Mathmatics 239 solutions to Homework for Chapter 4

§20.5 Prove by contradiction that consecutive integers cannot both be even.

Suppose for the sake of contradiction that there are two consecutive integers, $k, k+1$, such that both are even.

k is even, therefore there is an integer p such that $k = 2p$.

$k+1$ is even, therefore there is an integer q such that $k+1 = 2q$.

By substitution, $2p+1 = 2q$, or $2q - 2p = 1$. Factoring and dividing we find $q - p = \frac{1}{2}$. However, q and p are integers which are closed under subtraction, therefore $q-p$ must be an integer and $\frac{1}{2}$ is not. This is a contradiction. Therefore our last assumption is false, we have that there do not exist two consecutive integers, $k, k+1$, such that both are even. Hence, consecutive integers cannot both be even.

§20.7 Prove by contradiction: If the sum of two primes is prime, then one of the primes must be 2.

Assumption: we have already proven that the sum of two odd numbers is even.

Don't assume, but prove that a prime number greater than two is odd (suppose it were even, then it would be divisible by 2, which is less than the number so it is not prime). We will use these two facts.

Suppose that the sum of two primes, $p+q$, is prime.

Suppose for the sake of contradiction that both of the primes are not equal to 2, i.e. $p \neq 2$ and $q \neq 2$. By the above-stated assumption (and the definition that primes must be greater than one), p and q must be odd. Because p and q are both odd, $p+q$ is even. Because p and q are both larger than 2, $p+q$ is larger than 2. $p+q$ is even and larger than two, so by the above argument, $p+q$ is not prime. This is a contradiction with the assumption that $p+q$ is prime. Therefore, our last assumption must be false, namely that both of the primes are not equal to 2. Thus, one of the primes, p or q, is equal to 2, as desired.

Extra §20.11 This one is dull. I probably meant §20.10, but at least we did that in class. As directed, we will prove by contradiction. Suppose $4 \mid n$. Suppose for the sake of contradiction $4 \mid n+2$. Because if a number divides two numbers, it divides their difference, we have $4 \mid 2$, but this is false. Therefore the supposition $4 \mid n+2$ must be false. Hence $4 \nmid n+2$ is true.

Extra §20.13 We attempt to prove that $(A \backslash B) \cap (B \backslash A) = \emptyset$. Since the empty set is a subset of all sets, we only need to prove that the set on the left is a subset of the empty set. I.e. we only need to prove there are no elements in the set on the left. We will do this by contradiction. Suppose $x \in A \backslash B) \cap (B \backslash A)$. This says four things: $x \in A, x \notin B, x \in B, x \notin A$. There are two contradictions there. That is enough to have completed our proof.

§20.14 Let A and B be sets. Prove $A \cap B = \emptyset$ if and only if $(A \times B) \cap (B \times A) = \emptyset$.

As an "if and only if" statement, there are two parts of this proof.

($\Rightarrow$) If $A \cap B = \emptyset$ then $(A \times B) \cap (B \times A) = \emptyset$.

There are two ways I can think of to prove this. Let me present them both. The easier seems to be using the contrapositive. The contrapositive of this statement is:

If $(A \times B) \cap (B \times A) \neq \emptyset$ then $A \cap B \neq \emptyset$.

proof: Suppose that $(A \times B) \cap (B \times A) \neq \emptyset$.
Therefore there is some $(x, y) \in (A \times B) \cap (B \times A)$
(remember that $(A \times B)$ contains ordered pairs not elements)
So, $(x, y) \in (A \times B)$ and $(x, y) \in (B \times A)$.
Hence $x \in A$ and $y \in B$ and $x \in B$ and $y \in A$.
In particular, $x \in A$ and $x \in B$, therefore $x \in A \cap B$.
So, $A \cap B \neq \emptyset$, as desired.

An alternate way to prove this part is using contradiction.
The statement to prove is: If $A \cap B = \emptyset$ then $(A \times B) \cap (B \times A) = \emptyset$.
Suppose that $A \cap B = \emptyset$.
Suppose for the sake of contradiction that $(A \times B) \cap (B \times A) \neq \emptyset$.
Therefore there is some $(x, y) \in (A \times B) \cap (B \times A)$
So, $(x, y) \in (A \times B)$ and $(x, y) \in (B \times A)$.
Hence $x \in A$ and $y \in B$ and $x \in B$ and $y \in A$.
In particular, $x \in A$ and $x \in B$, therefore $x \in A \cap B$.
So, $A \cap B \neq \emptyset$. However we assumed that $A \cap B = \emptyset$. This is a contradiction, so our most recent assumption, that $(A \times B) \cap (B \times A) \neq \emptyset$, must be false and we have that $(A \times B) \cap (B \times A) = \emptyset$, as desired.

