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0010- Understand principles, properties, and relationships involving trigonometric functions and their associated geometric representations.

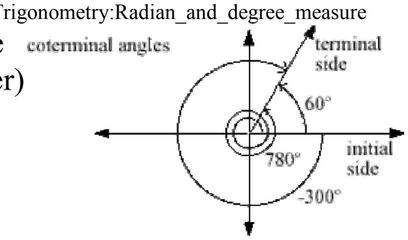
Let us review what makes and angle:

An angle is formed by the rotation of a ray about its endpoint. The *initial side* of the angle, is referred to as the starting position of the ray. The position the ray is in after it has made a rotation about the vertex (the endpoint of the ray) is called the *terminating side*. The rotation that is made from the initial side to the terminating side is referred to as an angle.

Coterminal Angles are angles that have the same terminal angle when they are graphed in standard positions (see right for example)

*These angles can be determined by adding k*360 to the original angle (where k is any positive or negative integer)

http://en.wikibooks.org/wiki/Trigonometry:Radian_and_degree_measure



There are two ways to measure angles (both are in respect to a circle):

- **Degrees**

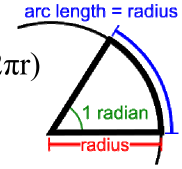
  The angle of measure equal to $\frac{1}{360}$ of a complete revolution

  i.e. 1 degree = $\frac{1}{360}$ revolutions (A circle is 360˚)

- **Radians**

  The angle of measurement created by wrapping the radius of a circle around its circumference

  i.e. $\Theta = \frac{s}{r}$, where **s** is arc length, **r** is radius of circle, and **Θ** is the angle of measurement (A circle is 2π radians, since the circumference of a circle is 2πr)



Degrees and Radians are directly related to each other in the following ways:

- Converting from Degrees to Radians:
  - Radians = $\frac{\pi}{180}$ * (Degree Measure)

    We derive this formula from knowing that 2π radians = 360˚

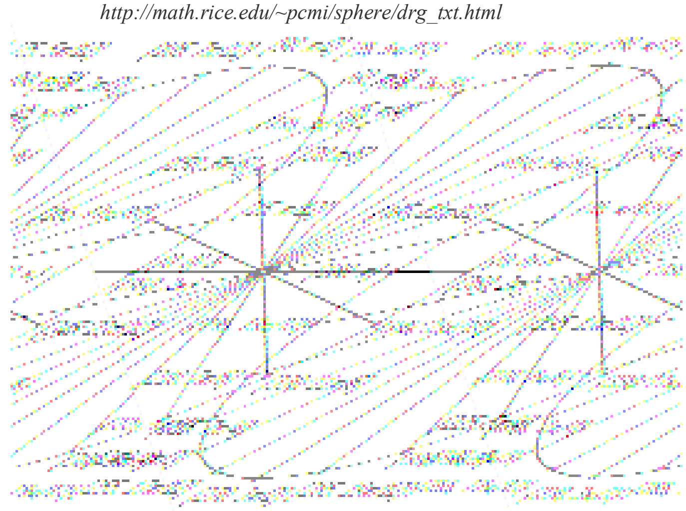
- Converting from Radians to Degrees:
  - Degrees = $\frac{180}{\pi}$ * (Radian Measure)

    We derive this formula by simply solving for degree measure in the previous formula

EQUIVALENCE BETWEEN

DEGREES AND RADIANS

http://math.rice.edu/~pcmi/sphere/drg_txt.html



- **Trigonometry** is "The branch of mathematics that deals with the relationships between the sides and the angles of triangles and the calculations based on them" (*The American Heritage Dictionary of the English Language, Fourth Edition)*

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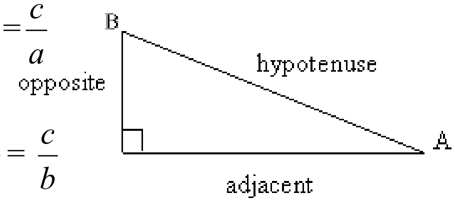
- **Trigonometric Functions:** functions of an angle formed by ratios of two sides of a right triangle
    - There are six common trigonometric functions. They are the following

$$* \textbf{ Sin A} = \frac{Opposite}{Hypotenuse} = \frac{a}{c} \qquad * \textbf{ Csc A} = \frac{1}{SinA} = \frac{Hypotenuse}{Opposite} = \frac{c}{a}$$



$$* \textbf{ Cos A} = \frac{Adjacent}{Hypotenuse} = \frac{b}{c} \qquad * \textbf{ Sec A} = \frac{1}{CosA} = \frac{Hypotenuse}{Adjacent} = \frac{c}{b}$$

$$* \textbf{ Tan A} = \frac{Opposite}{Adjacent} = \frac{a}{b} \qquad * \textbf{ Cot A} = \frac{1}{TanA} = \frac{Adjacent}{Opposite} = \frac{b}{a}$$

