Mathmatics 239 solutions to Homework for Chapter 2

Old version of 8.5 My compact disc player has space for 5 CDs; there are five trays numbered 1 through 5 into which I load the CDs. I own 100 CDs.

a) In how many ways can the CD player be loaded if all five trays are filled with CDs?

Assume the trays are numbered this seems standard to me; if they're not numbered, at least I believe that the intent of the problem is that switching CDs in adjacent trays would constitute a different way to load from 1 to 5. So, suppose we load them in numerical order. I have 100 choices for my first tray, then have used one up, so have 99 choices for the second tray, 98 for the third, 97 for the fourth, and 96 for the fifth. By the multiplication principle, we then have $100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = 9,034,502,400$ ways to load the player.

Note here: our author makes a comment about numerical answers that I agree with. Numerical answers that are uncomputed frequently give valuable information. Numerical answers that are computed less frequently give such information. Therefore, it is regularly useful to not compute out answers, and to leave them in factored form, e.g. $100 \cdot 99 \cdot 98 \cdot 97 \cdot 96$ might be a better answer than 9,034,502,400. Certainly this will be true on exams and quizzes. Please keep this in mind.

b) In how many ways can the CD player be loaded if only one CD is placed in the machine?

Two perspectives on this question . . . using the multiplication principle as follows: first we choose which tray to place a CD in 5 choices, 1 through 5and then we choose which CD to place in the tray 100 choices. Therefore there are $5 \cdot 100$ ways to load.

A different perspective which looks very similar, but might be helpful. There are several cases. We can either place a CD in the first tray, in which case there are 100 choices, or the second tray, where there are 100 choices, and so on. Because we have broken down into cases, we add these 100s, so $100 + 100 + 100 + 100 + 100$ ways to load. This is using the addition principle covered in section 10.

*NEW* 8.5 I want o create two play lists on my MP3 player from my collection of 500 songs. One play list is titled "Exercise" for listening in the gym, and the other is titled "Relaxing" for leisure time at home. I want 20 different songs on each of these lists.

In how many ways can I load songs onto my MP3 player if I allow a song to be on both play lists? We'll start with the first list, 500 choices for the first song, 499 for the second, down to 481 for the 20th. Then continuing to build the second list, we again have 500 choices for the first song (because some songs are great for exercising and relaxing?) down to 481 for the 20th. This produces $(500 \cdot 499 \cdot 498 \cdots 481)(500 \cdot 499 \cdot 498 \cdots 481)$. Scheinerman calls this $(500)_{20}^2$, I would probably call it $\left(\frac{500!}{480!}\right)^2$. Any of these answers are acceptable.

And how many ways can I load the songs if I want the two lists to have no overlap? Again, we'll start with the first list, 500 choices for the first song, 499 for the second, down to 481 for the 20th. Then continuing to build the second list, we now have 480 choices for the first song (because we can't use the first twenty) down to 461 for the 20th. This produces $(500 \cdot 499 \cdot 498 \cdots 481)(480 \cdot 479 \cdot 478 \cdots 461)$. Scheinerman calls this $(500)_{40}$, I would probably call it $\frac{500!}{460!}$. Any of these answers are acceptable.

8.12 A U.S. Social Security Number is a nine-digit number. The first digit or digits may be 0.

a) How many Social Security numbers are available?

There are ten choices for each digit, 0,1,2,3,4,5,6,7,8,9, and because there are nine digits the multiplication principle gives $10^9$ numbers.

b) How many of these are even?

Base ten numbers are even exactly when they end in an even digits. There are five even digits, 0,2,4,6,8. Therefore, there are five choices for the last digit, and still ten choices for the other digits. Therefore there are $10^8 \cdot 5$ even numbers.

c) How many have all their digits even?

If all the digits are even then we have five choices for each digit. Therefore there are $5^9$ numbers with all digits even.

d) How many read the same backward and forward?

Imagine building a number that reads the same backward and forward called a palindrome. In creating it, you are free to choose any digits for the first half, but the second half is determined. The middle digit is also free. Since there are nine digits total, there are five we can choose, and four that are determined by those choices. So we have $10^5 1^4$ palindromes.

e) How many have none of their digits equal to 8?

Consider part a) again. In this case, however, we have one fewer digit to choose from. Therefore, we have nine digits to choose from in each place. Therefore, there are $9^9$ numbers with no digits equal to 8.

f) How many have at least one digit equal to 8?

