Mathematics 239 extra solutions to Section 13

§13.2 is not combinatorial. It is algebra and the anti-point as far as we're concerned.

§13.3 The right hand side counts the number of lists of length n from three elements, with repetition, without one of them. Let's say the three elements are $\{0, 1, 2\}$. Then what we're looking at is three digit strings and we'll say the one that we're not counting is the one with all zeros. In other words we're looking at n-digit non-zero ternary strings (allowing the opening digits to be zeroes). There are 3^n total strings (each digit can be in each place), but if we don't count the one with all zeroes, then there are $3^n - 1$ such strings.

Now, what about the other side? We're looking at cases because of the sum. The decreasing exponent should give us a clue about decreasing length. Since zero already is a focus, we will think about cases based on how many zeroes the string starts with. If the string does not begin with zero, there are 2 other numbers it could begin with, and then the remaining $(n-1)$ digits can be anything, so we have $2 \cdot 3^{n-1}$. If the string begins with one zero, then there are two other numbers for the next digit, and the remaining $(n-2)$ digits can be anything, so we have $2 \cdot 3^{n-2}$. In the middle, if the string begins with $k-1$ zeros, there are two numbers for the next digit, and the remaining $(n-k)$ digits can be anything, so we have $2 \cdot 3^{n-k}$. This continues to the last case where all but the last digit is zero. The last digit then has two choices, and there a no remaining digits, but it is not wrong to say there are $3^0 = 1$ choices for the remaining digits and it does fit the pattern to say there are $2 \cdot 3^0$ strings with all but the last digit zero.

Putting this all together we have

$$3^n - 1 = \sum_{k=0}^{n-1} 2 \cdot 3^k = 2 \cdot 3^{n-1} + 2 \cdot 3^{n-2} + \cdots + 2 \cdot 3^{n-k} + \cdots + 2 \cdot 3^1 + 2 \cdot 3^0$$

(which is identical to the text, just commuting the sides and the addition).

The remaining part is mostly silly. So, for that case we're not as much writing a combinatorial proof, but just thinking about numbers. $10^n - 1$, if you think about it, is a string of n nines. Then in expanded notation, that string of n nines is just $9 \cdot 10^{n-1} + 9 \cdot 10^{n-2} + \cdots + 9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0$. You might notice that there are n places from 0 to $(n-1)$. This is not as important to us as the above combinatorial proof. Perhaps unsurprisingly.

§13.6 a. The number of two element lists (a, b) where $a \neq b$ is $(n+1)n$ (remember we are working with $0, 1, \ldots, n$, which, again, is $n+1$ numbers. In half of them $a < b$ and in half of them $a > b$. We want the first half, so there are $\frac{(n+1)n}{2}$, as desired.

b. is exactly the same as Proposition 17.5, putting $a < b$ allows us to interchange between lists and sets.

§13.7 The text is a little unclear, but it's trying to give us a hint. I will return to the hint, but start with easier side first. n^2 counts the number of two-element lists from our favourite n-element set.

Now we will approach the left-side. The hint seems to imply we should consider cases based on the largest element in the list. Ok, let's try that. There is only one two-element list where the largest element is 1, namely 11. There are three two-element lists where the largest element is 3, namely 21, 22, 12. And there are five two-element lists where the largest element is 5, namely 31, 32, 33, 23, 13. In fact, writing that list in an organised fashion gives some clues. Following that pattern we can see that the two-element lists where the largest element is a can be written as $a1, a2, a3, \dots a(a-1), aa, (a-1)a, (a-2)a, \dots 2a, 1a$. Looking at this we see there are a lists at the beginning (from $a1$ to aa) and a lists at the end (from aa to $1a$). That gives a total of $2a$, but the list aa is counted twice, so we have $2a-1$ two-element lists in which a is the largest. That should be the key step. So, using that reasoning, there are $2n-1$ lists where the largest element is n. That is the last case.

So, $n^2 = \sum_{a=1}^{n}(2a-1) = 1 + 3 + 5 + \cdots + (2n-1)$ as they both count the number of two-element lists from our favourite n-element set.