($\Leftarrow$) If $(A \times B) \cap (B \times A) = \emptyset$ then $A \cap B = \emptyset$.
A proof by contrapositive (adapt to contradiction as above if desired). The contrapositive statement is: If $A \cap B \neq \emptyset$ then $(A \times B) \cap (B \times A) \neq \emptyset$.
proof: Suppose that $A \cap B \neq \emptyset$.
Therefore there is some $z \in A \cap B$.
Hence $z \in A$ and $z \in B$.
So, $(z, z) \in (A \times B)$ and $(z, z) \in (B \times A)$.
We then have $(z, z) \in (A \times B) \cap (B \times A)$. Therefore, $(A \times B) \cap (B \times A) \neq \emptyset$, as desired.

§22.9 A group of people stand in line to purchase movie tickets. The first person in line is a woman and the last person in line is a man. Use proof by induction to show that somewhere in the line a woman is directly in front of a man.

Because this involves almost no mathematics, this will be a very verbal argument. Notice as said in the hints section, the point here is to practice formulating an induction argument. I will attempt to write it out as explicitly as possible.

In order to prove the assertion using induction we need a number to induct on. The only relevant number in the problem is the number of people in line (which is also what the hints tell us to use). So, to reformulate the problem in order to prove it by induction we have:

For any natural number n, any line with n people where the first person in line is a woman and the last person in line is a man will have the property that somewhere in the line a woman is directly in front of a man.

For the base case, we find the smallest natural number n that satisfies the given condition. Because we need to have a woman first and a man last, we need to have at least 2 people in line, therefore the base case should be $n = 2$.

Base case: any line with 2 people where the first person in line is a woman and the last person in line is a man will have the property that somewhere in the line a woman is directly in front of a man.

Suppose we have a line with 2 people where the first person in line is a woman and the last person in line is a man. This says that the line is nothing more than a woman then a man. Obviously in this two person line, there is a woman directly in front of a man, in fact, that is all there is.

Induction step: Assume any line with k people where the first person in line is a woman and the last person in line is a man will have the property that somewhere in the line a woman is directly in front of a man.

It is our goal to prove that any line with $k+1$ people where the first person in line is a woman and the last person in line is a man will have the property that somewhere in the line a woman is directly in front of a man.

To prove this, suppose that we have a line with $k+1$ people where the first person in line is a woman and the last person in line is a man. Somehow to proceed we need to use the induction hypothesis, and reduce to a line with k people. To do this, we will remove someone from the line, and then return them later. At this stage in the argument there is a great amount of choice in which person you remove from the line. One thing you must do is be precise about which person you remove. It becomes unclear if you just say "remove someone from the line" without being more specific. My choice is to remove the first person, the woman, but I have seen many other choices work out fine. So, let us proceed ...

We know the first person in line is a woman. Pull her out of the line (into the theatre if you will). We are then left with a line with one fewer, or k people. If the second person in line is a man, then originally we had a woman then a man at the front of the line, and we have already found our adjacent pair in the original line, and we have completed the quest.

Therefore, suppose that the second person in line is a woman. The new line we are left with then is a line with k people which has a woman in front and a man in back (he hasn't changed). The induction hypothesis then applies to this new line and says that we have somewhere in the line a woman directly in front of a man. If we now take the woman who we pushed into the theatre and bring her back to the front of the line, the consecutive pair that we found in the k person line will still be a consecutive pair in the $k+1$ person line and we will have found our pair as desired.

Therefore, for any natural number n, any line with n people where the first person in line is a woman and the last person in line is a man will have the property that somewhere in the line a woman is directly in front of a man.

Extra §22.12 Prove (by induction, of course, weak induction, if you care) for every positive integer n, the Tower of Hanoi puzzle (with n disks) can be solved in $2^n - 1$ moves.

Base case: $n = 1$. We solve it in one move by moving the single disk from its starting point to its desired ending point. $1 = 2^1 - 1, \checkmark$.