Consider the question rephrased as follows: How many do not have zero 8s? We know there are $10^9$ altogether, we know that $9^9$ have zero 8s. Therefore, it must be that $10^9 - 9^9$ do not have zero 8s, or with at least one digit equal to 8.

g)How many have exactly one 8?

In creating a number with exactly one 8, start by choosing a place to put 8. There are 9 choices of places to put 8. Then, put the 8 there this involves no choice. Then fill out the rest of the places. Each of the other places will not have an 8, so they each have 9 choices of digits to be placed. Therefore, we have $9 \cdot 9^8 = 9^9$ numbers with exactly one 8. When working this the first time, I wondered why the answers to e) and g) were the same. This is a coincidence. Consider, for example, if social security numbers had seven digits as the hints used to seem to imply. Then the answer to e)would be $9^7$, and the answer to g) would be $7 \cdot 9^6$ seven choices of where to put 8, and 9 choices for the six other digits.

8.18. A class contains ten boys and ten girls. In how many different ways can they stand in line if they must alternate in gender (no two boys and no two girls are standing next to one another)?

It seems to me there are many different perspectives on this question. This seems the most efficient: First arrange the boys in your desired order there are 10! ways, then arrange the girls in your desired order there are 10! ways, finally decide whether to start with boys or girls 2 choicesand weave the two arrangements together. Therefore there are 10!10!2 ways they can stand in an alternating line. A couple of other perspectives: choose to start boy or girl, and then fill spaces with 10, 10, 9, 9, 8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1 choices. A final idea would be you have a choice of 20 for the first, but then ten for the second and then 9, 9, 8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1 choices. All of these ideas are fine. Be sure to attend to the details of starting with boy or girl and how that affects the counting.

9.2 There are six different French books, eight different Russian books, and five different Spanish books.

a) In how many different ways can these books be placed on a bookshelf? As this question does not concern language, the answer is straightforward. There are 19 choices for first book, 18 for second, &c. Therefore, we have 19! ways.

b) In how many different ways can these books be placed on a bookshelf if all books in the same language are grouped together?

Start by asking how many ways can I order the three languages e.g. first Russian, then Spanish and finally French is one.There are three languages, so there are 3! ways to order the languages. Now, order the books within the languages, there are 6!, 8! and 5! ways to order the books in each language. So altogether there are 3!6!8!5! ways to place the books grouped by language note, far less than if they are ungrouped.

9.8. Calculate the following products:

(a) $\Pi_{k=1}^{4}(2k+1) = (2(1)+1)(2(2)+1)(2(3)+1)(2(4)+1) = 3 \cdot 5 \cdot 7 \cdot 9 = 945$

(b) $\Pi_{k=-3}^{4} k = -3 \cdot -2 \cdot -1 \cdot 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 0$

(c) $\Pi_{k=1}^{n} \frac{k+1}{k}$, where $n$ is a positive integer. We have $\Pi_{k=1}^{n} \frac{k+1}{k} = \frac{2}{1}\frac{3}{2}\frac{4}{3} \cdots \frac{n-1}{n-2}\frac{n}{n-1}\frac{n+1}{n}$, performing the visible cancellations leaves $\frac{n+1}{1} = n+1$.

(d) $\Pi_{k=1}^{n} \frac{1}{k}$, where $n$ is a positive integer.

We have $\Pi_{k=1}^{n} \frac{1}{k} = \frac{1}{1}\frac{1}{2}\frac{1}{3}\frac{1}{4} \cdots \frac{1}{n-2}\frac{1}{n-1}\frac{1}{n} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)n} = \frac{1}{n!}$.

10.3 Find the cardinality of the following sets:

(a) $\{x \in \mathbb{Z} \mid |x| \leq 10\}$

Here $x$ is a number, therefore $|x|$ means absolute value. This set can be written out explicitly as $\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. This set has 21 elements, so its cardinality is 21.

(b) $\{x \in \mathbb{Z} \mid 1 \leq x^2 \leq 2\}$

$1^2 = 1$, $(-1)^2 = 1$. If $\mid x \mid \geq 2$, then $x^2 \geq 4$. If $x = 0$, then $x^2 = 0$. Therefore, the only elements are 1 and -1. The set $\{1, -1\}$ has cardinality 2.

