

One of the main reasons we did all that work with trigonometry in §8.2 is that frequently it is valuable to introduce trigonometry into integrals without trigonometry in order to simplify them. Reread that sentence. It should surprise you. However, we have seen hints of this before - notice the two integrals $\int \frac{dx}{1+x^2}$, $\int \frac{dx}{\sqrt{1-x^2}}$ do not have trigonometry in the question, but they do in the answer.

Let's get started with $\int \frac{x^2}{\sqrt{1-x^2}} dx$. Notice that the square root is really inconvenient. (1) Why could we do the integral using substitution if the numerator were $2x$ instead of x^2 ? The idea here is to use trigonometry to address the square root problem. Remember $1 - \sin^2 \theta = \cos^2 \theta$, and so if we used $1 - x^2 = 1 - \sin^2 \theta = \cos^2 \theta$ then we could take the square root. That's our big idea for today, which we will see in different forms. One thing that is very nice about the substitution we do for these problems instead of the old u -substitutions is that we now make $x = \sin \theta$, which makes $dx = \cos \theta d\theta$ easier to deal with. Alas, it is also easier to forget. Make sure that you switch dx into your new variable, or things will go terribly wrong. Sometimes they will go so terribly wrong that you won't be able to continue. That's nice, in a way, because it's a good reminder to take care of dx . Ok, let's finish up this integral $\int \frac{x^2}{\sqrt{1-x^2}} dx$. (2) Make substitutions for x and dx . (3) Use $1 - \sin^2 \theta = \cos^2 \theta$ and take the square root you were going to. (4) Cancel in the numerator and denominator. I expect you're now at $\int \sin^2 \theta d\theta$. This should look like §8.2. (5) Finish the integral in terms of θ . This should be $\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$. But, this is not our answer - the question was in terms of x , not θ , so we need to translate back. Remember $x = \sin \theta$, so $\theta = \sin^{-1} x$. What about $\sin 2\theta$? Now we'll need a few things - our $\sin 2\theta$ identity, and our work with inverse trigonometry from §7.6. (6) Perform these steps to find $\int \frac{x^2}{\sqrt{1-x^2}} dx = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C$. That's a long story (and just our first example). To be sure and see that this all worked, (7) differentiate the answer to check. Yes, these are long.

Getting back to it. Consider $\int \frac{2}{\sqrt{9+x^2}} dx$. Again with the square root. But this time it is different. Remember $1 + \tan^2 \theta = \sec^2 \theta$. That should help, but also notice that we have $9 + x^2$. (8) So, what do we want for x ? (9) Make the substitution, don't forget dx , and then take the square root as you planned it. After canceling you should be left with $\int 2 \sec x dx$. Perhaps you recall from §8.2 that this is $2 \ln(\sec \theta + \tan \theta) + C$. Again we revert to inverse trigonometry, after solving for θ . (10) Complete this work to find $\int \frac{2}{\sqrt{9+x^2}} dx = 2 \ln\left(\frac{x+\sqrt{9+x^2}}{3}\right) + C$. An interesting comparison: consider $\int \frac{2x}{\sqrt{9+x^2}} dx$. While this integral can be done the same way, (11) what is a much shorter way?

Two more examples of methods. Consider $\int \frac{2}{x\sqrt{x^2-4}} dx$. Square roots again. Different again. Remember $1 + \tan^2 \theta = \sec^2 \theta$? Well this can also look like $\sec^2 \theta - 1 = \tan^2 \theta$. Remembering the 4, this time (12) what do we want for x ? (13) Make the substitution for x, dx , and simplify to get a quite nice integral $\int d\theta$. (14) Finish substitutions to find $\int \frac{2}{x\sqrt{x^2-4}} dx = \sec^{-1}\left(\frac{x}{2}\right) + C$.

Just one more $\int \sqrt{21+8x-4x^2} dx$. One more square root, but this is a step different. All because of the x term. Notice that it ends with $-x^2$. To use our methods we would like it to be number - something squared. So, we're back to completing the square. Here's the work, (15) make sure you follow it $21 + 8x - 4x^2 = 21 - 4(x^2 - 2x) = 21 + 4 - 4(x^2 - 2x + 1) = 25 - 4(x-1)^2$. So, now we want $4(x-1)^2 = 25 \sin^2 \theta$. To get this

to happen we use $2(x-1) = 5\sin\theta$ and (16) what is $dx = ?$ (Don't forget to divide by 2). (17) Make substitutions to get to $\int \frac{25}{2} \cos^2 \theta d\theta$. This should remind you of our first example. We tread much the same path. (18) Show all work to finish this out to get $\int \sqrt{21+8x-4x^2} dx = \frac{25}{4} \sin^{-1}\left(\frac{2(x-1)}{5}\right) + \frac{1}{4}(x-1)\sqrt{25-4(x-1)^2} + C$. Be careful that there are many equivalent forms for this answer.