3

Celestial Mechanics

3.1 The Calculus of Curves

We are not quite ready to prove that Newton's law of gravitational attraction implies Kepler's second law. We need to take a closer look at the Calculus of vector functions in the light of the vector algebra described in the last chapter.

As in Section 1.5, let $\vec{r}(t)$ denote the position of a moving particle at time t. The *derivative* of $\vec{r}(t)$ is defined as it is for scalar functions:

$$\frac{d}{dt}\vec{r}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}. \tag{3.1}$$

Observe that $\vec{r}(t+h) - \vec{r}(t)$ is the vector from $\vec{r}(t)$ to $\vec{r}(t+h)$, which can be viewed as a chord of the curve traced by our moving particle (Figure 3.1). In the limit, this becomes a tangent to the curve. The velocity vector is defined to be this derivative:

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t). \tag{3.2}$$

The acceleration vector is the derivative of the velocity:

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t). \tag{3.3}$$

Throughout this chapter, we shall assume that the first two derivatives of $\vec{r}(t)$ exist.

Rules of Differentiation

Derivatives of vector functions satisfy the same basic rules of differentiation that hold for scalar functions.

Theorem 3.1 *If $\vec{r}(t)$ is constant, then*

$$\frac{d}{dt}\vec{r}(t) = \vec{0}.$$

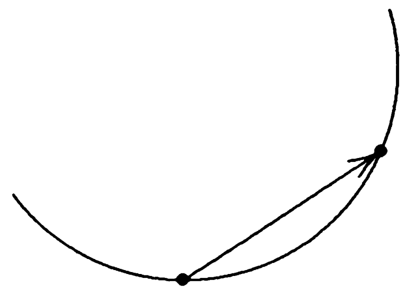

FIGURE 3.1. $\vec{r}(t+h) - \vec{r}(t)$.

Proof: If $\vec{r}(t)$ is constant, then

$$\vec{r}(t+h) - \vec{r}(t) = \vec{0},$$

so that

$$\frac{d}{dt}\vec{r}(t) = \lim_{h\to 0} \frac{\vec{0}}{h} = \vec{0}.$$

Q.E.D.

Theorem 3.2 *Let $\vec{r}(t)$ and $\vec{s}(t)$ be differentiable vector functions, $\lambda(t)$ a differentiable scalar function, and c a real constant. We have*

$$\frac{d}{dt}(c\vec{r}) = c\frac{d\vec{r}}{dt}, \tag{3.4}$$

$$\frac{d}{dt}(\vec{r}+\vec{s}) = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt}, \tag{3.5}$$

$$\frac{d}{dt}(\lambda\vec{r}) = \frac{d\lambda}{dt}\vec{r} + \lambda\frac{d\vec{r}}{dt}, \tag{3.6}$$

$$\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}, \tag{3.7}$$

$$\frac{d}{dt}(\vec{r}\times\vec{s}) = \frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt}. \tag{3.8}$$

Proof: The proofs exactly mimic those for scalar functions. To prove Equation (3.4) we observe that

$$\begin{aligned}\frac{d}{dt}(c\vec{r}) &= \lim_{h\to 0}\frac{c\vec{r}(t+h) - c\vec{r}(t)}{h} \\ &= \lim_{h\to 0} c\,\frac{\vec{r}(t+h) - \vec{r}(t)}{h}\end{aligned}$$

$$\begin{aligned} &= c \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\ &= c \frac{d\vec{r}}{dt}. \end{aligned}$$

The proof of Equation (3.5) is similarly straightforward and is left as an exercise. For Equation (3.6), we insert

$$-\lambda(t)\,\vec{r}(t+h) + \lambda(t)\,\vec{r}(t+h) = 0$$

into the numerator of the definition of the derivative:

$$\begin{aligned} &\frac{d}{dt}(\lambda\vec{r}) \\ &= \lim_{h \to 0} \frac{\lambda(t+h)\vec{r}(t+h) - \lambda(t)\vec{r}(t)}{h} \\ &= \lim_{h \to 0} \frac{\lambda(t+h)\vec{r}(t+h) - \lambda(t)\vec{r}(t+h) + \lambda(t)\vec{r}(t+h) - \lambda(t)\vec{r}(t)}{h} \\ &= \lim_{h \to 0} \left(\frac{\lambda(t+h) - \lambda(t)}{h} \vec{r}(t+h) + \lambda(t) \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right) \\ &= \left(\lim_{h \to 0} \frac{\lambda(t+h) - \lambda(t)}{h} \right) \left(\lim_{h \to 0} \vec{r}(t+h) \right) \\ &\quad + \lambda(t) \left(\lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right) \\ &= \frac{d\lambda}{dt}\,\vec{r} + \lambda\,\frac{d\vec{r}}{dt}. \end{aligned}$$