(c) $\{x \in \mathbb{Z} \mid x \in \emptyset\}$

There are no elements of the empty set, therefore there is no number such that it is in the empty set. Therefore this set is itself $\emptyset$. (Alternatively, this set is a subset of the empty set by definition, but the only subset of the empty set is itself $\emptyset$). Because $|\emptyset| = 0$, the cardinality here is 0. (Please remember for a set $|A|$ indicates the cardinality of A, or the number of elements in $A$).

(d) $\{x \in \mathbb{Z} \mid \emptyset \in x\}$

Here $x$ is a number. The question asks "how many integers have the empty set as an element?" Numbers do not have elements (at least the way we usually think of them) sets do. Therefore, no number contains the empty set as an element. Therefore, this set is empty and has cardinality 0.

(e) $\{x \in \mathbb{Z} \mid \emptyset \subset \{x\}\}$

$\{x\}$ is a set. The empty set is a subset of all sets. Therefore, for all x, $\emptyset \subset \{x\}$. How many $x$s are there? How many integers are there? Infinitely many. So the cardinality is infinite.

(f) $2^{2^{\{1,2,3\}}}$

This question and definitely the subsequent one are more difficult to read than to answer. The first question is how to associate the symbols. Does this mean $2^{(2^{\{1,2,3\}})}$ or $(2^2)^{\{1,2,3\}}$? Performing the first operation $(2^2)$ in the second interpretation we would then have $4^{\{1,2,3\}}$, which is meaningless to us. Therefore, let us assume the first notation is intended. What does $2^{(2^{\{1,2,3\}})}$ mean? The power set of the power set of $\{1, 2, 3\}$, or the set of all subsets of the set of all subsets of $\{1, 2, 3\}$. Lets start with $2^{\{1,2,3\}}$. The set of all subsets of $\{1, 2, 3\}$. This is the set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Now, the problem asks us to count the set of all subsets of this set. We could start . .

$$\{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1, 2\}\}, \{\{1, 3\}\}, \{\{2, 3\}\}, \{\{1, 2, 3\}\},$$
$$\{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{3\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\emptyset, \{1\}, \{2\}\}, \ldots\}$$

But it quickly becomes apparent that there are a *lot* of elements in this set. We need a better way to count them. (Note: I wrote out this start of this set for two reasons, first so that you would see the futility of counting it, but also second so that you would have an idea what set was being counted in this problem.) Remember we saw in class that $|2^A| = 2^{|A|}$ which reads the cardinality of the power set of $A$ is the number 2 raised to the power of the cardinality of $A$. Applying this where $A = \{1, 2, 3\}$ gives $|2^{\{1,2,3\}}| = 2^{|\{1,2,3\}|} = 2^3 = 8$, which is consistent with our first list, which has eight elements (count them!).

But, now we can also apply this where $A = 2^{\{1,2,3\}}$. This gives $|2^{2^{\{1,2,3\}}}| = 2^{|2^{\{1,2,3\}}|} = 2^{(2^3)} = 2^8 = 256$. Certainly it is a good thing we didn't try to list all the elements.

(g) $\{x \in 2^{\{1,2,3,4\}} \mid |x| = 1\}$

This question is actually very easy after you translate all the symbols. The difficult part here is reading the mathematics, not answering the question.

Translation 1: What is the number of $x$ in the power set of $\{1, 2, 3, 4\}$ such that $|x| = 1$?

Remember the power set of a set is the set of all subsets of the set.

Translation 2: What is the number of $x$ that are subsets of $\{1, 2, 3, 4\}$ such that $|x| = 1$?

So, we now see that $x$ is a set not a number, therefore $|x|$ means the number of elements in $x$, not the absolute value of $x$.

Translation 3: What is the number of $x$ that are subsets of $\{1, 2, 3, 4\}$ with one element?

Let's eliminate $x$ from the question

Translation 4: How many subsets of $\{1, 2, 3, 4\}$ have one element?

Ok, now the question is finally easy. The sets $\{1\}, \{2\}, \{3\}, \{4\}$ are the only one-element subsets of $\{1, 2, 3, 4\}$. There are four of them. Therefore the answer is 4. Sometimes the difficult part of mathematics is understanding the language, not understanding the actual concept. After unpacking all the mathematics, this turned into an easy question. Keep this principle in mind for future work.