We can derive Tan A = $\frac{SinA}{CosB}$ by knowing the definition of Tan A, and then dividing the numerator and denominator by *Hypotenuse*.

Mnemonic Device For Remembering Trig Functions

SIN = **S**ome **O**ld **H**orse — CSC is the inverse of Sin
COS = **C**aught **A**nother **H**orse — SEC is the inverse of Cos
TAN = **T**aking **O**ats **A**way — COT is the inverse of Tan

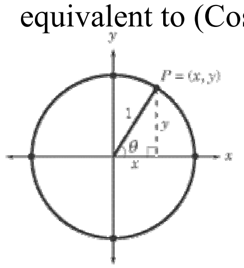
Relations between the Unit Circle and Trig Functions

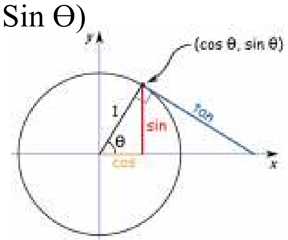
Unit Circle: A circle whose radius measure is 1 and centered at the origin (0, 0)

To see the relationship between the Unit Circle and the Trig Functions, we will arbitrary pick a point on the circle, say P= (x, y). Since we know that the radius is 1, we can determine that if we drop a perpendicular down from the point (x, y) to the x-axis, a right triangle is formed. Using the trig functions, we can see the relation that:

$$\sin\theta = \frac{y}{1} = y$$
$$\cos\theta = \frac{x}{1} = x$$
$$\tan\theta = \frac{y}{x}$$

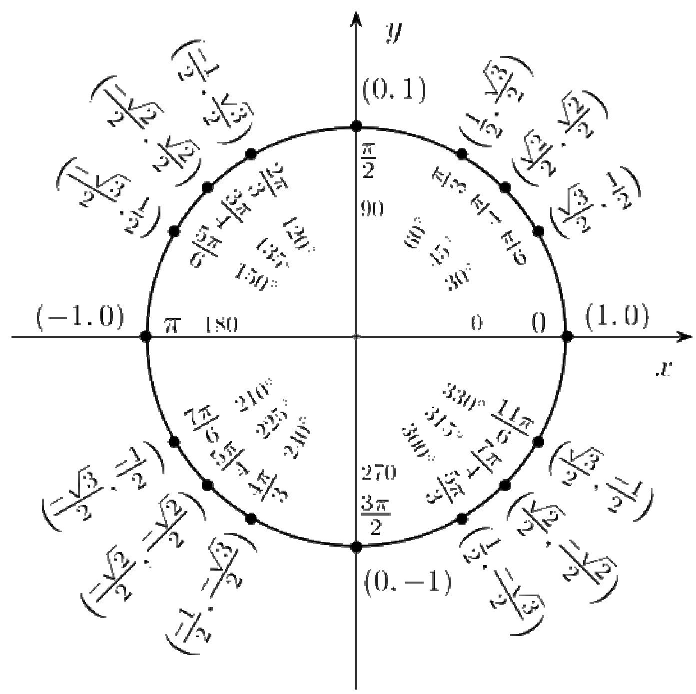
Hence, in the unit circle, the point (x, y) is equivalent to (Cos Θ, Sin Θ)





http://www.sparknotes.com/testprep/books/sat2/math2c/chapter9section6.rhtml
http://www.mathsisfun.com/geometry/unit-circle.html