What makes this proof work is the distributive law:

$$\begin{aligned} \lambda(t+h)\vec{r}(t+h) - \lambda(t)\vec{r}(t+h) &= (\lambda(t+h) - \lambda(t))\,\vec{r}(t+h), \\ \lambda(t)\vec{r}(t+h) - \lambda(t)\vec{r}(t) &= \lambda(t)\,(\vec{r}(t+h) - \vec{r}(t)). \end{aligned}$$

Notice that we need both forms of this law, Equations (2.3) and (2.4). We have proven that the distributive law also holds for dot products and cross products, and so exactly the same argument yields Equations (3.7) and (3.8).

Q.E.D.

One corollary to this theorem is a result we used in Section 1.5.

Corollary 3.1 *If* $\vec{r}(t) = (x(t), y(t), z(t))$*, then*

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right).$$

That is, differentiation of a vector function expressed in coordinate form is performed by differentiating each of the coordinates.

Proof: We use the additive rule [Equation (3.5)], the rule for the derivative of a product [Equation (3.6)], and the fact that $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ are constants so that

$$\frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{k}}{dt} = \vec{0}.$$

It follows that

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\vec{\imath}+y\vec{\jmath}+z\vec{k}) = \frac{dx}{dt}\vec{\imath}+\frac{dy}{dt}\vec{\jmath}+\frac{dz}{dt}\vec{k} = \left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right).$$

Q.E.D.

Corollary 3.2 *If $d\vec{r}/dt$ is zero for all t, then $\vec{r}(t)$ is a constant vector.*

Proof: We can write

$$\vec{r}(t) = (x(t), y(t), z(t)),$$

so that $d\vec{r}/dt = 0$ if and only if

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0.$$

From single variable calculus, this implies that x, y, and z are constants.

Q.E.D.

The next corollary may be slightly surprising at first glance, but a little consideration of what it means should convince you that you already know it.

Corollary 3.3 *If $|\vec{r}(t)|$ is a constant independent of t, then $\vec{r}(t)$ is perpendicular to $\vec{v}(t)$.*

Proof: By our hypothesis,

$$\vec{r}\cdot\vec{r} = |\vec{r}|^2 = c,$$

where c is a constant. Differentiating both sides with respect to t yields

$$0 = \frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d\vec{r}}{dt}\cdot\vec{r}+\vec{r}\cdot\frac{d\vec{r}}{dt} = 2\vec{r}\cdot\vec{v},$$

and therefore, $\vec{r}$ and $\vec{v}$ are perpendicular.

Q.E.D.

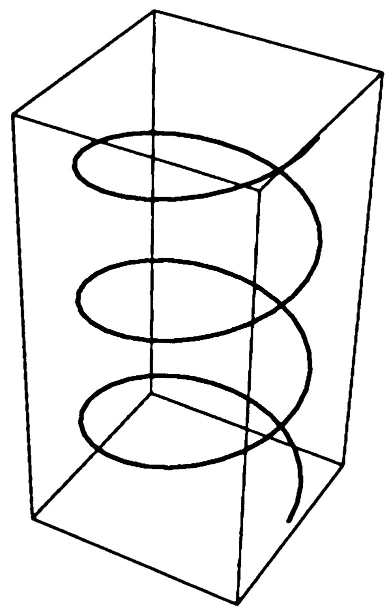

FIGURE 3.2. The curve $\vec{r}(t) = (\cos t, \sin t, t)$.

What we have just proved is that the tangent to a circle is always perpendicular to the radius.

Example

As an example, we take a path that spirals up a vertical cylinder of radius 1 (Figure 3.2):

$$\vec{r}(t) = (\cos t, \sin t, t).$$

The velocity is

$$\vec{v}(t) = (-\sin t, \cos t, 1).$$

The magnitude of this velocity is

$$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2},$$

a constant, and so Corollary 3.3 implies that the acceleration will always be perpendicular to the velocity. In fact, the acceleration is

$$\vec{a}(t) = (-\cos t, -\sin t, 0),$$

and it is easily seen that

$$\vec{v} \cdot \vec{a} = 0.$$

Arc Length and Tangents

We now define the following functions. The distance from the origin is

$$r(t) \;=\; |\vec{r}(t)|, \tag{3.9}$$

the speed is

$$v(t) \;=\; |\vec{v}(t)|, \tag{3.10}$$

the arc length is

$$s(t_1) \;=\; \int_0^{t_1} v(t)\,dt, \tag{3.11}$$

the unit tangent is

$$\vec{T}(t) \;=\; \frac{\vec{v}(t)}{v(t)}, \quad v(t) \neq 0, \tag{3.12}$$

and the principal normal is

$$\vec{N}(t) \;=\; \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \quad \vec{T}'(t) = \frac{d\vec{T}}{dt} \neq \vec{0}. \tag{3.13}$$