(h) $\{\{1, 2\}, \{3, 4, 5\}\}$

This set has 2 elements (note: not 5 or 7; see (f) and the way we counted $2^{\{1,2,3\}}$.

10.11 Generalize the previous problem. Let $A = \{x \in \mathbb{Z} \mid a \mid x\}$ and $B = \{x \in \mathbb{Z} \mid b \mid x\}$. Find and prove a necessary and sufficient condition for $A \subset B$. In other words, find and prove a theorem of the form "$A \subset B$ if and only if ____________."

The key step in #10 is underlined below:

if $x \in A$ then $4 \mid x$ then <u>because $2 \mid 4$</u> then $2 \mid x$ then $x \in B$.

This might lead us to try $b \mid a$ as our condition above. Before attempting to prove this theorem, we should check it for some examples of $b$ and $a$. We see that $3 \mid 15$ and multiples of fifteen are a subset of the multiples of three. We also see that $3 \nmid 14$ and multiples of fourteen are not a subset of the multiples of three. It seems to check.

Now let us prove this theorem: $A \subset B$ if and only if $b \mid a$.

($\Rightarrow$) If $A \subset B$ then $b \mid a$.

Suppose that $A \subset B$. That is, if $x \in A$ then $x \in B$. I assert that $a \in A$ (because $a$ is an integer and $a \mid a$ [see $a = a \cdot 1$]). Therefore, since $a \in A$, we also have that $a \in B$. The definition of $B$ says that $a \in B$ implies that $a$ is an integer (ok) and $b \mid a$. However, $b \mid a$ is exactly what we were hoping to prove.

($\Leftarrow$) If $b \mid a$ then $A \subset B$.

Suppose that $b \mid a$.

(Hm, now I want to prove that $A \subset B$. That is I want to prove that if $x \in A$ then $x \in B$. To do this, I will in effect be proving a theorem inside a theorem. Overall, we have assumed that $b \mid a$. We will furthermore assume $x \in A$ to prove the subtheorem if $x \in A$ then $x \in B$. To clarify, I will indent the entire proof of the subtheorem.)

> Suppose that $x \in A$. Then $x$ is an integer and $a \mid x$. We also know from the overriding assumption that $b \mid a$. Using one of the very first things we proved (4.3) we thus have that $b \mid x$. We already know that $x$ is an integer (see first line of subtheorem), therefore we now have the requirements to see that $x \in B$.

Therefore if $x \in A$ then $x \in B$ (because by assuming $x \in A$ we proved $x \in B$, or $A \subset B$, as desired.

11.5. For each of the following sentences write the negation of the sentence, but place the symbol as far to the right as possible. Then rewrite the negation in English.

(a) $\forall x \in \mathbb{Z}, x < 0$

$$\neg(\forall x \in \mathbb{Z}, x < 0) = \exists x \in \mathbb{Z} \mid \neg(x < 0) = \exists x \in \mathbb{Z} \mid (x \geq 0)$$

"There is an integer that is not less than zero" or
"There is an integer greater than or equal to zero"

(b) $\exists x \in \mathbb{Z} \mid x = x + 1$

$$\neg(\exists x \in \mathbb{Z} \mid x = x + 1) = \forall x \in \mathbb{Z}, \neg(x = x + 1) = \forall x \in \mathbb{Z}, x \neq x + 1$$

"Every integer does not have the property that it is equal to one greater than itself." or
"Every integer is not equal to one greater than itself."

(c) $\exists x \in \mathbb{N} \mid x > 10$

$$\neg(\exists x \in \mathbb{N} \mid x > 10) = \forall x \in \mathbb{N}, \neg(x > 10) = \forall x \in \mathbb{N}, x \leq 10$$

"Every natural number is not greater than 10" or
"Every natural number is less than or equal to 10"

(d) $\forall x \in \mathbb{N}, x + x = 2x$

$$\neg(\forall x \in \mathbb{N}, x + x = 2x) = \exists x \in \mathbb{N} \mid \neg(x + x = 2x) = \exists x \in \mathbb{N} \mid x + x \neq 2x$$

"There is a natural number that does not have the property that adding it to itself is the same as multiplying it by two." or

"There is a natural number such that adding it to itself is not the same as multiplying it by two."

