Honors Math III Review Guide 2

November 9, 2010

1 Definitions

1. **Transformation:**
   A transformation from a vector space V to a vector space W is denoted by $T : V \to W$ and is a function with domain V and codomain W
   If $\vec{a} \in V$ and $T(\vec{a}) = \vec{b} \in W$
   then $\vec{b}$ is the **image** of $\vec{a}$ under (T)
   and $\vec{a}$ is the **preimage** of $\vec{b}$

2. **Linear Transformation:**
   A transformation is linear if:
   (1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and
   (2) $T(c\vec{u}) = cT(\vec{u})$

3. **Identity Transformation:**
   $T : V \to V$ so that $T(\vec{x}) = \vec{x}$
   Notation: I, I_V, T_1

4. **Zero Transformation:**
   $T : V \to V$ so that $T(\vec{x}) = \vec{0}$
   Notation: $Z, 0, 0_V, T_0$

5. **Scalar Transformation:**
   $T : V \to V$ such that $T(\vec{x}) = c\vec{x}$
   Notation: T_c

6. **Kernal (nullspace):**
   Let $T : V \to W$ be linear, then the kernal of T is:

$$\begin{aligned} ker(T) &= null(T) \\ &= \{\vec{x} \in V | T(\vec{x}) = \vec{0}\} \\ &= \text{ set of things that } T \text{ eats or kills} \end{aligned}$$

7. **Image (range):**
   The image of $T : V \to W$ is:

$$\begin{aligned} Im(T) &= image(T) \\ &= \{\vec{y} \in W | \exists \vec{x} \in V \textit{ such that } T(\vec{x}) = \vec{y}\} \\ &= \text{ set of vectors in W that get hit by } T \\ Im(T) &= T(V) \end{aligned}$$

8. **Nullity:**
   $T : V \to W$ is linear, then:
   $dim(ker(T)) = nullity(T)$

9. **Rank:**
   $T : V \to W$ is linear, then:
   $dim(Im(T)) = rank(T)$

10. **Isomorphism:**
    A linear transformation $T : V \to W$ is called an isomorphism if $\exists$ a linear transformation $S : W \to V$ such that $S \circ T(\vec{x}) = \vec{x} \quad \forall \vec{x} \in V$ and $T \circ S(\vec{y}) = \vec{y} \quad \forall \vec{y} \in W$

11. **Isomorphic:**
    V is isomorphic to W when T is an isomorphism

12. **Injective:**
    T is one-to-one or injective if
    $$x \neq y \Rightarrow T(x) \neq T(y)$$
    or equivalently,
    $$T(x) = T(y) \Rightarrow x = y$$

13. **Surjective:**
    T is surjective if $Im(T) = W$

14. **Bijective:**
    T is bijective if it is both surjective and injective

15. **Ordered Basis:**
    An ordered basis is a basis with a specific order of the elements

16. **Matrix of T relative to basis E:**
    The matrix of T relative to basis $E = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the matrix whose i^{th} column is the coordinates of $T(\vec{e}_i)$ and it is denoted by
    $$[T]_E = [T(E)]$$

17. **Dot Product:**
    Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ be in $\mathbb{R}^3$. The dot product of $\vec{x}$ and $\vec{y}$ is the real number $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3$

18. **Matrix Product:**
    Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the matrix product of $AB = [c_{ij}]$ is an $m \times p$ matrix where c_{ij} is the dot product of i^{th} row of A and j^{th} column of B.

19. **Upper-Triangular Matrix:**
    $A = [a_{ij}]$ such that $a_{ij} = 0$ if $i > j$
    $$A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

20. **Lower-Triangular Matrix:**
$A = [a_{ij}]$ such that $a_{ij} = 0$ if $i < j$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 4 & 6 & 3 \end{pmatrix}$$

21. **Diagonal Matrix:**
A matrix that is both Upper and Lower Triangular

22. **Nilpotent:**
A matrix A is called nilpotent (with index k) if $A^k = 0$ for some positive integer k

23. **Idempotent:**
A matrix A is called idempotent if $A^2 = A$

24. **Nonsingular:**
A square matrix A is called nonsignular (or invertible) if there is a matrix B such that:

$$AB = I = BA$$

25. **Involution, Involutory:**
A matrix A is called an involution or involutory if $A = A^{-1}$

26. **Transpose:**
The transpose of matrix $A = [a_{ij}]$ is $A^{tr} = [a_{ji}]$
Denoted $A^{tr} = A^t = A^T = tranpose(A)$

27. **Symmetric:**
A matrix A is symmetric if $A = A^{tr}$

28. **Skew-Symmetric:**
A is skew-symmetric if $A = -A^{tr}$

29. **The Matrix of T relative to A and B:**
Let $T : V \to W$ be linear, and let A be an ordered basis of V with $dim(V) = n$, and let B be an ordered basis of W with $dim(W) = k$. The matrix of T relative to A and B is the $k \times n$ matrix $[T]_A^B$ whose columns are the coordinates of $T(A)$ relative to B.