We have already met $r(t)$. The speed is the absolute value of the velocity. Integrating the speed or distance traveled per unit time over an interval of time gives the total distance traveled in that time, denoted by $s(t)$. Starting the integral at 0 is an arbitrary convention since we usually treat the arc length not as a function of one time variable but of two: the arc length from $t = t_0$ to $t = t_1$:

$$s(t_1) - s(t_0) = \int_{t_0}^{t_1} v(x)\,dx. \tag{3.14}$$

The unit vector in the direction of $\vec{v}(t)$ is called the *unit tangent* or simply the *tangent*. The principal normal has two important properties given in the next theorem.

Theorem 3.3 *The principal normal, $\vec{N}(t)$, is perpendicular to the tangent, $\vec{T}(t)$, and, if $\vec{v}(t)$ and $\vec{a}(t)$ are not parallel, then $\vec{N}(t)$ lies in the plane spanned by $\vec{v}(t)$ and $\vec{a}(t)$.*

Proof: Since $|\vec{T}(t)| = 1$, Corollary 3.3 implies that $\vec{T}'(t)$, and thus $\vec{N}(t)$, is perpendicular to $\vec{T}(t)$. Since $\vec{v}(t)$ and $\vec{a}(t)$ are not parallel, neither of them is identically $\vec{0}$. We have

$$\vec{a}(t) \;=\; \frac{d}{dt}\vec{v} \;=\; \frac{d}{dt}(v\vec{T}) \;=\; v'\vec{T} + v\vec{T}' \;=\; \frac{v'}{v}\vec{v} + v|\vec{T}'|\vec{N}. \tag{3.15}$$

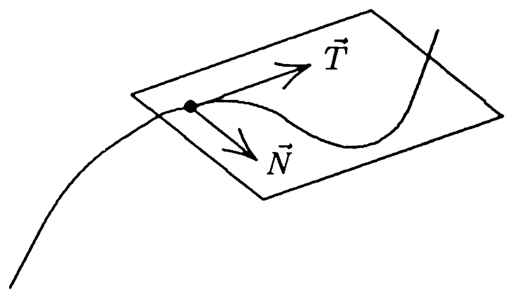

FIGURE 3.3. An osculating plane for $\vec{r}(t) = (\cos t, \sin t, t)$.

Since $\vec{a}$ and $\vec{v}$ are not parallel, $v|\vec{T}'|$ is not zero. Therefore, we have

$$\vec{N} = \frac{1}{v|\vec{T}'|}\vec{a} - \frac{v'}{v^2|\vec{T}'|}\vec{v}.$$

Q.E.D.

What we have demonstrated is that $\vec{T}$ and $\vec{N}$ are perpendicular unit vectors spanning the plane defined by $\vec{v}$ and $\vec{a}$, and thus $\vec{T}$ and $\vec{N}$ provide a convenient basis for describing points in this plane. If we translate this plane so that it passes through $\vec{r}(t)$,

$$\{\vec{r} + \alpha\vec{T} + \beta\vec{N} \mid \alpha, \beta \in \mathbf{R}\},$$

we obtain what is called the *osculating* (or kissing) *plane* (see Figure 3.3). Remember that $\vec{r}$, $\vec{T}$, and $\vec{N}$ are all functions of time, t, so that our plane changes over time. The significance of the osculating plane is that if our acceleration were constant, then our curve would lie in this plane. It thus provides us with a plane we can consider to be tangent to the curve.

Example

Returning to our spiral,

$$\vec{r}(t) = (\cos t, \sin t, t),$$

we have

$$r(t) = \sqrt{\cos^2 t + \sin^2 t + t^2} = \sqrt{1 + t^2},$$

$$\begin{aligned} v(t) &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}, \\ s(t) &= \int_0^t \sqrt{2}\, dx = t\sqrt{2}, \\ \vec{T}(t) &= \frac{\sqrt{2}}{2}(-\sin t, \cos t, 1), \\ \vec{N}(t) &= (-\cos t, -\sin t, 0). \end{aligned}$$