(e) $\exists x \in \mathbb{Z} \mid \forall y \in \mathbb{Z}, x > y$

$$\neg(\exists x \in \mathbb{Z} \mid \forall y \in \mathbb{Z}, x > y) = \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \mid \neg(x > y) = \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \mid x \leq y$$

"For every integer there is a second integer such that the first integer is not greater than the second integer." or

"For every integer there is a second integer such that the first integer is less than or equal to the second integer."

(f) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x = y$

$$\neg(\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x = y) = \exists x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} \mid \neg(x = y) = \exists x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} \mid x \neq y$$

"There exist two integers such that it is not the case that they are equal." or

"There exist two unequal integers."

(g) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \mid x + y = 0$

$$\neg(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \mid x + y = 0) = \exists x \in \mathbb{Z} \mid \forall y \in \mathbb{Z}, \neg(x + y = 0) = \exists x \in \mathbb{Z} \mid \forall y \in \mathbb{Z}, x + y \neq 0$$

"There exists an integer such that any other integer will not have the property that when added to the first integer will give zero". or

"There exists an integer such that any other integer added to it will not equal zero."

[extra 11.7] a. $\exists! x \in \mathbb{N}, x^2 = 4$ is true, the unique natural number is 2.
b. $\exists! x \in \mathbb{Z}, x^2 = 4$ is false, there are two integers, 2 and $-2$.
c. $\exists! x \in \mathbb{N}, x^2 = 3$ is false, there are no such natural numbers ($1^2 < 3 < 2^2$)
d. $\exists! x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = x$ is true, the unique integer is 0, i.e. $\forall y \in \mathbb{Z}, 0y = 0$.
e. $\exists! x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = y$ is true, the unique integer is 1, i.e. $\forall y \in \mathbb{Z}, 1y = y$.

[extra] 11.8 a. $\forall x \in R, \forall y \in R, \forall z \in L(x, y), z \in R$. This is pretty much a literal symbolic interpretation of what is written. For every two points (one at a time) in the region, every point on the line that connects them is again in the region.

b. Remember to negate $\forall P$, we have $\exists \neg P$. So, this is rather direct, too. Aside from there being a good number of words in this problem, it's not that bad. So, the negation here is $\exists x \in R | \exists y \in R | \exists z \in L(x, y), z \notin R$.

c. There are two points in the region such that the line segment joining them contains a point that is not in the region.

d. Drawing is not so easy for me here, but consider a region R that is U-shaped. Consider two points, one on each side of the U, the line segment that connects them crosses over the middle of the U, and is therefore not in the U. If you don't see this, please ask me, and I will happily show you.

12.18 Let $A$ and $B$ be sets. We have $A \backslash B = B \backslash A$ if and only $A = B$.

Suppose $A = B$, $A \backslash B = A \backslash A$ by substitution $= \emptyset = B \backslash B = B \backslash A$ by substitution again. Therefore $A \backslash B = B \backslash A$.

Suppose $A \backslash B = B \backslash A$. Consider an $x \in A \backslash B = B \backslash A$. This would mean $x \in A, x \notin B, x \in B$, and $x \notin A$. Because $x \in A$ and $x \notin A$ is impossible we have that $A \backslash B = B \backslash A = \emptyset$. 12.16 says $A \subset B$ iff $A \backslash B = \emptyset$. If we use this we get both that $A \subset B$ and that $B \subset A$, which produces $A = B$ as desired.

If you protest that we didn't do 12.16, here is the direction we need: if $A \backslash B = \emptyset$ then $A \subset B$. To prove, suppose $x \in A$. Is $x \in B$? If it isn't, then it is in $A \backslash B$, but $A \backslash B$ is empty, so it must be that $x \in B$. Technically this is a proof by contradiction.

Here's a better argument: We wish to prove $A = B$, so we prove $A \subset B$ and $B \subset A$. To prove $A \subset B$:

Suppose $x \in A$. Therefore $x \notin B \backslash A$, but $B \backslash A = A \backslash B$. So, $x \notin A \backslash B$. We now have $x \in A$ and $x \notin A \backslash B$. By definition of $A \backslash B$, the only way this happens is if $x \in B$. This is what we wanted to show, so $A \subset B$. The proof that $B \subset A$ is identical by switching letters. Yes, this is a technical thing to say in a proof, but only when it is true. Notice in this case, if I exchange $A$ and $B$ we get exactly the desired proof for $B \subset A$. Don't use it carelessly, but it is a nice way to avoid being redundant.