Points of Interest on the UNIT CIRCLE



http://en.wikipedia.org/wiki/Trigonometric_function

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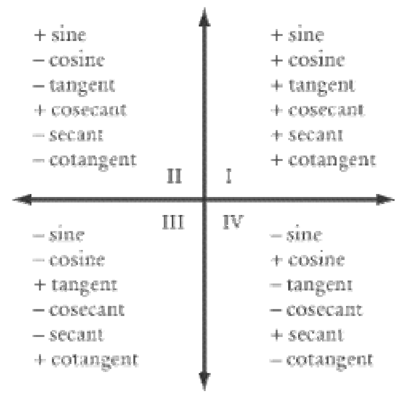
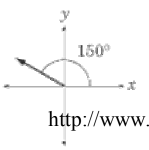
Simplifying Angles for Measurement

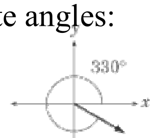
Angles that are larger than 90˚ cannot be measured on a regular calculator. Therefore, we must think of the angle on the coordinate plane, and then "simplify" the angle into the acute angle its quadrant, to calculate it. The measurement of the acute angle will be the same as the angle larger than 90˚, however the sign of the angle will change depending on what quadrant we are in.

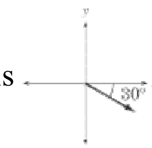


Example of angles "simplified" into acute angles:



simplified is


OR


simplified is


http://www.sparknotes.com/testprep/books/sat2/math2c/chapter9section1.rhtml

To determine the sign of the angle based on the quadrant it lies, we use the chart to the right.

*Easy way to remember: Beginning in quadrant I, and going counterclockwise, the following trig functions are positive I- **all** trig function, II- **Sin** and its inverse III-**Tan** and its inverse IV- **Cos** and its inverse i.e. **A**ll **S**tudents **T**ake **C**alc

- **CO-FUNCTIONS**
  - The trigonometric function of the complement of an angle

    Observe that in the right triangle ABC, with C being the right angle, angle A and B are complements of each other, and therefore total 90°. To calculate the measurement of angle B, we would do B=(90-A).

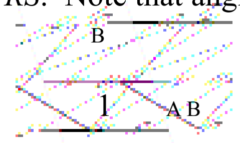
  We note that sinA = $\frac{a}{c}$ is equivalent to cosB. Hence sinA and cosB are cofunctions, and thus can be expressed as **sinA = cos(90-A).**
  We can similarly note other cofunctions as follows:

  tanA = cot(90-A) **secA = csc(90-A**) (*Note, $\frac{\pi}{2}$ can be interchanged with 90°)

- **TRIGONOMETRIC FORMULAS**
  - **Functions of the sum of TWO angles**

    sin (A + B) = sinAcosB + cosAsinB

Proof: "Let positive angles *A* and *B* be given, whose sum is less than 90 degrees. Construct segment *PU* with length 1. Construct triangle *TPU* so that angle *TPU* is equal to angle *A*, and angle *TUP* is equal to the complement of *A*. Construct the circumscribed rectangle *PQRS* so that angle *QPT* is equal to angle *B*, angle *QPU* is equal to the sum of angles *A* and *B*, *T* is on segment *QR* and *U* is on segment *RS*. Note that angle *RTU* is also equal to angle *B*." (http://staff.jccc.edu/swilson/trig/anglesumidentities.htm)



Using the trigonometric ratios, we claim that:
Sin(A +B) = UV/PU = UV (since PU is length 1), from the right triange UVP
Since UV = RT + TQ, we will solve for RT and TQ
From triangle TRU, we know that cosB = RT/UT, and hence RT = TUcosB
Similarly, from triangle TQP, we know that sinB = TQ/TP, and hence TQ = TPsinB
Therefore, by substitution, UV = TUcosB + TPsinB. Again, using the trigonometric ratios, we can see that from triangle UTP, sinA = TU/PU = TU and cosA = PT/PU = PT. Hence, we have that UV = sinAsinB + cosAsinB. Therefore, Sin (A +B) = sinAcosB + cosAsinB.