A perpendicular to the osculating plane is given by

$$\vec{v} \times \vec{a} = (\sin t, -\cos t, 1).$$

The dot product of this perpendicular with $\vec{r}$ is

$$\vec{v} \times \vec{a} \cdot \vec{r} = t,$$

and so the equation of the osculating plane passing through $\vec{r}$ is

$$\vec{v} \times \vec{a} \cdot (x, y, z) - \vec{v} \times \vec{a} \cdot \vec{r} = 0,$$

which is

$$(\sin t)x - (\cos t)y + z - t = 0.$$

For example, the osculating planes at $t = 0, \pi/2$, and $5\pi/4$ are, respectively,

$$\begin{aligned} -y + z &= 0, \\ x + z - \pi/2 &= 0, \\ -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y + z - \frac{5\pi}{4} &= 0. \end{aligned}$$

Curvature

We next investigate the notion of *curvature*, finding the radius of the circle that best approximates our curve. Let us start by assuming that the curve traced by $\vec{r}(t)$ is in fact an arc of a circle of radius ρ lying in some plane. We specify some fixed direction in that plane and let $\alpha(t)$ be the angle between the tangent, $\vec{T}$, and our fixed direction (Figure 3.4). We view our plane so that the path travels counterclockwise around the center of the circle.

We consider the derivative $d\alpha/ds$, the rate at which α changes with respect to the arc length. On a circle, this is constant, and the value of this constant can be determined by considering what happens if we go

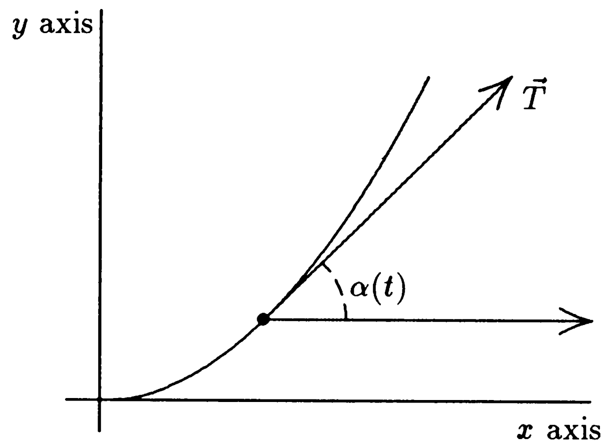

FIGURE 3.4. The angle $\alpha(t)$ between $\vec{T}$ and the chosen direction.

completely around the circle: α will have changed by 2π while the arc length traversed is $2\pi\rho$,

$$\frac{d\alpha}{ds} = \frac{2\pi}{2\pi\rho} = \frac{1}{\rho}. \tag{3.16}$$

For a curve lying in a plane, $d\alpha/ds$ is well defined, and we can define ρ as the reciprocal of this derivative at the point in question. To calculate ρ, we observe that

$$\frac{d\alpha}{ds} = \frac{d\alpha/dt}{ds/dt}.$$

From Equation (3.11), we see that

$$\frac{ds}{dt} = v(t).$$

If we standardize our plane so that $\vec{\imath}$ is the unit vector in our chosen direction and $\vec{\jmath}$ is the perpendicular unit vector in the plane, then

$$\vec{T}(t) = (\cos\alpha)\,\vec{\imath} + (\sin\alpha)\,\vec{\jmath}.$$

Differentiating both sides with respect to t gives us

$$\vec{T}'(t) = \alpha'(t)\,[-(\sin\alpha)\,\vec{\imath} + (\cos\alpha)\,\vec{\jmath}].$$

The vector in parentheses is a unit vector and so

$$|\vec{T}'(t)| = |\alpha'(t)|.$$

Since the particle is moving counterclockwise, the angle $\alpha(t)$ is increasing, and so $\alpha'(t)$ is positive:

$$\frac{d\alpha}{dt} = |\vec{T}'(t)|.$$

We have shown that

$$\frac{d\alpha}{ds} = \frac{d\alpha/dt}{ds/dt} = \frac{|\vec{T}'(t)|}{v(t)}.$$

This gives us a definition of curvature that is valid for any curve for which $\vec{T}'$ is well defined.

Definition: The *curvature* of the path $\vec{r}(t)$, denoted by $\kappa(t)$, is defined to be the reciprocal of the radius of the circle that best approximates the curve at $\vec{r}(t)$. Specifically, this is defined to be

$$\kappa(t) = \frac{|\vec{T}'(t)|}{v(t)}. \tag{3.17}$$

The radius $\rho(t) = 1/\kappa(t)$ is called the *radius of curvature.* Note that the curvature is zero if and only if the path is a straight line.

Computing $|\vec{T}'(t)|$ from the definition is often difficult. The following theorem provides us with a more direct approach to computing κ.