Induction step: Suppose the puzzle with n disks can be solved in $2^n - 1$ moves. Consider that we have a puzzle with $n+1$ disks. Move the top n disks from the starting point to the free dowel (the one that you do not desire the stack to be moved to. This takes $(2^n - 1)$ moves by hypothesis. Next move the bottom disk to the desired location (which is free because it is not where we moved the top disks). This takes one move. Finally, move the stack from its current place to on top of the bottom disk. This takes $(2^n - 1)$ moves by hypothesis. Altogether we have $2^n - 1 + 1 + 2^n - 1 = 2(2^n) - 1 = 2^{n+1} - 1$ moves for $n+1$ disks, as desired.

Extra §22.14 Let $A_1, A_2, \ldots, A_n$ be sets (where $n \geq 2$). Suppose for any two sets either $A_i \subset A_j$ or $A_j \subset A_i$. We will prove by induction that given any such n sets, one of them is a subset of all of them.

Base case: Given any such 2 sets. Either $A_1 \subset A_2$ or $A_2 \subset A_1$ is given. Because $A_1 \subset A_1$ and $A_2 \subset A_2$, the one set that is a subset of the other is therefore a subset of both sets, as desired.

Induction step: Suppose given any k such sets, one of them is a subset of all of them. Our goal is to prove that given any $k+1$ such sets, one of them is a subset of all of them. To this end we suppose that we have $A_1, A_2, \ldots, A_{k+1}$ sets (where $k \geq 3$), and that for any two sets either $A_i \subset A_j$ or $A_j \subset A_i$. Consider the first k sets, $A_1, A_2, \ldots, A_k$. The induction hypothesis apples to them. So there is a set A_s that is a subset of each of the first k sets. Now we consider how A_s relates to A_{k+1}. By our given about these sets, either $A_s \subset A_{k+1}$ or $A_{k+1} \subset A_s$. In the first case A_s is subset of all of the $A_1, A_2, \ldots, A_{k+1}$ sets because it is a subset of the first k by induction hypothesis, and a subset of the last one as given above. In the second case $A_{k+1} \subset A_s \subset A_i$ for all $1 \leq i \leq k$; therefore A_{k+1} is a subset of all of the first k sets, and as it is a subset of itself, it is also therefore a subset of all sets. In either case, there is a set (either A_s or A_{k+1}) that is a subset of all sets, as desired.

Therefore, for any n such sets, one of them is a subset of all of them.

§22.16a Let $a_0 = 1$ and, for $n > 0$, let $a_n = 2a_{n-1} + 1$. The first few terms of the sequence $a_0, a_1, a_2, a_3 \ldots$ are $1, 3, 7, 15, \ldots$ What are the next three terms?

31, 63, 127.

Prove $a_n = 2^{n+1} - 1$.

We will use weak induction.

Base case: $n = 0$. $a_0 = 1 = 2^1 - 1\checkmark$

Induction step: Suppose that $a_k = 2^{k+1} - 1$.

It is our goal to prove that $a_{k+1} = 2^{(k+1)+1} - 1$.

We know by the recursion definition that $a_{k+1} = 2a_k + 1$, and we can use the induction hypothesis to find that $a_{k+1} = 2(2^{k+1} - 1) + 1 = (2^{k+2} - 2) + 1 = 2^{(k+1)+1} - 1$ as desired.

We have thus shown that for every $n, a_n = 2^{n+1} - 1$.

§22.16b Let $b_0 = 1$ and, for $n > 0$, let $b_n = 3b_{n-1} - 1$. What are the first five terms of the sequence $b_0, b_1, b_2 \ldots$?

1, 2, 5, 14, 41

Prove: $b_n = \frac{3^n+1}{2}$.

We will use weak induction.

Base case: $n = 0. b_0 = 1 = \frac{3^0+1}{2}\checkmark$

Induction step: Suppose that $b_k = \frac{3^k+1}{2}$.

It is our goal to prove that $b_{k+1} = \frac{3^{k+1}+1}{2}$.

We know by the recursion definition that $b_{k+1} = 3b_k - 1$, and we can use the induction hypothesis to find that $b_{k+1} = 3\left(\frac{3^k+1}{2}\right) - 1 = \frac{3^{k+1}+3}{2} - 1 = \frac{3^{k+1}+1}{2}$ as desired.

We have thus shown that for every $n, b_n = \frac{3^n+1}{2}$.

