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INTD 301 Final Project
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0017- Understanding and Using Vector and Transformational Geometries

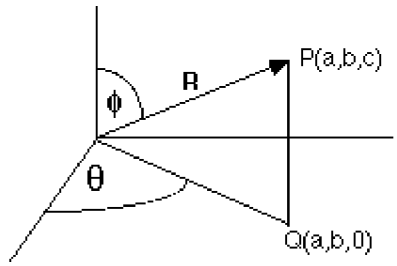
Vector geometry:

3-D Cartesian coordinate representation:

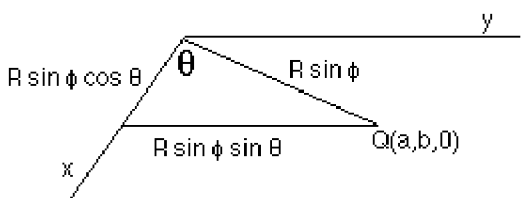
- A vector **v** is written as **v** = a**i** + b**j** + c**k** or <a, b, c>, where **i, j**, and **k** indicate the *x*, *y*, and *z* direction, respectively, of the vector with the beginning of the vector at the origin and the endpoint of the vector at (a, b, c) in three-space. For example, a vector **v** = 2**i** + 3**j** + **k** would be represented in the Cartesian plane as a vector starting at the origin and ending at the point (2, 3, 1).
- Vector equations are determined by putting the starting point of the vector at the origin and finding the (a, b, c) coordinates of the endpoint. This vector may then be translated by shifting its position, but its equation remains the same. So, the vector from (0, 0, 0) to (-2, 4, 1) and the vector from (3, -2, 3) to (1, 2, 4) are the same vector, namely **v** = -2**i** + 4**j** + 1**k**.
- The length of the vector, or magnitude written |**v**|, can be found using the distance formula: |**v**| = (a^2 + b^2 + c^2) ^ (1/2)
- Having a unit vector (vector of length one) may be useful in some situations. To change a vector into a unit vector simply multiply the vector by the scalar 1 / |**v**|. Thus, **v / |v|** is a unit vector.
- Adding/ subtracting vectors:
  - Algebraically- Complete this in a piecewise fashion:
    (a**i** + b**j** + c**k**) + (d**i** + e**j** + f**k**) = (a + d)**i +** (b + e)**j** + (c + f)**k**
  - Geometry of vector addition- Two methods
    - 1) Start with one vector. Put the beginning of the second vector at the end of the first vector. Now draw the sum vector from the beginning of the first vector to the end of the second vector
    - 2) Put the tails of both vectors together, this forms half of a parallelogram. Complete the parallelogram with copies of the vectors making up the other side. Now, draw the sum vector from the intersection of the tails of the original vectors to the intersection of the ends of the copied vectors (the diagonal of the parallelogram).
- Scalar multiplication of vectors:
  - Algebraically- Use the distributive law (similar to polynomials)
    t (a**i** + b**j** + c**k**) = ta**i** + tb**j** + tc**k**
  - Geometry of scalar multiplication- Extending or shrinking the length of the vector by the scalar. *This does not change the direction of the vector.

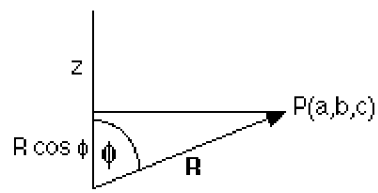
Spherical Coordinate representation (Mukherjee, 1995):

- a vector **v** is named by its length or magnitude (R), the angle between **v** and the z-axis (Φ), and the angle between **v**'s projection onto the xy plane and the x-axis (_). This is illustrated by the following picture:



- "By looking at the right triangles formed by the vector, its projection, and the axes, the relationship between the Spherical coordinates and Cartesian ones can be determined. It turns out that x = R sin(Φ) cos(_); y = R sin(Φ) sin(_); and z = R cos(Φ)" (Mukherjee).





How can we use this? Force Problems (bring on the physics!)

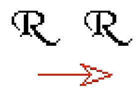
Say we are given an object that has many forces being applied to it. We can construct coordinates around the object, with it being at the origin. We can then use vector addition and scalar multiplication to find the total force being applied to the object. You may then be able to find (depending on the problem) the ending position of the object, velocity of the object, or other related data.