30. **Projection:**
A linear transformation $T : V \to V$ is a projection iff $T \circ T = T$

31. **Nilpotent Transformation:**
$T : V \to V$ is nilpotent of index k is $T^k = 0_V$ and $T^{k-1} \neq O_V$ for some k. (So k is the smallest such value)

32. **Cyclic:**
$T : V \to V$ is cyclic if $\exists \vec{x} \in V$ such that $\{\vec{x}, T(\vec{x}), T^2(\vec{x}), \ldots\}$ spans all of V and $\vec{x}$ is called a cyclic vector of T

2 Theorems

1. **Proposition 8.3.1:**
   If $T : V \to W$ is a linear transformation, then $T(\vec{0}) = \vec{0}$ $(T(\vec{0}_V) = \vec{0}_W)$

2. **Proposition 8.3.2:**
   If $T : V \to W$ is linear, then $T(a_1\vec{v}_1 + \ldots + a_n\vec{v}_n) = a_1T(\vec{v}_1) + \ldots + a_nT(\vec{v}_n)$

3. **Proposition 8.3.3:**
   If $T : V \to W$ is linear and U is a subspace of V and
   $T(U) = \{\vec{y} \in W | \exists \vec{x} \in U \ with \ T(\vec{x}) = \vec{y}\}$ then $T(U)$ is a subspace of W.

4. **Proposition 8.3.4:**
   Let $T : V \to W$ be linear, then if E is a subset of V then:

   $$T(Span(E)) = Span(T(E))$$

5. **Proposition 8.3.6:**
   Let $T : V \to W$ and $S : W \to U$ be linear.
   Then the composition:

   $$S \circ T : V \to W \to U$$

   is linear also.

6. **Proposition 8.4.1:**
   $T : V \to W$ is linear then:
   (a) $ker(T)$ is a subspace of V
   (b) $Im(T)$ is a subspace of W

7. **Dimension Theorem:**
   If $T : V \to W$ is linear and V is finite-dimensional, then:

   $$dim(V) = rank(T) + nullity(T)$$

8. **Proposition:**
   T is injective $\Leftrightarrow ker(T) = \{\vec{0}\}$

9. **Proposition 8.6.1:**
   Let $T : V \to W$ be linear.
   Then T is an isomorphism $\Leftrightarrow T$ is injective and surjective
   $\Leftrightarrow ker(T) = \{\vec{0}\}$ and $Im(T) = W$

10. **Theorem 8.6.4:**
    Let V and W be finite dimenstional vector spaces. then:

    $$V \cong (isomorphic) W \Leftrightarrow dim(V) = dim(W)$$

11. **Proposition:**
    If A is invertible, then A^{-1} is unique.

12. **Proposition 10.4.1:**
If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then A is invertible $\Leftrightarrow ad - bc \neq 0$
If A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

13. **Theorem:**
The set of all linear transformations $T : V \to W$ is a vector space and it is denoted $L(V, W)$

14. **Theorem:**
Let A be a finite ordered basis for V and let B be a finite ordered basis for W.
Let $T, U : V \to W$ be linear

(a) $[T + U]_A^B = [T]_A^B + [U]_A^B$
(b) $[cT]_A^B = c[T]_A^B$

15. **Theorem 11.2.1:**
Let V and W be vector spaces with $dim(V) = n$ and $dim(W) = k$ with ordered bases B and C respectively.
Then $\Phi : L(V, W) \to M_{k \times n}$ when $\Phi(T) = [T]_B^C$ is an isomorphism.

16. **Cor. 11.2.2:**
If V, W are finite dimensional vector spaces then $dim(L(V, W)) = dim(V)dim(W)$

17. **Proposition 11.2.3:**
Let V, W, U be finite dimensional vector spaces with ordered bases A, B, C respectively. If $T : V \to W$ and $S : W \to U$ are linear, then:

$$[S \cdot T]_A^C = [S]_B^C \cdot [T]_A^B$$

18. **Cor.:**
Let V, W be finite dimensional vector spaces with $dim(V) = n$ and $dim(W) = k$ and with ordered bases A, B respectively. If $T : V \to W$ is linear and the coordinates of $\vec{x} \in V$ relative to A are $x_1, x_2, \ldots, x_n$
Then the coordinates of $T(\vec{x})$ relative to B are $c_1, \ldots, c_k$ where

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = [T]_A^B \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

19. **Cor 11.2.4:**
$T : V \to W$ is an isomorphism $\Leftrightarrow [T]_A^B$ with A as a basis for V, B as a basis for W is invertible.

20. **Cor 11.2.5:**
If A, B are $n \times n$ matrices and $AB = I$ then $BA = I$

21. **Theorem 11.3.1:**
Let A, B be $k \times n$ matrices. Let V and W be vector spaces with $dim(V) = n$, $dim(W) = k$.
A and B represent the same transformation $T : V \to W$ relative to some ordered basis pairs $\Leftrightarrow A = PBQ^{-1}$ for some invertible matrices P and Q

22. **Theorem 12.2.1:**
    If $T : V \to V$ is nilpotent with index k and $\vec{x} \in V$ is a vector such that $T^{k-1}(\vec{x}) \neq 0$, then $\{\vec{x}, T(\vec{x}), T^2(\vec{x}), \ldots, T^{k-1}(\vec{x})\}$ is linearly independent.

23. **Cor.:**
    If $dim(V) = n$ and $T : V \to V$ is nilpotent with index k, then $k \leq n$. If $k = n$, then $\{\vec{x}, T(\vec{x}), \ldots, T^{k-1}(\vec{x})\}$ is a basis for V.

24. **Proposition 12.3.1:**
    If $T : V \to V$ is cyclic, $dim(V) = n$, and $\vec{x}$ is a cyclic vector of T then
    $\{\vec{x}, T(\vec{x}), T^2(\vec{x}), \ldots, T^{n-1}(\vec{x})\}$ is a basis for V.