§22.16c Let $c_0 = 3$ and, for $n > 0$, let $c_n = c_{n-1} + n$. What are the first five terms of the sequence $c_0, c_1, c_2 \ldots$?

3, 4, 6, 9, 13

Prove: $c_n = \frac{n^2+n+6}{2}$.

We will use weak induction.

Base case: $n = 0$. $c_0 = 3 = \frac{0^2+0+6}{2}\checkmark$

Induction step: Suppose that $c_k = \frac{k^2+k+6}{2}$.

Our goal is to prove that $c_{k+1} = \frac{(k+1)^2+(k+1)+6}{2} = \frac{k^2+3k+8}{2}$.

We know by the recursion definition that $c_{k+1} = c_k + k + 1$, and we can use the induction hypothesis to find that $c_{k+1} = \frac{k^2+k+6}{2} + k + 1 = \frac{k^2+3k+8}{2}$ as desired.

We have thus shown that for every $n, c_n = \frac{n^2+n+6}{2}$.

Here's a cute alternate proof. We know that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$. This sequence is merely this series with three added, so equals $\frac{n(n+1)}{2} + 3 = \frac{n^2+n+6}{2}$. Seemingly no induction needed, but could be used to prove our known fact. It also could be proven combinatorially as we did in §17.

§22.16d Let $d_0 = 2, d_1 = 5$ and, for $n > 1$, let $d_n = 5d_{n-1} - 6d_{n-2}$. Why did we give two basis definitions?

Because the recursion definition doesn't start until $n > 1$ and needs two steps to look back upon.

What are the first five terms of the sequence $d_0, d_1, d_2, \ldots$?

2, 5, 13, 35, 97

Prove: $d_n = 2^n + 3^n$.

Because the recursion definition looks two steps backward, weak induction will not suffice. Hence we will use strong induction.

Base cases: $n = 0$. $d_0 = 2 = 2^0 + 3^0\checkmark$ $n = 1$. $d_1 = 5 = 2^1 + 3^1\checkmark$

Induction step: Suppose that $d_j = 2^j + 3^j$ for all $j \leq k$.

Our goal is to prove that $d_{k+1} = 2^{k+1} + 3^{k+1}$.

We know by the recursion definition that $d_{k+1} = 5d_k - 6d_{k-1}$, and we can use the induction hypothesis to find that $d_{k+1} = 5(2^k+3^k) - 6(2^{k-1}+3^{k-1}) = 5(2\cdot 2^{k-1}+3\cdot 3^{k-1}) - 6(2^{k-1}+3^{k-1}) = (5\cdot 2\cdot 2^{k-1} - 6\cdot 2^{k-1}) + (5\cdot 3\cdot 3^{k-1} - 6\cdot 3^{k-1}) = (10-6)2^{k-1} + (15-6)3^{k-1} = 4\cdot 2^{k-1} + 9\cdot 3^{k-1} = 2^2\cdot 2^{k-1} + 3^2\cdot 3^{k-1} = 2^{k+1} + 3^{k+1}$ as desired.

§22.16e Let $e_0 = 1, e_1 = 4$ and, for $n > 1$, let $e_n = 4(e_{n-1} - e_{n-2})$. What are the first five terms of the sequence $e_0, e_1, e_2, \ldots$?

1, 4, 12, 32, 80

Prove $e_n = (n+1)2^n$

We will use strong induction.

Base cases: $n = 0$. $e_0 = 1 = (0+1)2^0\checkmark$ $n = 1$. $e_1 = 4 = (1+1)2^1\checkmark$

Induction step: Suppose that $e_j = (j+1)2^j$ for all $j \leq k$.

Our goal is to prove that $e_{k+1} = ((k+1)+1)2^{k+1} = (k+2)2^{k+1}$.

We know by the recursion definition that $e_{k+1} = 4(e_k - e_{k-1})$, and we can use the induction hypothesis to find that $e_{k+1} = 4((k+1)2^k - k2^{k-1}) = 4((k+1)2\cdot 2^{k-1} - k(2^{k-1})) =$

$2^{k-1}(4(2(k+1)-k)) = 2^{k-1}(4(2k+2-k)) = 2^{k-1}(4(k+2)) = 2^{k-1}2^2(k+2) = (k+2)2^{k+1}$ as desired.