Theorem 3.4 *If $\vec{r}(t)$ is twice differentiable and the first derivative is not $\vec{0}$, then the curvature of the path traced by $\vec{r}(t)$ is given by*

$$\kappa = \frac{|\vec{a} \times \vec{v}|}{v^3}. \tag{3.18}$$

Proof: From Equations (3.15) and (3.17) we see that

$$\vec{a}(t) = v'\vec{T} + v|\vec{T}'|\vec{N} = v'\vec{T} + v^2\kappa\vec{N},$$

and so, since $\vec{T} \times \vec{v} = \vec{0}$,

$$\vec{a} \times \vec{v} = v'\vec{T} \times \vec{v} + v^2\kappa\vec{N} \times \vec{v} = v^2\kappa\vec{N} \times \vec{v} = v^3\kappa\vec{N} \times \vec{T}.$$

Since $|\vec{N} \times \vec{T}| = 1$, we finally arrive at

$$|\vec{a} \times \vec{v}| = v^3\kappa.$$

Q.E.D.

Examples

The curvature of $\vec{r}(t) = (\cos t, \sin t, t)$ is

$$\begin{aligned} \kappa &= \frac{|(-\sin t, \cos t, 1) \times (-\cos t, -\sin t, 0)|}{2^{3/2}} \\ &= \frac{|(\sin t, -\cos t, 1)|}{2^{3/2}} = \frac{\sqrt{2}}{2^{3/2}} = \frac{1}{2}, \end{aligned}$$

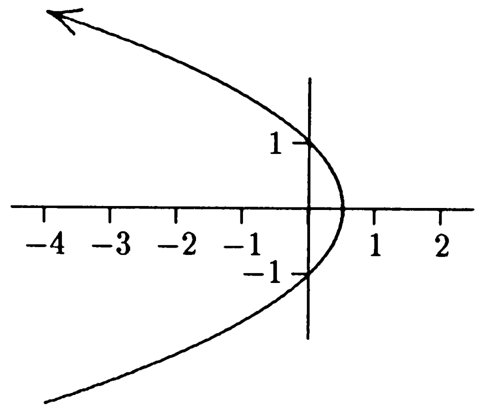

FIGURE 3.5. The curve $r + r\cos\theta = 1$.

and the radius of curvature is 2.

As a second example, consider the problem of finding the velocity, acceleration, and curvature of the path of a particle following the parabola given in polar coordinates by

$$r(1+\cos\theta) = 1, \tag{3.19}$$

which is traversed at a constant speed,

$$v(t) = 2,$$

in a counterclockwise direction about the origin (Figure 3.5).

If we differentiate both sides of Equation (3.19) with respect to t and use the relationship

$$1+\cos\theta = r^{-1}, \tag{3.20}$$

we see that

$$\frac{dr}{dt}(1+\cos\theta) - r\sin\theta\,\frac{d\theta}{dt} = 0,$$

$$\frac{dr}{dt}r^{-1} - r\sin\theta\,\frac{d\theta}{dt} = 0,$$

$$\frac{d\theta}{dt} = \frac{1}{r^2\sin\theta}\frac{dr}{dt}. \tag{3.21}$$

We now recall from Equation (1.15) that

$$\vec{v} = \frac{dr}{dt}\,\vec{u}_r + r\frac{d\theta}{dt}\,\vec{u}_\theta,$$

where $\vec{u}_r$ and $\vec{u}_\theta$ are perpendicular unit vectors, and so

$$v = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\,\frac{d\theta}{dt}\right)^2}. \tag{3.22}$$

Combining this with the expression for $d\theta/dt$ given in Equation (3.21) and recalling that we have a constant speed of 2, we see that

$$2 = v = \left|\frac{dr}{dt}\right|\sqrt{1+\frac{1}{r^2\sin^2\theta}} = \left|\frac{dr}{dt}\right|\frac{\sqrt{1+r^2\sin^2\theta}}{|r\sin\theta|}.$$

Using the fact that our curve is traversed counterclockwise, we can choose the proper sign, and we see that

$$\begin{aligned}\frac{dr}{dt} &= \frac{2r\sin\theta}{\sqrt{1+r^2\sin^2\theta}},\\ \frac{d\theta}{dt} &= \frac{2}{r\sqrt{1+r^2\sin^2\theta}}.\end{aligned}$$

We can simplify the square root by using the fact that $\sin^2\theta = 1-\cos^2\theta$ and, from Equation (3.20), that $\cos\theta = r^{-1}-1$:

$$1+r^2\sin^2\theta = 1+r^2-r^2\cos^2\theta = 1+r^2-r^2(r^{-1}-1)^2 = 2r.$$

We have shown that

$$\frac{dr}{dt} = \sin\theta\sqrt{2r}, \tag{3.23}$$

$$\frac{d\theta}{dt} = r^{-3/2}\sqrt{2}, \tag{3.24}$$

$$\vec{v} = \sqrt{2}\left(r^{1/2}\sin\theta\,\vec{u}_r + r^{-1/2}\vec{u}_\theta\right). \tag{3.25}$$