Ex. You are playing soccer with a friend. You and your friend kick the ball simultaneously. You kick it with a force equal to 20ft north and 6ft west and your friend kicks it with a force of 4ft north and 32ft east. At the same time the wind pushes the ball with a force equal to 5ft south. How far did the ball move?

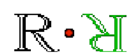
Solutions: Turn each force into vectors with the ball placed at the origin of a coordinate plane. We will have **i** as the *x* direction and **j** as the *y* direction. So we have the vectors –6**i** + 20**j**, 32**i** + 4**j**, -5**j**. Now take their sum: (–6**i** + 20**j**) + (32**i** + 4**j**) + (-5**j**) = 26**i** + 19**j**. So, the ball with end 19ft north and 26ft east of where it started.

Transformations:

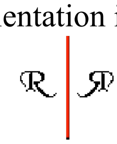
"A transformation is a one-to-one mapping on a set of points. The most common transformations map the points of the plane onto themselves, in a way that keeps all lengths the same. These transformations are called isometries. Another common sort of transformation which does not preserve lengths are dilatations. There are four isometries in the plane: translations, rotations, reflections, and glide reflections" (*Transformations*).

- Translations: This is sliding the figure. This does not change the shape, size, or angle of the figure. See example: 

  - How to translate a figure with points (*x, y*) by a vector <a, b>:
    - Graphically, you slide the points of the figure on the vector.
    - Add piecewise the vector coefficients to the coordinates of the points of the figure: (*x, y*) _ (a + *x*, b + *y*).

- Rotations: This is rotating the figure around an axis. This does not change the shape or size of the figure but it does change the figure's angle. See example where the red dot is the axis and there is a 180 degree rotation: 

  - How to rotate a figure with points (*x, y*) by _ degrees:
    - Graphically, construct a line segment from your point (*x, y*) and the axis point. Rotate the line segment by _. Now, the end of the line segment is your transformed point.
    - You can also do this using matrix multiplication. Represent the points of your figure as the 1 by 2 matrix [[x], [y]] and multiply it on the right by the 2 by 2 matrix [[cos _, -sin _][sin _, cos _]] (*Geometry of Linear Transformations of the Plane*).

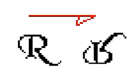
- Reflections: This is flipping an object over a line of reflection. This does not change the shape or size of the figure but it does change its orientation in the plan. See example where the red line is the line of reflection: 

  - How to reflect a figure with points (*x, y*) over a line of reflection *y* = m*x* + b
    - Graphically, construct a line, *l*, perpendicular to the line of reflection through your point (*x, y*). Now, measure the distance, *d*, from the intersection of *l* and the line of reflection (call it (*x*1, *y*2)) to the point on the figure (*x, y*). Draw your

    new transformed point on line *l*, *d* distance from the (*x*1, *y*2) on the opposite side of the line of reflection.

    - This can also be accomplished using matrix multiplication. First, represent the points in your figure as the 1 by 2 matrix [[x], [y]]. Using a few points in your figure, find the 2 by 2 matrix that you multiply [[x], [y]] on the right by to get your new point [[x1, y1]]. This can be accomplished by using specific example points, observation, guessing and checking, or solving the matrix equation. For example, if you wanted to rotate a figure over the *x*-axis you would multiply by the matrix [[1, 0][0, -1]].

- Glide reflections: This is a combination of a translation and a reflection over a line parallel to the direction of the translation. This does not change the size or shape of the object, but it does change the object's orientation. See example with the red vector being the direction of translation:



  - How to glide reflect a figure with points (*x, y*) and a translation vector of <a, b, c>:
    - First translate the figure by the given vector and then reflect it over a line parallel to the translation vector going through the center of the figure (both processes are described above).

- Dilations: This makes a figure larger or smaller. This does not change the shape or orientation of the figure, but it does change the size of the figure.

  - How to dilate a figure with points (*x, y*) and a scale factor *k*:
    - Graphically, construct a line *l* from the origin going through your point (*x, y*). Find the distance between (0, 0) and (*x, y*) with the distance formula. Multiply this distance by the scale factor, obtaining a new distance. Draw the new transformed point on *l* that is this newly found distance from the origin.
    - Multiply the points by the scale factor: (*x, y*) __ (*kx, ky*)
    - You may also multiply the point in matrix form by the matrix [[*k*, 0][0, *k*]] on the right.