§22.16f Let F_n denote the nth Fibonacci number. Prove:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

Before we go on, let us note two facts that will be useful later: $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$ and $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$

Because the Fibonacci definition uses a two step induction, we will use strong induction.

Base cases: $n = 0$. $F_0 = 1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} \checkmark$

$n = 1$. $F_1 = 1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} \checkmark$

Now, to proceed with the induction step. Suppose that $F_j = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{j+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{j+1}}{\sqrt{5}}$ for all $j \le k$.

It is our goal to prove that $F_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}}{\sqrt{5}}$

We know by the recursion definition that $F_{k+1} = F_k + F_{k-1}$, and we can use the induction hypothesis to find that $F_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

$= \frac{\left(\left(\frac{1+\sqrt{5}}{2}\right)+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\left(\frac{1-\sqrt{5}}{2}\right)+1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

$= \frac{\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

(using the facts noted above) $= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^2\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}}{\sqrt{5}}$, as desired.

<u>Extra Problem</u>

Prove: If n is a positive integer, then the smallest perfect square that exceeds n^2 is $n^2 + 2n + 1$.

Prove this by contradiction. Suppose there is a perfect square, say z^2 between n^2 and $n^2 + 2n + 1 = (n+1)^2$. That is suppose $n^2 < z^2 < (n+1)^2$ for some integer z. This would seem to say there is an integer between n and $n+1$, which there clearly isn't.

Do we have all the details here? Not quite. Do we know if $n^2 < z^2$ that $n < z$? I don't think we assumed that. Let's prove it.

In fact, I'll prove this by contrapositive. Suppose $n \ge z$, prove $n^2 \ge z^2$.

Suppose $n \geq z$. Both n and z are positive integers, so multiplying both sides by n produces $n^2 \geq nz$ and multiplying both sides by z produces $nz \geq z^2$. By transitivity of greater than or equal to, $n^2 \geq z^2$, as claimed.

I think this fills all our gaps and we have $n < z < n + 1$ for an integer z, which is a contradiction.

Bonus Solutions

§21.8 Calculate the sum of the first n Fibonacci numbers for $n = 0, 1, 2, \ldots$. In other words calculate $F_0 + F_1 + \cdots + F_n = S_n$ for several values of n. Formulate a conjecture about these sums and prove it.

Either by our own work, or from the hints section of the text, we have

n	F_n	S_n
0	1	1
1	1	2
2	2	4
3	3	7
4	5	12
5	8	20
6	13	33
7	21	54
8	34	88
9	55	143
10	89	232

And we set off hunting for patterns. Here are some that I discovered with various people:

a. $S_{n+1} = S_n + F_{n+1}$

b. $S_n = 2F_n + S_{n-3}$

c. $S_n = F_{n+2} - 1$

The first is probably a bit *too* obvious, and is more just a definition of S_{n+1}. (See: $S_{n+1} = F_0 + F_1 + \cdots + F_n + F_{n+1} = (F_0 + F_1 + \cdots + F_n) + F_{n+1} = S_n + Fn + 1$.)

he second can probably be reduced to definitions, but it's interesting enough to call a conjecture and write up a proof. $S_n = F_0 + F_1 + \cdots + F_{n-3} + F_{n-2} + F_{n-1} + F_n = (F_0 + F_1 + \cdots + F_{n-3}) + (F_{n-2} + F_{n-1}) + F_n = S_{n-3} + F_n + F_n = S_{n-3} + 2F_n$

The third conjecture is probably what the book intended and the most interesting of the conjectures that I've seen. This conjecture can be proven by induction.

Base case: $n = 0$. Looking at the above table we see that $S_0 = 1$ and $F_2 = 2$, and we have $1 = 2 - 1$. ✓

Induction step. Suppose $S_k = F_{k+2} - 1$. It is our goal to prove $S_{k+1} = F_{k+3} - 1$.

Start with $S_{k+1} = S_k + F_{k+1}$ (see conjecture a. above) then use the induction hypothesis to get $S_{k+1} = (F_{k+2} - 1) + F_{k+1} = (F_{k+1} + F_{k+2}) - 1 = F_{k+3} - 1$ (the final step uses the recursion for the Fibonacci sequence). By weak induction we have proven that for all $n, S_n = F_{n+2} - 1$. There are probably other conjectures out there, but these are the only three I've thought about.

