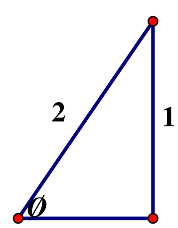
Most of this should be review from high school trigonometry. If you need more review of it, please seek it out. Inverse trigonometry is surprisingly strange. Start with the simplest thing we could imagine. What about inverse sine? $\sin^{-1} x$ asks what angle gives the ratio of x for sine. Even the simplest questions have strange answers: what angle has sine equal to 0? (Let's not get into the silly degree system; at least resolve that problem and say we're doing calculus so we're using radians.) Well, an angle of zero radians gives a sine of zero, but so does an angle of π radians and 2π and $-\pi$ and -7564π. Hm. Just asking an innocent question we have infinitely many answers. Does it get any better if we ask what angle has sine equal to 1? No. (1) What are the infinitely many answers? And, what if we ask: (2) what angle has sine equal to 2? According to what we discussed in §7.1 we would say that sine doesn't have an inverse function. And that is right. And true. And ... inconvenient. It is useful to find angles in triangles when you know the sides of the triangles. But, in that setting, especially if we're dealing with right triangles, we have limitations on the angles. So, instead of thinking of the function $\sin : \mathbb{R} \to \mathbb{R}$, we consider a restriction $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$. (3) Make the graph of sine on this window on your calculator. Notice that it is only increasing so perfectly invertible. For practice, now what are (4) $\sin^{-1}(\frac{\sqrt{3}}{2}), \sin^{-1}(-\frac{\sqrt{2}}{2})$?

There are some simple questions one can ask which have simple answers. (5) $\sin(\sin^{-1} \frac{3}{4})$ asks the question "what is the sine of the angle whose sine is $\frac{3}{4}$?" In fact, (6) what is $\sin(\sin^{-1} x)$? Similar questions can be interesting, largely because of the restriction put in place above. (7) What is $\sin^{-1}(\sin(3\pi/2))$?

Here is a combination that is even more intriguing and turns out to be both useful and important: $\cos(\sin^{-1}(\frac{1}{2}))$. This asks "what is the cosine of the angle whose sine is $\frac{1}{2}$?" What is interesting is that we don't need to find the angle to do this, although we could in this case. (8) Complete this by finding the angle and then finding its cosine. Now, let's look at this in a more interesting and powerful way. Let's label a triangle that is as we desire. We want $\sin(\theta) = \frac{1}{2}$. So, we'll label our triangle this way:



(9) Now, to finish, find the length of the third side, and then find $\cos(\theta)$. Notice that we can repeat these steps for $\cos(\sin^{-1} x)$. (10) Do so.

Now, we need to get to calculus here. Remember that all of the above can, and needs to, be done for each of the trigonometry functions. (11) For each of the five other inverse trigonometry functions, what are the appropriate domain and range?

Above we saw that $\sin(\sin^{-1} x) = x$; we can either use this or the formula derived in §7.1 to find the derivative of $\sin^{-1} x$. Using this conclusion, (12) start by differentiating implicitly, don't forget the chain rule on the left. (13) solve for $\frac{d}{dx} \sin^{-1} x$ and get $\frac{1}{\cos(\sin^{-1} x)}$ and (14) use your work from before to find $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Now, repeat. (15) Do this again for $\cos^{-1} x$, then for $\tan^{-1} x$ and finally for $\sec^{-1} x$. And then (16) apply this to find $\int \frac{dx}{\sqrt{1-x^2}}$, $\int \frac{dx}{1+x^2}$, and $\frac{d}{dx} \sec^{-1}(\ln x)$.

To wrap things up here, let's extend some of these integrals. First let's try $\int \frac{dx}{9+x^2}$. (17) What is the notable difference? (18) Try handling it by factoring it out of the integral. (19) Then what is now in the place where x^2 was? (20) To get a feel for this, try a substitution, letting u^2 equal what is now in the place. (21) Find u and complete the integration by substitution. I expect your final answer to be $\frac{1}{3} \tan^{-1}(\frac{x}{3}) + C$.

One more final extension. Consider $\int \frac{dx}{5+4x+x^2}$. (22) What is the main difference this time? This time we need to resolve by completing the square. It happens to be simple here, for our last question, but we'll consider in class how it could be more complicated. (23) What goes in the place of x^2 this time? (24) Complete the integral to find an answer of $\tan^{-1}(x+2) + C$.