- Similary, we can derive the following sum and difference of TWO angle functions:
  cos (A + B) = cosAcosB + sinAsinB
  sin (A – B) = sinAcosB – cosAsinB
  cos (A – B) = cosAcosB + sinAsinB
- Knowing the proof for sin (A +B), we can use that to formulate other functions such as the double angle formulas and the half angle formulas
- It is also important to note some of the Pythagorean Identities:

  i.e. $\sin^2 A + \cos^2 A = 1$

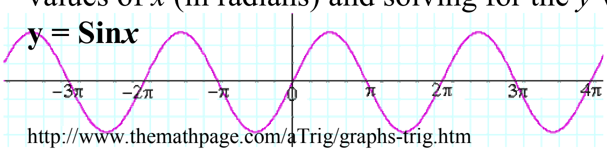
  Proof: By the Pythagorean theorem, we know $a^2 + b^2 = c^2$. Dividing everything by c^2, we have $(\frac{a}{c})^2 + (\frac{b}{c})^2 = 1$. From the trigonometric functions, we can conclude $\sin^2 A + \cos^2 A = 1$.
- Other Pythagorean Identities are easy to derive once we have the above formula. Some include

  $\tan^2 A + 1 = \sec^2 A$ and $\cot^2 A + 1 = \csc^2 A$

- **GRAPHING Trigonometric Functions**

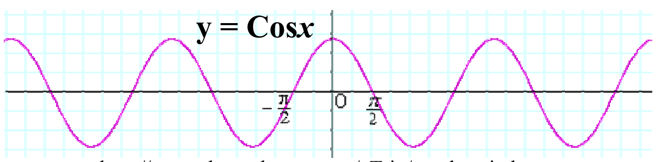
Trig Functions can be graphed by writing the function as an equation (i.e. *y* = sin*x*), and the substituting values of *x* (in radians) and solving for the *y* values

y = Sin*x*



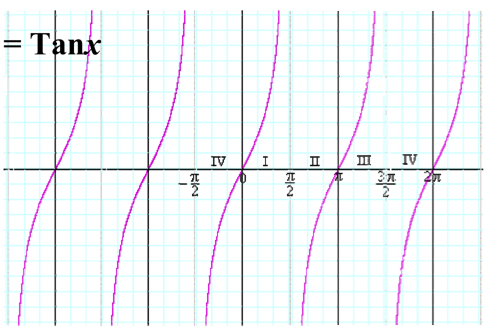
http://www.themathpage.com/aTrig/graphs-trig.htm

y = Cos*x*



http://www.themathpage.com/aTrig/graphs-trig.htm

y = Tan*x*



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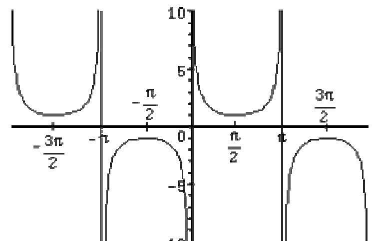
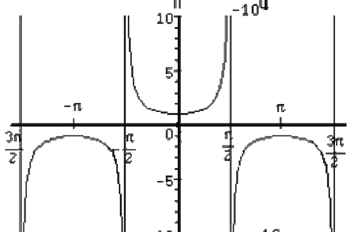
y = Csc*x*



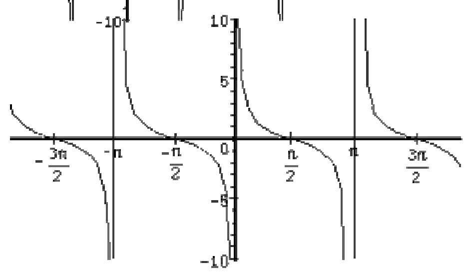
http://math.usask.ca/emr/examples/motf_graphs.html

y = Sec*x*



http://math.usask.ca/emr/examples/motf_graphs.html

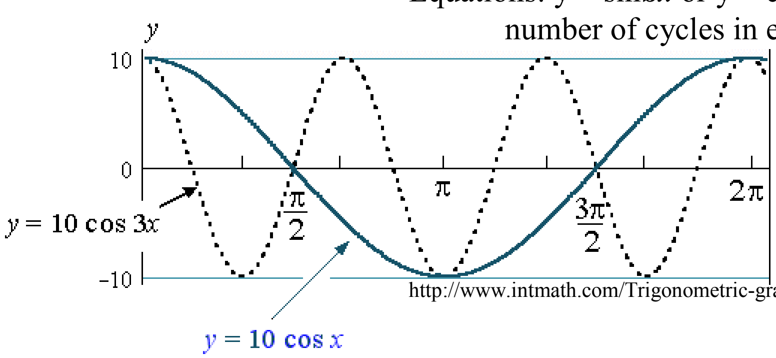
y = Cot*x*



http://math.usask.ca/emr/examples/motf_graphs.html

- The following are characteristics graphed trigonometric functions
  - **PERIOD:** the length of each cycle
  - PERIODIC: when the waves of a graph regularly repeat themselves
    - Sin and Cos curves are periodic with period 2π

