

We saw that sometimes we need to be careful with infinity. We've seen that before, and in chapter ten it will be demonstrated more clearly, too. We are eventually heading for infinite polynomials. These are infinite sums. How will we get to infinity? Step-by-step.

Let's start with a simple example: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$. What we want to know is what happens at the end, at infinity. We've seen things like this before, we're asking for $\lim_{n \rightarrow \infty} \frac{1}{n}$.

(1) What is it? Let's just gather some examples here to get a feel for things:

(2) $\lim_{n \rightarrow \infty} \sqrt{n}$

(3) $\lim_{n \rightarrow \infty} \frac{n+4}{n+1}$

(4) $\lim_{n \rightarrow \infty} \frac{n+\ln n}{n^2}$

(5) $\lim_{n \rightarrow \infty} (-1)^n n$

(6) $\lim_{n \rightarrow \infty} 5 + (-1)^n 2$

(7) $\lim_{n \rightarrow \infty} \frac{5^n}{n!}$

Notice something unusual - in this context the letter we use strangely *does* matter. We use n to indicate that we are only considering *natural* numbers going to infinity. In particular 4-6 are not defined for all real numbers (8) Why not? The language we use for this topic is that (1) converges to 0, (2) diverges to infinity, and (5) diverges by oscillation.

Consider a sequence that is stuck between 2 and 7. It can't get any smaller than 2, and it can't get any bigger than 7. (9) Draw some graphs that stay between 2 and 7. Include graphs that do not converge. (10) What kind of divergence does your sequence demonstrate? (11) So, if a sequence is *bounded* (i.e. stuck between two values) and *monotonic* (i.e. either always increasing or always decreasing), what must happen? (12) Draw an example of such a sequence that is increasing. (13) Draw an example of such a sequence that is decreasing. (14) Draw an example of such a sequence that does not converge to either the upper or lower bound - draw the bounds in the picture.

How can we use this? Here's an example: Consider $a_n = \sqrt{n+1} - \sqrt{n}$. Is it increasing or decreasing? How can you test? (15) Do so. I hope you find that it is decreasing. (16) Because of this, what is the largest it will ever be? (17) Can you explain why $a_n > 0$ for all n ? (18) Therefore a_n is bounded, and monotonic, so must converge. Interestingly, this doesn't say what it converges to. Thus we see one of the uses and weakness of this approach.

Let's consider a more involved example. Warning: this example will have some open-endedness to it at the end. Consider a sequence that starts at $a_1 = 1$ and then the terms after that are determined by $a_n = \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-1}})$. (19) What are a_2, a_3, a_4 ? (20) Explain why a_n will always be positive. Based on the few examples, it looks as if this is decreasing after the first term. If we can show that the sequence is always decreasing (after 1) we will only need to find a lower bound. Thinking about decreasing, if we could show that for any term a , the next term is larger then we would be in a great place. The next term after a is $\frac{1}{2}(a + \frac{2}{a})$. We would like to show that $\frac{1}{2}(a + \frac{2}{a}) < a$, or at least ask when is this true? (21) Start by multiplying by 2, continue by "solving for" the 2 on the left hand side. (22) Eventually (assuming $a > 0$, which we already established), find $2 < a^2$. Then notice that all these algebra steps are reversible to see that if $a > \sqrt{2}$ that the sequence is decreasing. This leaves us with one important question to settle convergence - do we know the sequence is always bounded by $\sqrt{2}$? That seems to be tricky, although there is something tempting to notice: if we assume that the sequence has a limit, we can find what it is. So, if the sequence

has a limit we can find it. Curious. How does this work out? Well, if it has a limit, since $a_\infty = a_{\infty+1}$, we can call it L and see that $L = \frac{1}{2}(L + \frac{2}{L})$ and solve this for L . (23) Do so - you should get $\pm\sqrt{2}$. We can rule out $-\sqrt{2}$ because we saw $a_n > 0$. So that leaves us with the conclusion that if this sequence converges, it converges to $\sqrt{2}$. How could it not? Well, we know it decreases for higher values, so it could jump *under* $\sqrt{2}$ and then back over, and then under, and never quite make up it's mind, like your picture for (9). What's the truth? Looks like it converges, but we don't have a proof yet. (24) Can you devise one?