§22.17 A flagpole is n feet tall. On this pole we display flags of the following types: red flags that are 1 foot tall, blue flags that are 2 feet tall, and green flags that are 2 feet tall. The sum of the heights of the flags is exactly n feet.

Prove that there are $\frac{2}{3}2^n + \frac{1}{3}(-1)^n$ ways to do this . The hints section says "Let a_n denote the number of possible solutions. Find a recurrence relation for a_n." Let's list out the first few examples:

n	a_n	flag arrangements
0	1	no flags
1	1	R
2	3	B, G, RR
3	5	RB, RG, BR, GR, RRR
4	11	BB, BG, GB, GG, RRB, RRG, RBR, RGR, BRR, GRR, RRRR
5	21	too many to list

My next step was to examine the an numbers and look for a pattern of how I could get one number from previous numbers. Some people discovered a numeric pattern, that $a_n = 2a_{n-1} \pm 1$, where $\pm$ is determined by whether n is even or odd. I see the numeric pattern, but cannot justify it in terms of the flags. (I am *not* saying that this cannot be done, merely that I do not see the argument now.) The numeric pattern I saw is: $a_n = 2a_{n-2} + a_{n-1}$, for $n \geq 2$. This can be seen in the flags as listed above (in fact, I organized the arrangements to make this more clear). We can construct flag arrangements of n feet in two ways, by looking back two feet and adding either a blue or green flag on the bottom, or by looking back one foot and adding a red flag on the bottom. This can be seen in the examples of 2, 3 and 4 feet above. Examining 3 in detail: we look back to the 1 foot arrangements and see there is one, R, to that we can either add blue or green to get RB and RG, the first two arrangements on the 3 feet list; then we look back to the 2 feet arrangements and see there are three, B, G, and RR, to that we can add red flags to get the last three possible 3 feet arrangements, BR, GR, RRR. I suggest you examine the 2 and 4 cases to further understand how this recurrence relation works.

So far we have proven that $a_0 = 1, a_1 = 1$, and $a_n = 2a_{n-2} + a_{n-1}$, for $n \geq 2$. Our goal is to prove that $a_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n$. We will prove this by strong induction.

Base cases: $n = 0$. $a_0 = 1 = \frac{2}{3}2^0 + \frac{1}{3}(-1)^0 = \frac{2}{3} + \frac{1}{3}\checkmark$

$n = 1$. $a_1 = 1 = \frac{2}{3}2^1 + \frac{1}{3}(-1)^1 = \frac{2}{3}2 - \frac{1}{3} = \frac{4}{3} - \frac{1}{3}\checkmark$

Induction step: Suppose that $a_j = \frac{2}{3}2^j + \frac{1}{3}(-1)^j$ for all $j \leq k$. Our goal is to prove that $a_{k+1} = \frac{2}{3}2^{k+1} + \frac{1}{3}(-1)^{k+1}$.

We know by the recursion definition that $a_{k+1} = 2a_{k-1} + a_k$, and we can use the induction hypothesis to find that $a_{k+1} = 2\left(\frac{2}{3}2^{k-1} + \frac{1}{3}(-1)^{k-1}\right) + \left(\frac{2}{3}2^k + \frac{1}{3}(-1)^k\right)$

$= 2\left(\frac{2}{3}2^{k-1} + \frac{1}{3}(-1)^{k-1}\right) + \left(\frac{2}{3}(2)2^{k-1} + \frac{1}{3}(-1)(-1)^{k-1}\right)$

$= (2)\frac{2}{3}2^{k-1} + \frac{2}{3}(2)2^{k-1} + (2)\frac{1}{3}(-1)^{k-1} + \frac{1}{3}(-1)(-1)^{k-1}$

$= 2^{k-1}\left((2)\frac{2}{3} + \frac{2}{3}(2)\right) + (-1)^{k-1}\left((2)\frac{1}{3} + \frac{1}{3}(-1)\right)$

$= 2^{k-1}\left((4)\frac{2}{3}\right) + (-1)^{k-1}\left(\frac{1}{3}\right)$ (remembering that $(-1)^2 = 1$)

$= 2^{k-1}(2^2)\frac{2}{3} + (-1)^{k-1}(-1)^2\frac{1}{3}$

$= 2^{k+1}\frac{2}{3} + (-1)^{k+1}\frac{1}{3} = \frac{2}{3}2^{k+1} + \frac{1}{3}(-1)^{k+1}$, as desired.