In the case $p=1$, we get

$$0 \to H_1(S^n) \to \widetilde{H}_0(S^{n-1}) \to \widetilde{H}_0(B^n) \times \widetilde{H}_0(B^n) \to \cdots$$

$$\Rightarrow \quad 0 \to H_1(S^n) \xrightarrow{\cong} 0 \to 0$$

$$\Rightarrow \quad \boxed{H_1(S^n) \cong 0} \quad \text{for } n \geq 3 \quad (\text{also } n=2)$$

Know: $H_0(S^n) \cong \mathbb{Z}$
$H_1(S^n) \cong 0$
$H_n(S^n) \cong \mathbb{Z}$
$H_p(S^n) \cong 0$ for $p > n$
$H_p(S^n) \cong H_{p-1}(S^{n-1})$ for $p \geq 2$

So we have:

$H_2(S^n) \cong H_1(S^{n-1}) \cong 0$ if $n \geq 3$ ($\mathbb{Z}$ if $n=2$)

$H_3(S^n) \cong H_2(S^{n-1}) \cong 0$ if $n \geq 4$ ($\mathbb{Z}$ if $n=3$)

$H_4(S^n) \cong H_3(S^{n-1}) \cong 0$ if $n \geq 5$

$\vdots$

Needs induction $\to$ $H_p(S^n) \cong H_{p-1}(S^{n-1}) \cong 0$ if $n \geq p+1$

$$\Rightarrow H_p(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } p=0,n \\ 0 & \text{otherwise} \end{cases}$$

**Cor.** $S^n$ and $S^m$ are not homeomorphic if $n \neq m$.

Prove $H_p(S^n) \cong 0$ for $n \geq p+1$ by induction on $p$.

$p=1$: $H_1(S^n) \cong 0$ for $n \geq 2$ ✓

$k \Rightarrow k+1$:

Assume $H_k(S^n) \cong 0$ for $n \geq k+1$

Prove $H_{k+1}(S^n) \cong 0$ for $n \geq k+2$

We know for $p \geq 2$ that

$$H_p(S^n) \cong H_{p-1}(S^{n-1}).$$

If $k \geq 1$, then $k+1 \geq 2$ and

$$H_{k+1}(S^n) \cong H_k(S^{n-1})$$

Since $n \geq k+2$, we have

$$n-1 \geq (k+2)-1 = k+1$$

$\Rightarrow H_k(S^{n-1}) \cong 0$ by induct. hypoth.

$\Rightarrow H_{k+1}(S^n) \cong 0$.

∎

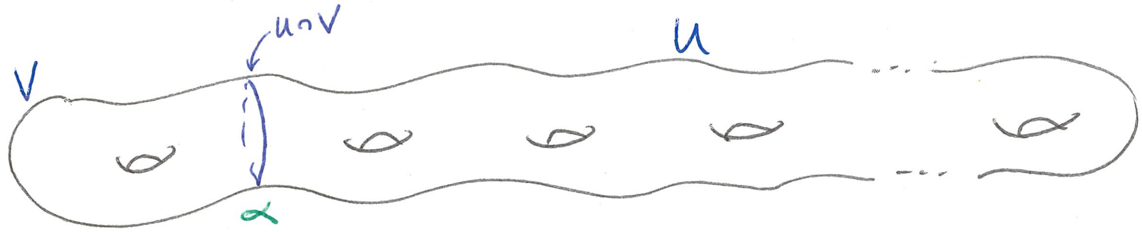
**Ex.** Let $M_g$ be a surface of genus $g$. Determine $H_p(M_g)$.

---

Notice $M_g = N \# T^2$, where $N$ has genus $g-1$.

Let $U = \underbrace{N - D^2}_{\approx \vee^{2g-2} S^1}$, $V = \underbrace{T^2 - D^2}_{\approx S^1 \vee S^1}$, then

$M_g = U \cup V$ and $U \cap V = S^1$




Mayer-Vietoris

$$0 \xrightarrow{\partial} \cancel{H_2(S^1)}^{\,0} \xrightarrow{\phi} \cancel{H_2(U)}^{\,0} \times \cancel{H_2(V)}^{\,0} \xrightarrow{\psi} H_2(M_g)$$

$$\xrightarrow{\partial} \underbrace{H_1(S^1)}_{\mathbb{Z}} \xrightarrow{\phi} \underbrace{H_1(U)}_{\mathbb{Z}^{2g-2}} \times \underbrace{H_1(V)}_{\mathbb{Z}^2} \xrightarrow{\psi} H_1(M_g) \to 0$$

$$\Rightarrow \quad 0 \xrightarrow{\psi} H_2(M_g) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}^{2g-2} \times \mathbb{Z}^2 \xrightarrow{\psi} H_1(M_g) \to 0$$

Since this is exact, $\psi$ is surjective

$$\Rightarrow \boxed{H_1(M_g)} \cong \frac{\mathbb{Z}^{2g-2} \times \mathbb{Z}^2}{\ker \psi} \cong \frac{\mathbb{Z}^{2g}}{\operatorname{im} \phi} \cong \boxed{\frac{\mathbb{Z}^{2g}}{0}}$$

If $H_1(S^1) \cong \mathbb{Z} \cong \langle \alpha \rangle$, then $\phi(\alpha) = (i_1(\alpha), \underbrace{-i_2(\alpha)}_{-(a+b-a-b)}) = (0,0)$

since $\alpha$ bounds in $U$ and in $V$.

We know $\boxed{\partial \text{ injective}}$ and $\operatorname{im} \phi = 0$

$\Rightarrow \ker \phi = \mathbb{Z} \Rightarrow \operatorname{im} \partial = \mathbb{Z} \Rightarrow \boxed{\partial \text{ is surj.}}$

$\Rightarrow \partial$ is an isom. $\Rightarrow \boxed{H_2(M_g) \cong \mathbb{Z}}$

Helpful example to help you when you compute $H_p(P^2 \# P^2)$ on homework.

(You cannot use this to compute $H_1(P^2\#P^2)$ on hw. You must use M-V sequence)

Ex. $\pi_1(\overset{g}{\#} P^2) = \langle a_1, \ldots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 = 1 \rangle$

$\cong \langle a_1, a_2, \ldots, a_{g-1}, a_1 a_2 \cdots a_g \mid a_1^2 a_2^2 \cdots a_g^2 = 1 \rangle$

When we abelianize, we get $\dfrac{\langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_{g-1} \rangle \times \langle a_1 \cdots a_g \rangle}{(a_1 a_2 \cdots a_g)^2}$

$$\Rightarrow \boxed{H_1(\overset{g}{\#} P^2) \cong \mathbb{Z}^{g-1} \times \mathbb{Z}_2}$$

When computing $H_1(P^2 \# P^2)$ using Mayer-Vietoris, you will need to consider a similar "change of generators".