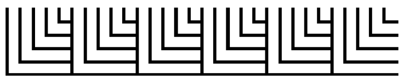
These transformations can be used in succession to solve problems. To do this you can use any of the previously described methods in succession. Be careful to do the transformations in the correct order, as these transformations are not commutative. This can be seen if you use the matrix multiplication representation since we know that matrix multiplication is not commutative.

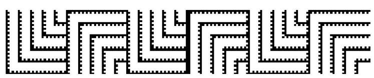
For example: Lets take the point (1,1).

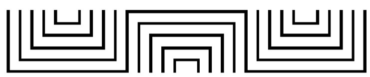
- Case 1: Reflect (1, 1) over the line x = -y and get (-1, 1). Then, rotate (-1, 1) 90-degrees around the origin. This gives us (1, -1).
- Case 2: Rotate (1, 1) 90-degrees around the origin and get (-1, 1). Then, reflect (-1, 1) over the line x = -y and get (-1, 1).
    - As you can see even with this very simple example these transformations are not commutative.

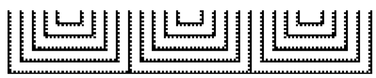
Where are transformations used? Frieze Patterns (art comes in to the picture!)

Frieze patterns are linear patterns created using multiple transformations on a design or pattern in succession. They are commonly seen in art and architecture (such as in wallpaper borders or molding). Lets look at some examples (taken from *Frieze Patterns*):

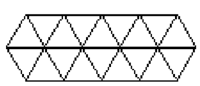
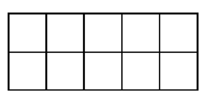
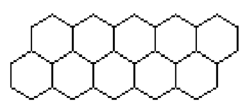
Translation: 

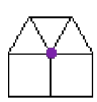
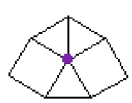
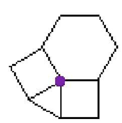
Glide reflection: 

A reflection and a half turn: 

Two parallel reflections: 

Transformations are also seen in tiling or tessellation (a pattern created by repeated a shape, or shapes, over and over again covering a plane without any gaps or overlaps). Here are some examples (taken from *What is a Tessellation?*):

Regular Tessellations:   

Semi-regular Tessellations: 
3.3.3.4.4

3.3.4.3.4

3.4.6.4

For more information on Vector and Transformational Geometry check the following sources that I have referenced throughout this project:

Erzberger, Andria. *Vectors in Particle Physics*. http://quarknet.fnal.gov/toolkits/ati/vectors.html

Geometry of Linear Transformations of the Plane. http://www.math.hmc.edu/calculus/tutorials/lineartransformations/

Mukherjee, Subrata. *Geometry of Vectors*. http://omega.albany.edu:8008/calc3/3d-geom-dir/cornell-lecture.html

Reid, David A. *Frieze Patterns.* http://plato.acadiau.ca/courses/educ/reid/Geometry/Symmetry/frieze.html. April 4, 2003.

Transformations. http://plato.acadiau.ca/courses/educ/reid/Geometry/Symmetry/Transformations.html.

What Is a Tessellation? http://mathforum.org/sum95/suzanne/whattess.html. 1994-2007 Drexel University.

Problems:

1. Two forces, one 120lbs and the other 200lbs, act on a body and make a 52-degree angle with each other. What's the magnitude of the resultant of the forces and what is the measure, to the nearest degree, of the angle that it makes with the 200lb force?

2. A flock of birds is preparing to migrate south for the winter. The winds at the altitude they fly at are from the northwest (125 degrees from due east) with wind velocities averaging at 10 miles/hr. What direction should they actually fly in order to end up due south if they fly at 19 miles/hr?

3. From the previous problem: How long will it take them to get to their destination that is 2,000 miles away?

4. Show that for any parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of the sides.

5. Take a triangle with corners at (1,2), (3,4), and (3, 1). Using algebra or a graph do the following transformations in order:

i. Dilation of scalar 3
ii. Rotation of 90 degrees about the origin
iii. Reflection over y = x.
iv. Translation through the vector <-5, 20>

6. Take the function y = x^2 and perform the following transformations in order using matrix multiplication.

i. Rotation of 180 degrees
ii. Reflection over $y = -x$
iii. Dilation of _

7. You have an equilateral triangle with numbered corners in front of you. Suppose that the only thing you are allowed to do with it is rotate the triangle until it is unto itself again. How is this activity similar to addition in base 3?