If we differentiate Equation (3.25), remembering that r, θ, $\vec{u}_r$, and $\vec{u}_\theta$ are all functions of t, we get the acceleration

$$\begin{aligned}\vec{a} &= \sqrt{2}\left(\frac{1}{2}r^{-1/2}\frac{dr}{dt}\sin\theta\,\vec{u}_r + r^{1/2}\cos\theta\frac{d\theta}{dt}\vec{u}_r + r^{1/2}\sin\theta\frac{d\theta}{dt}\vec{u}_\theta\right.\\ &\quad\left.-\frac{1}{2}r^{-3/2}\frac{dr}{dt}\vec{u}_\theta - r^{-1/2}\frac{d\theta}{dt}\vec{u}_r\right).\end{aligned}$$

Substituting our values of dr/dt and $d\theta/dt$ from Equations (3.23) and (3.24) yields

$$\begin{aligned}\vec{a} &= (\sin^2\theta)\,\vec{u}_r + 2r^{-1}(\cos\theta)\,\vec{u}_r + 2r^{-1}(\sin\theta)\,\vec{u}_\theta\\ &\quad - r^{-1}(\sin\theta)\,\vec{u}_\theta - 2r^{-2}\vec{u}_r\\ &= \left(\sin^2\theta + 2r^{-1}(\cos\theta - r^{-1})\right)\vec{u}_r + r^{-1}(\sin\theta)\,\vec{u}_\theta\\ &= \left(\sin^2\theta + 2r^{-1}\right)\vec{u}_r + r^{-1}(\sin\theta)\,\vec{u}_\theta,\end{aligned} \tag{3.26}$$

where in the last line we have used Equation (3.20) once again. It follows that

$$\begin{aligned} \vec{v} \times \vec{a} &= \left(\sqrt{2}r^{-1/2}\sin^2\theta - \sqrt{2}r^{-1/2}(\sin^2\theta + 2r^{-1})\right)\vec{u}_r \times \vec{u}_\theta \\ &= -2\sqrt{2}r^{-3/2}\,\vec{u}_r \times \vec{u}_\theta, \\ |\vec{v} \times \vec{a}| &= 2\sqrt{2}r^{-3/2}, \\ \kappa &= \frac{2\sqrt{2}r^{-3/2}}{8} = (2r)^{-3/2}. \end{aligned}$$

3.2 Exercises

1. Prove that
$$\frac{d}{dt}(\vec{r} + \vec{s}) = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt}.$$

2. Prove that
$$\frac{d}{dt}(\vec{r} \cdot \vec{s}) = \frac{d\vec{r}}{dt} \cdot \vec{s} + \vec{r} \cdot \frac{d\vec{s}}{dt}.$$

3. Prove that
$$\frac{d}{dt}(\vec{r} \times \vec{s}) = \frac{d\vec{r}}{dt} \times \vec{s} + \vec{r} \times \frac{d\vec{s}}{dt}.$$

In Exercises 4 through 12, use each of the following trajectories:

(a) $\vec{r}(t) = t^2\vec{\imath} - 4t\vec{\jmath} - t^2\vec{k}$,

(b) $\vec{r}(t) = (\cosh t)\,\vec{\imath} + (\sinh t)\,\vec{\jmath} + t\vec{k}$,
$(\cosh t = (e^t + e^{-t})/2, \sinh t = (e^t - e^{-t})/2)$

(c) $\vec{r}(t) = t\cos t\,\vec{\imath} + t\sin t\,\vec{\jmath} + \vec{k}$.

4. Sketch the curve traced over the interval from $t = 0$ to 2.

5. Find $\vec{v}(t)$ and $\vec{a}(t)$.

6. Find $r(t)$ and $v(t)$.

7. Find the cosine of the angle between $\vec{r}$ and $\vec{v}$. For what values of t is $\vec{r}$ perpendicular to $\vec{v}$? When is it parallel to $\vec{v}$?

8. Find the cosine of the angle between $\vec{v}$ and $\vec{a}$. For what values of t is $\vec{v}$ perpendicular to $\vec{a}$? When is it parallel to $\vec{a}$?

9. Find the definite integral that expresses the arc length from $t = 0$ to 2.

10. Find $\vec{v} \times \vec{a}$.

11. Find the equation of the osculating plane at time t.

12. Find the curvature at time t.

13. Prove that
$$\vec{N} = \frac{\vec{a}_{v\perp}}{|\vec{a}_{v\perp}|}.$$

14. Does $\vec{r}(t) \cdot \vec{v}(t) = 0$ for all t imply that $r(t)$ is constant?

15. Consider a particle whose path is the ellipse
$$r(2 + \cos\theta) = 2,$$
traversed in a counterclockwise direction about the origin, and that sweeps out one unit of area per unit time:
$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = 1.$$
Find the velocity and acceleration expressed in terms of the local coordinates $\vec{u}_r$ and $\vec{u}_\theta$.