      Equations: y = sin**b***x* or y = cos**b***x*, where **b** represents the number of cycles in each 2π

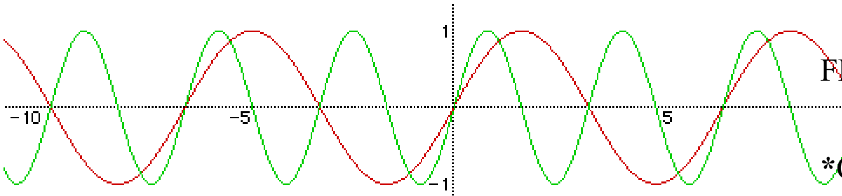


http://www.intmath.com/Trigonometric-graphs/2_Graphs-sine-cosine-period.php

$$\text{PERIOD} = \frac{2\pi}{b}$$

*Changing PERIOD on a cos curve

○ **FREQUENCY**: the number of completed cycles during a specific interval



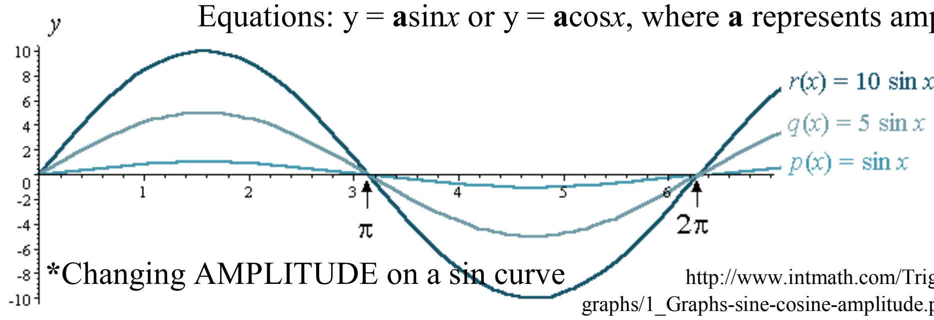
jwilson.coe.uga.edu/.../writeup1/writeup1.html

FREQUENCY = $\frac{1}{Period}$ *or* $\frac{b}{2\pi}$

*Changing FREQUENCY on a sin curve

○ **AMPLITUDE:** the maximum absolute value of a curve from the horizontal axis (the height of the graph)

Equations: y = **a**sin$x$ or y = **a**cos$x$, where **a** represents amplitude

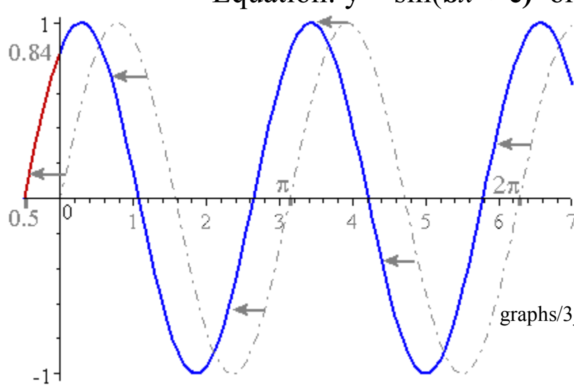


*Changing AMPLITUDE on a sin curve

http://www.intmath.com/Trigonometric-graphs/1_Graphs-sine-cosine-amplitude.php

○ **PHASE SHIFT**: the horizontal slide (left or right) of a graph from and original starting point

Equation: y = sin(**b**x + **c**) or y = cos(**b**x + **c**), where **b** and **c** represent the phase shift



PHASE SHIFT = $\frac{-c}{b}$

*PHASE SHIFT on a curve

http://www.intmath.com/Trigonometric-graphs/3_Graphs-sin-cos-phase-shift.php

- **A power series** with one variable is an infinite series of the following form:

$$f(x)\sum_{n=o}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

- Power series and trigonometric functions are connected in such a way that we can use power series to calculate the values of a trigonometric function (i.e. sin($\frac{3\pi}{17}$)), without the use of a calculator
- Power series representation of trigonometric function, where x is a number is radians (NOT DEGREES!) are as follows:

