha} B \xrightarrow{\phi} C \xrightarrow{\beta} D$$

Prove that the following statements are equivalent.

(a) α is surjective.

(b) ϕ is the zero homomorphism.

(c) β is injective.

Problem 2. Let G be a group, and let ϕ and ψ be homomorphisms so that the following sequence is exact.

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\phi} \mathbb{Z}_4 \xrightarrow{\psi} G \longrightarrow 0$$

Determine all of the possibilities for the group G. How many different exact sequences of this form are possible? Explain your reasoning.

Problem 3. Let G be a group, and let ϕ and ψ be homomorphisms so that the following sequence is exact.

$$0 \longrightarrow \mathbb{Z} \times \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \xrightarrow{\psi} G \longrightarrow 0$$

If $\phi(x, y) = (0, y, 3x)$, determine the group G. Explain your reasoning.

Problem 4. For each sequence below, determine if the sequence could be exact. Explain your reasoning.

(a) $0 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$

(b) $0 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow 0.$

Problem 5. Use the Mayer-Vietoris sequence to determine the homology groups of $P^2 \# P^2$, the connected sum of two projective planes. Use extreme caution when computing H_1.