16. Consider a particle whose path is the spiral
$$r = e^{3\theta},$$
sweeping out one unit of area per unit time. Find the velocity and acceleration expressed in terms of the local coordinates $\vec{u}_r$ and $\vec{u}_\theta$.

17. A missile traveling at constant speed is homing in on a target at the origin. Due to an error in its circuitry, it is consistently misdirected by a constant angle α. Find its path. Show that if $|\alpha| < 90°$, then it will eventually hit its target, taking $1/\cos\alpha$ times as long as if it were correctly aimed. (Hint: use local coordinates $\vec{u}_r$ and $\vec{u}_\theta$.)

3.3 Orbital Mechanics

Equipped with calculus and vector algebra, we can now make short work of Newton's result that the law of gravity implies Kepler's second law.

Lemma 3.1 *Let $\vec{r}(t)$ be the position of a particle at time t, $\vec{v}(t)$ its velocity, and $\vec{a}(t)$ its acceleration. If $\vec{a}$ is radial (always parallel to $\vec{r}$), then*
$$\vec{r} \times \vec{v} = \vec{K}, \tag{3.27}$$

where $\vec{K}$ is a constant vector of magnitude

$$K = |\vec{K}| = 2\frac{dA}{dt} \tag{3.28}$$

$$= rv\sin\phi, \tag{3.29}$$

where dA/dt is the rate at which area is swept out and ϕ is the angle between $\vec{r}$ and $\vec{v}$.

Proof: From Equation (3.8) and the fact that $\vec{r}$ and $\vec{a}$ are parallel, we have

$$\frac{d}{dt}(\vec{r}\times\vec{v}) = \vec{v}\times\vec{v}+\vec{r}\times\vec{a} = \vec{0}+\vec{0} = \vec{0},$$

and thus, by Corollary 3.2, $\vec{r}\times\vec{v}$ is a constant vector that we shall call $\vec{K}$.

Equation (3.29) follows from Equation (3.27) and the definition of the cross product. To prove Equation (3.28), we use the representations of $\vec{r}$ and $\vec{v}$ in terms of local coordinates [Equations (1.14) and (1.15)]:

$$\vec{K} = \vec{r}\times\vec{v} = r\vec{u}_r \times \left(\frac{dr}{dt}\,\vec{u}_r + r\,\frac{d\theta}{dt}\,\vec{u}_\theta\right) = r^2\frac{d\theta}{dt}(\vec{u}_r\times\vec{u}_\theta), \tag{3.30}$$

so that

$$K = r^2\,\frac{d\theta}{dt}. \tag{3.31}$$

Lemma 1.9 now concludes the proof.

Q.E.D.

Kepler's Second Law

The full law of gravity says that the force of gravitational attraction is inversely proportional to the square of the distance and directly proportional to each of the masses:

$$\vec{F} = -G\frac{Mm}{r^2}\vec{u}_r, \tag{3.32}$$

where G is a gravitational constant, M and m are the respective masses, and r is the distance. If m is the mass of our orbiting particle, then its acceleration satisfies

$$\vec{F} = m\vec{a}, \tag{3.33}$$

or

$$\vec{a} = -\frac{GM}{r^2}\vec{u}_r. \tag{3.34}$$

Theorem 3.5 *Let $\vec{r}(t)$ denote the position at time t of a moving particle whose acceleration is given by Equation (3.34) and that sweeps out area at the constant rate $K/2$. There then exists a constant vector $\vec{\varepsilon}$ such that*

$$|\vec{r}| + \vec{r}\cdot\vec{\varepsilon} = \frac{K^2}{GM}. \tag{3.35}$$

Equivalently, if (r, θ) is the position in polar coordinates, then

$$r(1 + \varepsilon\cos\theta) = \frac{K^2}{GM}. \tag{3.36}$$

We recognize Equation (3.36) as the equation of a conic section: an ellipse, parabola, or hyperbola (Lemma 1.10 and Exercise 12 of Section 1.5). In particular, if $|\vec{\varepsilon}| < 1$, then it is the equation of an ellipse with one focus at the origin.