[extra] 12.20 a. This is not true. So, we need a counterexample. Consider two disks that intersect in a point. The disks are both convex, but the union is not. This isn't entirely obvious to prove, so I'll prove a simpler example.

Consider $A = \{(a, 0) | a \in \mathbb{R}, -3 < a < -1\}$ and $B = \{(b, 0) | b \in \mathbb{R}, 1 < a < 3\}$ and $A \cup B$. Both sets $A$ and $B$ are convex because the line segment joining any two points is still part of the $x$-axis which is what the two points are made of. But, consider $(-2, 0) \in A$ and $(2, 0) \in B$. Because they are each in one set, $(-2, 0), (2, 0) \in A \cup B$. The line segment joining these two points contains the origin, $(0, 0)$, which is not in $A$ and not in $B$, so not in $A \cup B$. Therefore the union is not convex.

b. This is true. So we need a general proof. Suppose $A$ is convex and suppose $B$ is convex. We wish to prove that $A \cap B$ is convex. To prove this we must show something about any two points in $A \cap B$. So, suppose $x, y \in A \cap B$. In particular, because $A \cap B \subset A$, we have $x, y \in A$. $A$ is convex, so for any point $z \in L(x, y)$, we have that $z \in A$. The same can be said about $B$ (because $A \cap B \subset B$, we have $x, y \in B$. $B$ is convex, so for any point $z \in L(x, y)$, we have that $z \in B$). Therefore $z \in A \cap B$. We have proven that for any two points $x, y \in A \cap B$, and for any $z \in L(x, y)$, $z \in A \cap B$. Therefore by definition $A \cap B$ is convex, as desired.

12.21 True or False. For each of the following statements, determine whether the statement is true or false and the prove your assertion.

(a) $A \backslash (B \backslash C) \neq (A \backslash B) \backslash C$
Consider $A = \{1, 2\}, B = \{1\}, C = \{1\}$. Compute:
$A \backslash (B \backslash C) = A \backslash \emptyset = A = \{1, 2\}$
$(A \backslash B) \backslash C = (\{2\}) \backslash \{1\} = \{2\}$

(b) $(A \backslash B) \backslash C = (A \backslash C) \backslash B$
*Proof*: Suppose that $x \in (A \backslash B) \backslash C$
Then $x \in (A \backslash B) \wedge x \notin C$.
Then $x \in A \wedge x \notin B \wedge x \notin C$. Since $\wedge$ is associative and commutative (5.2) we have

So, $(x \in A \wedge x \notin C) \wedge x \notin B$.

Hence $x \in (A \backslash C) \wedge x \notin B$.

So, $x \in (A \backslash C) \backslash B$.

Therefore $(A \backslash B) \backslash C \subset (A \backslash B) \backslash C$

The proof of the other implication is identical with $B$ and $C$ switched. Together we get $(A \backslash B) \backslash C = (A \backslash C) \backslash B$.

(c) $(A \cup B) \backslash C \neq (A \backslash C) \cap (B \backslash C)$

Consider $A = \{1\}, B = \{2\}, C = \emptyset$. Compute:

$(A \cup B) \backslash C = \{1, 2\} \backslash \emptyset = \{1, 2\}$

$(A \backslash C) \cap (B \backslash C) = \{1\} \cap \{2\} = \emptyset$

(d) $B \neq (B \backslash C) \cup C$

Let $B = \emptyset, C = \{2\}$. Compute:

$(B \backslash C) \cup C = \emptyset \cup \{2\} = \{2\} \neq \emptyset = B$.

(e) $A \neq (A \cup C) \backslash C$

Let $A = \{1\}, C = \{1\}$. Compute:

$(A \cup C) \backslash C = \{1\} \backslash \{1\} = \emptyset \neq \{1\} = A$.

(f) $\mid A \backslash B \mid \neq \mid A \mid - \mid B \mid$

Let $A = \emptyset, B = \{2\}$. Compute:

$A \backslash B = \emptyset, \mid A \backslash B \mid = 0$.

$\mid A \mid = 0, \mid B \mid = 1, \mid A \mid - \mid B \mid = 0 - 1 = -1 \neq 0$.

(g) $(A \backslash B) \cup B \neq A$. See (d)

(h) $(A \cup B) \backslash B \neq A$. See (e)

*Extra 12.22. Prove the following about set complements in a universe $U$.