8. Look in art, history, or your everyday life and find at least 7 examples of frieze patterns. Determine what transformations were used to make them.

Solutions to Review Problems:

1. Solution: Start with a coordinate plane. Position the 120lbs force with a direction that is purely in the *x*-direction. Thus, this vector is **v** = 120**i**. The second vector's beginning is then placed at the end of the first vector. This vector, when you observe the triangle it forms, is **u** = 200 cos(52)**i** + 200 sin(52)**j**. Now, find the resultant by adding the vectors: **v** + **u** = (120 + 200 cos(52))**i** + 200 sin(52)**j** = 243.132**i** + 157.602**j**. The magnitude of the vector is Sqrt(243.132^2 +157.602^2) = 289.744. The angle between **v** + **u** and **v** can be found by observing that cos _ = (120 + 200 cos(52))/289.744 = .839 or _ = 32.95.

2. Solution: Translate the wind and the bird's flight into vectors. Let **w** be the wind vector with |**w**| = 10 and it has an angle of 125 degrees. We then find that the end point has coordinates *x* = 10 cos(55) = 5.73 and *y* = -10 sin(55) = -8.19. This means that **w** = 5.73**i** – 8.19**j**. Now, let **b** be the bird's actual flight vector. We know that |**b**| = 19 and has an angle of _. Similarly, we can find that **b** = -19 cos _ **i** –19 sin _ **j**. Now, we want the resulting vector to have an **i** component of 0 so we add the **i** components of **w** and **b** and set it equal to 0. So, we want 5.73 -19 cos _ = 0 _ 5.73 = 19 cos _ _ .301 = cos _ _ _ = 72.48. Looking at a picture we see that the angle we want is actually 180 + 72.48 = 252.48. So the birds must fly 252.48 degrees from due east to fly directly south.

3. Solution: We know that the birds are flying with an angle of 252.48 degrees. So, the **j** component of **b** is 18.1186. So the sum vector of **w** and **b** has **j** component of 8.19 + 18.1186 = 26.3086. We have found that the birds travel directly south at a rate of 26.3 miles/ hour. Now we need to know how long it would take to go 2,000 miles. To do this simply divide 2,000 miles / (26.3 miles/ hour) = 76 hours.

4. Solution: Let vectors **v** = a**i** + b**j** and **u** = c**i** + d**j**.
So |**v**| = Sqrt(a^2 + b^2) and |**u**| = Sqrt(c^2 +d^2).
Lets add these vectors using the geometric parallelogram method. This way **v** and **u** are the sides of the parallelogram and **u** + **v** and **u** – **v** are the diagonals.
So, we have that **u** + **v** = (a + c)**i** + (b + d)**j**, which is one of the diagonals.
Also, |**u** + **v**| = Sqrt((a + c)^2 + (b + d)^2) = Sqrt(a^2 + 2ac + c^2 + b^2 + 2bd + d^2)
The other diagonal is **u** - **v** = (a - c)**i** + (b - d)**j**
It follows that |**u** - **v|** = Sqrt((a - c)^2 + (b - d)^2) = Sqrt(a^2 - 2ac + c^2 + b^2 - 2bd + d^2)
So, the sum of the squares of the diagonals is |**u** + **v**|^2 + |**u** - **v**|^2
= [Sqrt(a^2 + 2ac + c^2 + b^2 + 2bd + d^2)]^2 + [Sqrt(a^2 - 2ac + c^2 + b^2 - 2bd + d^2)]^2
= a^2 + 2ac + c^2 + b^2 + 2bd + d^2 + a^2 - 2ac + c^2 + b^2 - 2bd + d^2
=2a^2 + 2c^2 + 2b^2 + 2d^2
Now, the sum of the squares of the sides is 2[|**v**|^2] + 2[|**u**|^2]
=2[Sqrt(a^2 + b^2)]^2 + 2[Sqrt(c^2 +d^2)]^2

