Unions and Intersections of Indexed Sets

A nice idea of work for me . . . write up some notes on unions and intersections of indexed sets to supplement the lack in the text.

Here's the basic idea. Given some indexed sets, I_a, the intersection $\bigcap_{a\in A} I_a$ is the set of elements in *every one* of the sets I_a. On the other hand, the union $\bigcup_{a\in A} I_a$ is the set of elements in *at least one* of the sets I_a.

And now for several examples. Most of our examples will be intervals in $\mathbb{R}$. Remember $(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.

Consider $S_n = (-5, n]$, where $n \in E$, where E is the set of positive even integers. Let's think of this a listing sets first, with intersections we have:

$$(-5,2] \cap (-5,4] \cap (-5,6] \cap (-5,8] \cap \cdots$$

In this example the sets are all "nested", i.e. one inside the other. So, the intersection, the elements that are in all of them is the smallest one. Therefore $\bigcap_n S_n = (-5, 2]$. We could say that the union is the largest one, but there is no largest one. That gives us the right idea, though, and as the numbers grow larger this leads us to $\bigcup_n S_n = (-5, \infty)$.

Next we'll look at $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, where n is a positive integer. Again the sets are nested. Listing we have:

$$\left(-\frac{1}{1},\frac{1}{1}\right)\bigcap\left(-\frac{1}{2},\frac{1}{2}\right)\bigcap\left(-\frac{1}{3},\frac{1}{3}\right)\bigcap\left(-\frac{1}{4},\frac{1}{4}\right)\bigcap\left(-\frac{1}{5},\frac{1}{5}\right)\bigcap\cdots$$

This time, however, the sets are getting smaller. Here, the union is the biggest one, $\bigcup_n A_n = (-1, 1)$, but the intersection requires a little more thought because of the sets getting smaller. The only number that is in all of these sets is zero, therefore the intersection contains only zero, $\bigcap_n A_n = [0, 0] = \{0\}$.

Here are some more similar examples of nested sets where the indexing set is the positive integers:

$$B_n = \left[2 - \tfrac{1}{n^2}, 4 + \tfrac{1}{n^2}\right], \bigcap_n B_n = [2,4], \bigcup_n B_n = [1,5]$$
$$C_n = [n^3, \infty), \bigcap_n C_n = \emptyset, \bigcup_n C_n = [1, \infty)$$
$$D_n = (0, \tfrac{1}{n}), \bigcap_n D_n = \emptyset, \bigcup_n D_n = (0,1)$$

And some different examples: Let $E_n = (n, n+1)$, where n is an integer. These sets are disjoint. So, $\bigcap_n E_n = \emptyset$, and their union misses the integers, so $\bigcup_n E_n = \mathbb{R}\backslash\mathbb{Z}$.

What if we have a set of elements rather than intervals? Let $F_n = \{n, n-1, n^2\}$ for $n \in \mathbb{N}$. In this case the sets are "spread out"; we can find two that don't overlap, $F_1 = \{1, 0, 1\}, F_5 = \{5, 4, 25\}$, so $\bigcap_n F_n = \emptyset$, and elements get repeated, leaving us with $\bigcup_n F_n = \{-1\} \cup \mathbb{N}$.

One last example. Let $G_n = \left(\frac{1}{n+3}, \frac{1}{n+1}\right)$ for $n \in \mathbb{N}$. Again, we can separate the sets $G_0 = \left(\frac{1}{3}, \frac{1}{1}\right)$ and $G_4 = \left(\frac{1}{6}, \frac{1}{4}\right)$, so $\bigcap_n G_n = \emptyset$. They do overlap though, and together give us $\bigcup_n G_n = (0, 1)$.