233 Problem Set 1 Solutions

§1.1 P 12

$$\begin{bmatrix} 1 & 5 & 2 & -5 & 1 \\ -1 & -4 & 1 & 4 & 6 \\ 1 & 3 & -3 & 3 & -9 \end{bmatrix}$$

We're reducing. The first pivot element is the first entry. We'll add the first to the second row and place it in the second row (I like our Maple notation): Rowop($r_1 + r_2, r_2$). Next we'll subtract row 1 from row 3: Rowop($r_3 - r_1, r_3$). These two moves leave us with:

$$\begin{bmatrix} 1 & 5 & 2 & -5 & 1 \\ 0 & 1 & 3 & -1 & 7 \\ 0 & -2 & -5 & 8 & -10 \end{bmatrix}$$

Now we pivot on the (2,2) entry. We'll add two times the second row to the third row: Rowop($2r_2 + r_3, r_3$). We get:

$$\begin{bmatrix} 1 & 5 & 2 & -5 & 1 \\ 0 & 1 & 3 & -1 & 7 \\ 0 & 0 & 1 & 6 & 4 \end{bmatrix}$$

Now we are in row echelon form. The pivot columns are the first three. The system is consistent (there's not 0= nonzero in the last row). It has an infinite number of solutions because there's no pivot in the fourth column. The leading variables are the first three and the free variable is the fourth, i.e. perhaps x_1, x_2, x_3 are leading and x_4 is free.

§1.1 P 21 Quadratic polynomials produce parabolas. When do three points not produce a parabola? When they lie on a a straight line. Well, except that's not quite right, because we can get the straight line back by just having $c = 0$ in our quadratic polynomial. So, when then? Well, if any two are on a vertical line. So, let's pick three points with two on a vertical line. How about $(3, 2), (3, 8)$, and $(-7, 6)$? What is the corresponding augmented matrix? First the equations: $8 = a + 3b + 9c$, $2 = a + 3b + 9c$, and $6 = a - 7b + 49c$. The augmented matrix is then:

$$\begin{bmatrix} 1 & 3 & 9 & 8 \\ 1 & 3 & 9 & 2 \\ 1 & -7 & 49 & 6 \end{bmatrix}$$

Now to reduce. Well, first we subtract the first row from the two others, i.e. Rowop($r_2 - r_1, r_2$) and Rowop($r_3 - r_1, r_3$):

$$\begin{bmatrix} 1 & 3 & 9 & 8 \\ 0 & 0 & 0 & 6 \\ 0 & -10 & 40 & 14 \end{bmatrix}$$

Finally we merely interchange the last two row, i.e. Rowop(r_2, r_3):

$$\begin{bmatrix} 1 & 3 & 9 & 8 \\ 0 & -10 & 40 & 14 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

§2.1 P 15 We addressed much of this when looking at 14 in class. There are several ways to do it. Two points on this line are $(0, 1)$ and $(2, -2)$. The vector between them, from their difference, gives a direction vector: $(2, -3)$. We now have a point and a direction vector so one answer is $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. (I'm trusting you to draw this - please ask if this is a problem.) A third point on the line is $(-2,4)$. The vector between this and $(2,-2)$, from their difference, is $(4,-6)$, so another form for the same line is $\begin{bmatrix} 2 \\ -2 \end{bmatrix} + t\begin{bmatrix} 4 \\ -6 \end{bmatrix}$. The position vectors are all on the line, by definition. And the direction vectors are all multiples of each other.

§2.2 P 6 (2,4) and (-1,-2) are two nonzero vectors in $\mathbb{R}^2$, yet (1,3) isn't in their span. We can check this: $a(2,4)+b(-1,-2)=(1,3)$ produces the augmented matrix:

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix}$$

One row reduction move, Rowop(r_2-2r_1, r_2) produces

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence no solution. The problem with my example is that the two vectors were multiples $-2(-1,-2) = (2,4)$. So, if we add this in our condition to "If **u** and **v** are two nonzero vectors in $\mathbb{R}^2$ that are not scalar multiples, then ..." we should be fine.

§2.2 P 11 a. i. Since $a+b=1$, if $a=0$, then $b=1$. That point is merely the vector **u** itself. Similarly, if $a=1$, $b=0$ and we get the vector **v** itself. The point $a=0.5$ has $b=0.5$ and is the vector to the midpoint between **u** and **v**.

ii. The entire set **N** is merely the segment joining **u** and **v**.

iii. $\mathbf{N}=\{a\mathbf{u}+b\mathbf{v}\}$. Solving our constraint for b we get $b=1-a$, and since $0\le b$, $a\le 1$. We then have $\mathbf{N}=\{a\mathbf{u}+(1-a)\mathbf{v}\}$ for $0\le a\le 1$, which is a standard presentation of the segment from **u** to **v**. (Notice none of this really matters what **u** and **v** are.)

b. This is the set $\mathbf{P}=\{a\mathbf{u}+b\mathbf{v}|0\le a\le 1 \text{ and } 0\le b\le 1\}$.

c. This is closely related to part a., which is the boundary of this triangle. So, we only modify **N** slightly to get this triangle $\mathbf{T}=\{a\mathbf{u}+b\mathbf{v}|a+b\le 1 \text{ and } 0\le a \text{ and } 0\le b\}$.

d. The above triangle **T** is the triangle with **u**, **v** and the origin, **0** as vertices. We want **u**, **v** and **w** as vertices. The tricky thing here is that while **u** is the vector from **0** to **u**, it isn't the vector from **u** to **w**. We will need these vectors. $\mathbf{w}-\mathbf{u}$ *is* the vector from **u** to **w** and $\mathbf{v}-\mathbf{u}$ is the vector from **u** to **v**. So our second triangle $\mathbf{S}=\{\mathbf{u}+a(\mathbf{v}-\mathbf{u})+b(\mathbf{w}-\mathbf{u})|a+b\le 1 \text{ and } 0\le a \text{ and } 0\le b\}$. To check this, notice that when $a=b=0$ we get **u**, when $a=1, b=0$ we get **v**, and when $a=0, b=1$ we get **w**. Since those are the extreme values, all other points must be inside the desired triangle.

§2.3 P 6 This is emphasising the difference between homogeneous solutions and non-homogeneous solutions. The first conclusion is correct, but the second is not. Let's see why not. Two solutions are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Clearly their sum is in their span: $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, but this is not a solution to $x+2y-z=3$. What went wrong? A position vector is missing. What is correct? This should have the same direction vectors as the homogenous solution in the first part, with a new position vector. Solving for z produces $z=x+2y-3$ and so we have a solution set of $x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$. Without the position vector at the end, we get incorrect solutions as above.