○ $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ for all x

○ $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ for all x

○ $\tan x = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3!} + \frac{2x^5}{15!} + \ldots$ for $|x| < \frac{\pi}{2}$

*Note: The further we continue the series, the more precise we will be with our calculations of the trigonometric functions

- The representation of these trigonometric functions is derived from Taylor expansion of these functions
- **Taylor Series** is of the following form:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$, where $f^{(n)}(a)$ denotes the nth derivative

- The **Complex Exponential Function** is expressed as e^z, where z represents the real value *x* in the power series, by a complex number
  - This can be represented as the following:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z - \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} - \frac{z^6}{6!} \ldots$$

  - Similarly, we can represent the trigonometric functions in complex form:

○ $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \frac{e^{ix} - e^{-ix}}{2i}$

○ $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \frac{e^{ix} + e^{-ix}}{2i}$

- From these functions, we can derive **Euler's Formula**, which says, for any real number *x*,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

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Questions:

1) What is 195° equivalent to in radians? What is 1.2π radians equivalent to in degrees? Show work!

2) Suppose ΔABC is a right triangle with $\angle C$ at the right angle. Let side AC=10in. and AB=17in. Find the measure of $\angle A$ using the sin ratio. Find the measure of $\angle B$ using the cos ratio. Put your answers to the nearest tenth.

3) If sinA<0 and tanA=$\frac{4}{5}$, which quadrant does the terminating side of A lie? Explain.

4) Prove the following is an identity:

$$\cot x = \frac{\sin 2x}{1 - \cos^2 x}$$

5) Prove that the following functions are cofunctions:

$$\csc A = \sec(\frac{\pi}{2} - A)$$

6) a) Sketch the graph of the equations y = sin$x$ in the interval $0 \le x \le 2\pi$
   b) Sketch the graph of the equation y = 2sin(4x) $0 \le x \le 2\pi$
   c) Describe the changes that have been made from the first graph to the second graph

7) Calculate $\sin\frac{\pi}{6}$ using the power series for n=3. Compare this value to the value of $\sin\frac{\pi}{6}$ given by the calculator.

8) Substituting π for θ in Euler's Formula, what value do we get? Solving this equation for 0, explain why these numbers are the 5 most important mathematical numbers.

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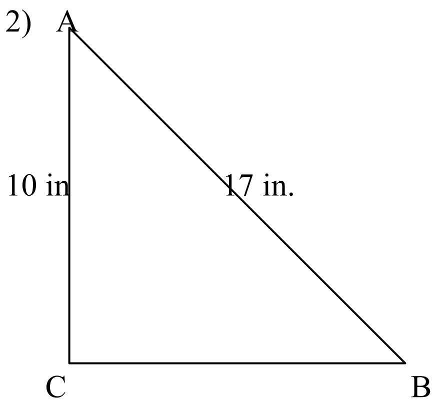
Answers:

1) Radians = $\frac{\pi}{180}(195°) = \boxed{\frac{13\pi}{12}\ Radians}$

Degrees = $\frac{180}{\pi}(1.2\pi) = \boxed{216°}$

2)



Using Pythagorean Theorem:

$AB^2 = AC^2 + CB^2$

$CB^2 = AB^2 - AC^2$

$CB = \sqrt{AB^2 - AC^2}$

$CB = \sqrt{17^2 - 10^2} = 13.747727$

sinA= $\frac{opposite}{hypotenuse} = \frac{13.747727}{17}$

A = $\sin^{-1}(.808689823) = \boxed{53.97°}$

cosB= $\frac{adjacent}{hypotenuse} = \frac{13.747727}{17}$

B = $\cos^{-1}(.808689823) = \boxed{36.03°}$

3) Quadrant III. Since sinA < 0, we know that the terminating angle will either lie in Quadrants III or IV, since those are the only two quadrants where sin in negative. Similarly, since tanA is positive, the terminating angle will either lie in Quadrants I or III. Since both must be true, we can determine that the terminating angle will lie in Quadrant III.

