In the beginning was nothing. And so we start. Let's go back to basics, very basics. First let's think about numbers and addition. (1) What number has the property that when added to any other number doesn't change the number? If you prefer, what is the special number s such that $a+s=s+a=a$? We call this number the additive identity because it leaves the other number identically the same.

Two numbers are called additive inverses if they add together to get the additive identity. (2) If $a+b=b+a=s$ and s is the special number from (1), how are a and b related? What is another name for additive inverses? Great, that's addition. No surprises there.

What about multiplication? (3) What number has the property that when multiplied to any other number doesn't change the number? If you prefer, what is the extra-special number e such that $a \cdot e = e \cdot a = a$? We call this number the multiplicative identity because it leaves the other number identically the same. Similarly, two numbers are called multiplicative inverses if they multiply together to get the multiplicative identity. (4) If $a \cdot b = b \cdot a = e$ and e is the extra-special number from (3), how are a and b related? What is another name for multiplicative inverses? There's multiplication. Still no surprises.

None of that truly concerns us. This is calculus - we care about functions. The most important operation on functions isn't anything that can be done with numbers. It isn't adding, subtracting, multiplication, or division - it is *composition*. So, now we ask the same questions again. What *function* has the property that when composed with any other function doesn't change the function? This is asking what is the *function* i such that $f \circ i = i \circ f = f$? The answer is a rather boring, but very important, function. The function is $i(x) = x$, the identity function, which gives identically the same output as the input. Once we know this, then, as above, two functions are inverses if they compose together to get the function identity, i.e. f and g are inverses if $f \circ g = g \circ f = i$, i.e. if $f(g(x)) = g(f(x)) = i(x) = x$. We tend to denote the inverse of $f(x)$ by $f^{-1}(x)$, but please remember this is not the same as the reciprocal. So, using this notation, our standard definition of inverses is that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. Below we see how to use the first of these.

In high school you studied some things about inverse functions. In particular you were taught a method for finding inverses. What you're actually doing is starting with $f(f^{-1}(x)) = x$ and solving for $f^{-1}(x)$. Let's consider this for a simple example. Start with $f(x) = x+3$. The undoing function for adding three should be pretty apparent: (5) what is it? Let's use this method to find it. If $f(x) = x+3$, $f(f^{-1}(x)) = f^{-1}(x)+3$, we want this to equal x, so we have $f^{-1}(x)+3 = x$, which we may solve for $f^{-1}(x)$. (6) What do you get? (7) Apply this method or the high school method (or both) to find an inverse function for $f(x) = \frac{3x+2}{x+4}$. [Consider the two previous examples and check that they are consistent with your answers to the two following questions.] (8) If we made a table for a function and its inverse, how would the tables be related? (9) If we made graphs for a function and its inverse, how would the graphs be related?

All numbers have additive inverses. Not all numbers have multiplicative inverses: (10) which do not? The same goes for functions. (11) What is one function that does not have a an inverse? (12) Why not?

You might notice that we haven't done any calculus yet. So, all of the above might be review (I hope). Now, finally, what about calculus? Happily the calculus bit is pretty short.

Let's start again with $f(f^{-1}(x)) = x$ and differentiate implicitly. Start with the right side, it should be easy. (13) What is the derivative of the right side? (14) What is the derivative of the left side (don't forget the chain rule, please)? Notice the end of the chain rule of the left side should have produced a factor of $\frac{d}{dx}f^{-1}(x)$. Solve the equation produced by implicit differentiation to get $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$. Notice the meaning here. This says the slope of the inverse is the reciprocal of the slope of the original function - as long as you look at the right place. This seems to fit well with the idea of reflecting over the $y = x$ line.

You might wonder why this would be useful. Here is one example. Consider $f(x) = x^5 + 4x^3 + 3x^2 + 7x + 5$. Notice that this function is increasing so it does have an inverse. If you think "let's just find the inverse and take its derivative", you might notice that finding the inverse is not easy (in fact, it is mathematically impossible). But it turns out to be possible (and not too difficult) to find a derivative of the inverse, say $\frac{d}{dx}|_{x=5}f^{-1}(x)$.

$\frac{d}{dx}|_{x=5}f^{-1}(x) = \frac{1}{f'(f^{-1}(5))}$. It happens (not an accident) that $f^{-1}(5)$ is rather easy to spot. (15) What is it? It is also rather simple to compute $f'(x)$. (16) What is that? Combining those two should yield $\frac{d}{dx}|_{x=5}f^{-1}(x) = \frac{1}{7}$. Not too bad, eh?