(a) $A = B$ if and only if $\overline{A} = \overline{B}$

($\Rightarrow$) Suppose that $A = B$.

Very short version of the argument: start with $\overline{A}$, substitute $B$ in for $A$, we then get $\overline{B}$, thus they are equal.

Here's a longer version: To show that $\overline{A} = \overline{B}$ we will first show that $\overline{A} \subset \overline{B}$.

Suppose that $x \in \overline{A}$ . Then $x \in U \backslash A$. Therefore $x \in U$ and $x \notin A$. Because $A = B$, if $x \notin A$ then $x \notin B$. So, $x \in U$ and $x \notin B$. Therefore $x \in U \backslash B$, or $x \in \overline{B}$. Therefore $\overline{A} \subset \overline{B}$. To show $\overline{A} \supset \overline{B}$ interchange $A$ and $B$ in the previous argument.

($\Leftarrow$) Suppose that $\overline{A} = \overline{B}$. To show that $A = B$ we will first show that $A \subset B$.

Suppose that $x \in A$. Then $x \notin U \backslash A$ (because it is not the case that $x \notin A$). So, $x \notin \overline{A}$. Because $\overline{A} = \overline{B}$, $x \notin \overline{B}$, or $x \notin U \backslash B$. This says it is not the case that $x \in U$ and $x \notin B$. By DeMorgans laws (5.2) we then have that either $x \notin U$ or $x \in B$. We know from the beginning that $x \in A$ which is a subset of $U$, therefore we know that $x \in U$. Since the option $x \notin U$ is therefore not possible, we must have that $x \in B$. Therefore $A \subset B$. To show $A \supset B$ interchange $A$ and $B$ in the argument.

(b) $\overline{\overline{A}} = A$

Suppose that $x \in A$. If $y \in U \backslash A$ then $y \notin A$. Therefore $x \notin U \backslash A$, but $x \in U$ (because $x \in A \subset U$). Consequently $x \in U \backslash (U \backslash A) = U \backslash \overline{A} = \overline{\overline{A}}$. Therefore we have shown that $A \subset \overline{\overline{A}}$.

Suppose that $x \in \overline{\overline{A}}$. Therefore $x \in U \backslash (U \backslash A)$. So, $x \in U$ and $x \notin U \backslash A$. $x \notin U \backslash A$ says that it is not the case that $x \in U$ and $x \notin A$. Applying DeMorgans laws (5.2) again, this says that either $x \notin U$ or it is not the case that $x \notin A$. We know that $x \in U$, so it must not be the case that $x \notin A$. Once again, using a logic fact from 5.2, this gives us that $x \in A$, as desired. Therefore we have shown that $\overline{\overline{A}} \subset A$, and finally $\overline{\overline{A}} = A$.

(c) $\overline{(A \cup B \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$

*Proof*: $\overline{(A \cup B \cup C)} = U \backslash (A \cup B \cup C)$ which by DeMorgans laws 10.12

$= (U \backslash A) \cap (U \backslash B) \cap (U \backslash C) = \overline{A} \cap \overline{B} \cap \overline{C}$

(This could be proven in a longer element method like (b), but this is much nicer.)

*One more extra:

Use the results in 11.17 to prove $\overline{A \cap (\overline{A} \cup B)} \cup B = U$

*Lemma 1*: For any set $A \subset U$, $A \cup \overline{A} = U$.

*Lemma 2*: For any set $A \subset U$, $A \cap \overline{A} = \emptyset$.

*Lemma 3*: For any sets $A, B \subset U$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Those are proven exactly as the other results in 11.17. The first relies on the result from (6.2) of $x \vee (\neg x) =$ TRUE, and the second on $x \wedge (\neg x) =$ FALSE. I will assume and freely use both of these results.

(Note: to read the following, the reasons are *before* the equal signs.)

Here we go: $\overline{A \cap (\overline{A} \cup B)} \cup B$ by distributive properties 11.3

$= \overline{(A \cap \overline{A}) \cup (A \cap B)} \cup B$ by Lemma 2

$= \overline{\emptyset \cup (A \cap B)} \cup B$ by another part of 11.3

$= \overline{A \cap B} \cup B$ by Lemma 3

$= \overline{A} \cup \overline{B} \cup B$ by Lemma 1

$= \overline{A} \cup U$ because $A \subset U$

$= U$.