4) $\frac{\sin 2x}{1-\cos^2 x} = \frac{2\sin x\cos x}{1-(1-2\sin^2 x)}$ (by using the double angle formulas)

$= \frac{2\sin x\cos x}{2\sin^2 x}$

$= \frac{2\sin x\cos x}{2\sin x\sin x}$

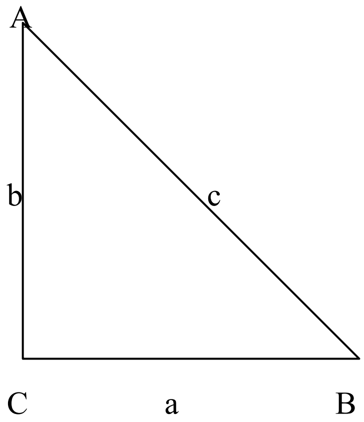
$= \frac{2\sin x\cos x}{2\sin x\sin x}$

$= \frac{\cos x}{\sin x}$

$= \cot x$

Hence, $\cot x = \frac{\sin 2x}{1-\cos^2 x}$

5)



Since ABC is a right triangle, with C being the right angle, we know angle A + angle B = $\frac{\pi}{2}$. Hence,

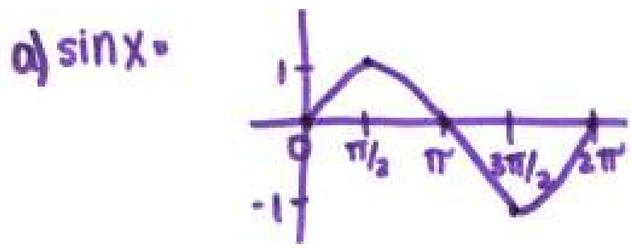
angle B = $\frac{\pi}{2}$ - angle A. We note that

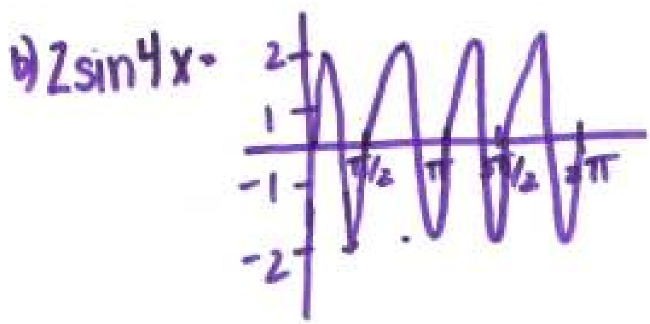
$\csc A = \frac{1}{\sin A} = \frac{1}{\frac{a}{c}} = \frac{c}{a}$. Similarly, we note

$\sec B = \frac{1}{\cos B} = \frac{1}{\frac{a}{c}} = \frac{c}{a}$. Since cscA=secB, we have

proven that cscA = ($\frac{\pi}{2}$-A).

6)





In the first graph the amplitude is 1, where in the second graph the amplitude is 2. Also in the first graph the period is 2π since b=1. In the second graph, the period changes to $\frac{\pi}{2}$, since b=4 and Period = $\frac{2\pi}{b}$

7) $\sin\frac{\pi}{6} = \sum_{n=0}^{3} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$$= (\frac{\pi}{6}) - (\frac{(\frac{\pi}{6})^3}{3!}) + (\frac{(\frac{\pi}{6})^5}{5!}) - (\frac{(\frac{\pi}{6})^7}{7!})$$

$$= (\frac{\pi}{6}) - (\frac{\pi^3}{1296}) + (\frac{\pi^5}{933120}) - (\frac{\pi^7}{1410877440})$$

$=.499999991869$

According to the calculator, $\sin\frac{\pi}{6} = .5$

By carrying the Power Series out 4 times, we got a very accurate answer to the true value of $\sin\frac{\pi}{6}$. If we were continue the series farther, out answer would become even more precise.

8) $e^{i\theta} = \cos\theta + i\sin\theta$

Substituting π, we get $e^{i\pi} = \cos\pi + i\sin\pi$. Since we know that $\cos\pi = -1$ and $\sin\pi = 0$ from the unit circle, we have that $e^{i\pi} = -1$. Solving this equation for 0, we get $e^{i\pi} + 1 = 0$. Hence, we have the 5 most important mathematical numbers.

e is important since it is the base of log
i is important since it is $\sqrt{-1}$
π is important since it is the ration of circumference to diameter
1 is important since it is the multiplicative identity
0 is important since it is the additive inverse

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References:

Webpages:

http://en.wikibooks.org/wiki/Trigonometry:Radian_and_degree_measure

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