=2[(a^2 + b^2)] + 2[(c^2 +d^2)]
=2a^2 + 2c^2 + 2b^2 + 2d^2

Finally, we have that 2[|**v**|^2] + 2[|**u**|^2] = 2a^2 + 2c^2 + 2b^2 + 2d^2 = |**u** + **v**|^2 **+** |**u** - **v**|^2

5. Solution:

i. Multiply each point by the scalar 3: (1,2) _ (3,6); (3,4) _ (9,12); (3,1) _ (9,3)
ii. Draw a picture. We see that if we pivot this triangle about the origin 90 degrees the *x* and *y* coordinates of the points switch and the new *x* coordinate is negated. Thus, we now have (3,6) _(-6,3); (9,12) _(-12,9); (9,3) _(-3,9)
iii. To reflect over this line we simply switch the *x* and *y* coordinate: (-6,3) _ (3,-6); (-12,9) _ (9,-12); (-3,9) _(9,-3)
iv. Add the vector coefficients to the corresponding coordinates: (3,-6) _(-2,26); (9,-12) _(4, 8); (9,-3) _(4, 17)

6. Solution:

i. We have that y = x^(2) can be represented as [[x][y]]. We want to multiply this on the right with [[cos 180, -sin 180][sin 180, cos 180]] or [[-1,0][0,-1]]. This yields –y = (-x)^2 or y = -(x^2).
ii. Now, multiply on the right by the matrix [[0, -1][1,0]] and get –x = -(y^2) or x = y^2.
iii. Multiply on the right by the matrix [[1/2,0][0,1/2]] and get x/2 = (y/2)^2 or x/2 = (1/4)y^2 or x = (1/2)x^2

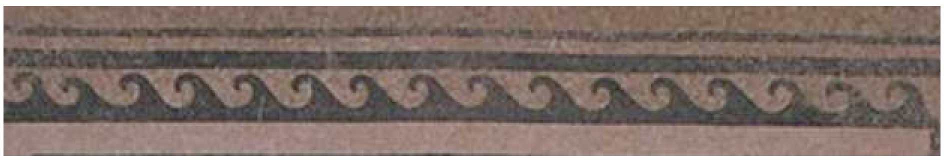
7. Solution: Answers may vary but here are some that I came up with…

- Given these rules you only have 3 unique moves: rotate the triangle 0, 120, or 240 degrees. In base 3 we only have 3 unique elements as well, namely 0, 1, 2.
- There is a one to one correspondence between these two systems. We can see this when we map 0 degrees to 0, 120 degrees to 1, and 240 degrees to 2. If we follow the rules of each system the are identical
    - 2 + 2 = 1 in base 3 and rotation of 240 followed by a rotation of 240 is equivalent to a rotation of 120.
    - 0 in base 3 and a 0 degree rotation are the additive identities of the systems.
    - This works with all additive combinations, which can be seen in a multiplication chart.

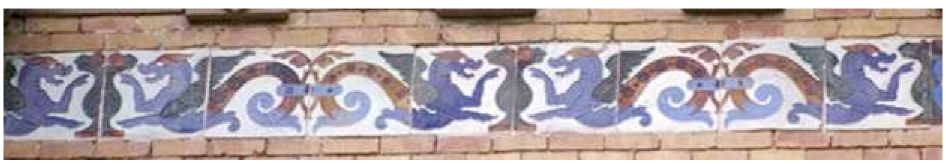
Base 3, +	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Rotations in succession	0	120	240
0	0	120	240
120	120	240	0
240	240	0	120

8. Solution: Obviously this will vary. Most important is the correct determining of the transformations used. Here are some I found:

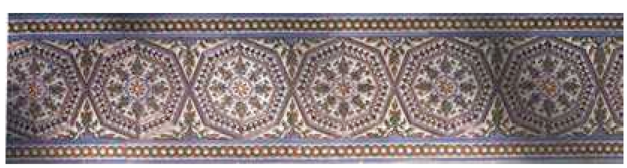


Translation

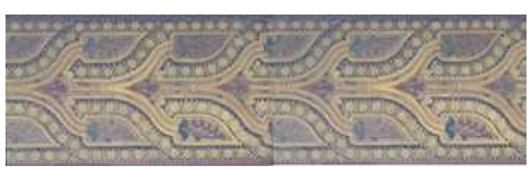


Reflection

Translation with a 90-degree rotation



Reflection or Translation



Reflection over a horizontal line and translation