Proof: We shall prove this by using the identity for scalar triple products:

$$\vec{r}\times\vec{v}\cdot\vec{K} = \vec{v}\times\vec{K}\cdot\vec{r}. \tag{3.37}$$

By Equation (3.27), the left side is

$$\vec{r}\times\vec{v}\cdot\vec{K} = \vec{K}\cdot\vec{K} = K^2. \tag{3.38}$$

To evaluate the right side, we use our definition of $\vec{a}$ [Equation (3.34)], the representation of $\vec{K}$ in local coordinates [Equation (3.30)], and the fact that $(d/dt)\vec{u}_r = (d\theta/dt)\vec{u}_\theta$ [Equation (1.11)]:

$$\begin{aligned}\frac{d}{dt}(\vec{v}\times\vec{K}) &= \vec{a}\times\vec{K} = \left(-\frac{GM}{r^2}\vec{u}_r\right)\times\left(r^2\frac{d\theta}{dt}\vec{u}_r\times\vec{u}_\theta\right)\\ &= -GM\frac{d\theta}{dt}\left[\vec{u}_r\times(\vec{u}_r\times\vec{u}_\theta)\right] = GM\frac{d\theta}{dt}\vec{u}_\theta = \frac{d}{dt}(GM\vec{u}_r).\end{aligned}$$

This means that the derivative of $\vec{v}\times\vec{K} - GM\vec{u}_r$ is $\vec{0}$, and so $\vec{v}\times\vec{K} - GM\vec{u}_r$ is a constant vector which we shall denote by $GM\vec{\varepsilon}$:

$$\vec{v}\times\vec{K} = GM(\vec{u}_r + \vec{\varepsilon}), \tag{3.39}$$

$$\vec{v}\times\vec{K}\cdot\vec{r} = GM(\vec{u}_r + \vec{\varepsilon})\cdot\vec{r} = GM(|\vec{r}| + \vec{r}\cdot\vec{\varepsilon}). \tag{3.40}$$

Combining this result with Equations (3.37) and (3.38), we see that

$$\begin{aligned}K^2 = \vec{r}\times\vec{v}\cdot\vec{K} &= \vec{v}\times\vec{K}\cdot\vec{r} = GM(|\vec{r}| + \vec{r}\cdot\vec{\varepsilon}),\\ |\vec{r}| + \vec{r}\cdot\vec{\varepsilon} &= \frac{K^2}{GM}.\end{aligned}$$

Equation (3.36) follows from the equalities $|\vec{r}| = r$ and $\vec{r} \cdot \vec{\varepsilon} = r\varepsilon\cos\theta$.

Q.E.D.

I challenge the reader to return to Newton's original proof of Kepler's second law (Proposition XVII) and work through it, comparing it to this proof.

Equation of the Orbit

If we define the positive x axis to be parallel to $\vec{\varepsilon} = \varepsilon\vec{\imath}$ and set

$$\gamma = \frac{K^2}{GM}, \tag{3.41}$$

then we have an elliptic orbit precisely when $|\varepsilon| < 1$, and the equation of this orbit is, by Lemma 1.10,

$$(1-\varepsilon^2)(x+\alpha)^2 + y^2 = \frac{\gamma^2}{1-\varepsilon^2}, \tag{3.42}$$

where

$$\alpha = \frac{\varepsilon\gamma}{1-\varepsilon^2}. \tag{3.43}$$

The *apogee*, or farthest distance, is

$$|\alpha| + \frac{\gamma}{1-\varepsilon^2} = \frac{|\varepsilon|\gamma}{1-\varepsilon^2} + \frac{\gamma}{1-\varepsilon^2} = \frac{(1+|\varepsilon|)\gamma}{(1-|\varepsilon|)(1+|\varepsilon|)} = \frac{\gamma}{1-|\varepsilon|}. \tag{3.44}$$

The *perigee*, or nearest distance, is

$$-|\alpha| + \frac{\gamma}{1-\varepsilon^2} = \frac{-|\varepsilon|\gamma}{1-\varepsilon^2} + \frac{\gamma}{1-\varepsilon^2} = \frac{(1-|\varepsilon|)\gamma}{(1-|\varepsilon|)(1+|\varepsilon|)} = \frac{\gamma}{1+|\varepsilon|}. \tag{3.45}$$

The *mean distance* is the semimajor axis:

$$a = \frac{\gamma}{1-\varepsilon^2}. \tag{3.46}$$

Note also that if ε is positive, then most of the ellipse lies to the *left* of the y axis, and the orbiting particle reaches its perigee when it crosses the positive x axis. If ε is negative, then most of the ellipse lies to the *right* of the y axis, and the orbiting particle reaches its apogee when it crosses the positive x axis (Figure 3.6).

If the absolute value of ε is 1, then our orbit is a parabola. If it is greater than 1, then the orbit is a hyperbola. What is significant about these cases is that they are *nonperiodic*: our particle sweeps in close to the object it is orbiting and then heads off, never to return. A satellite circling the earth

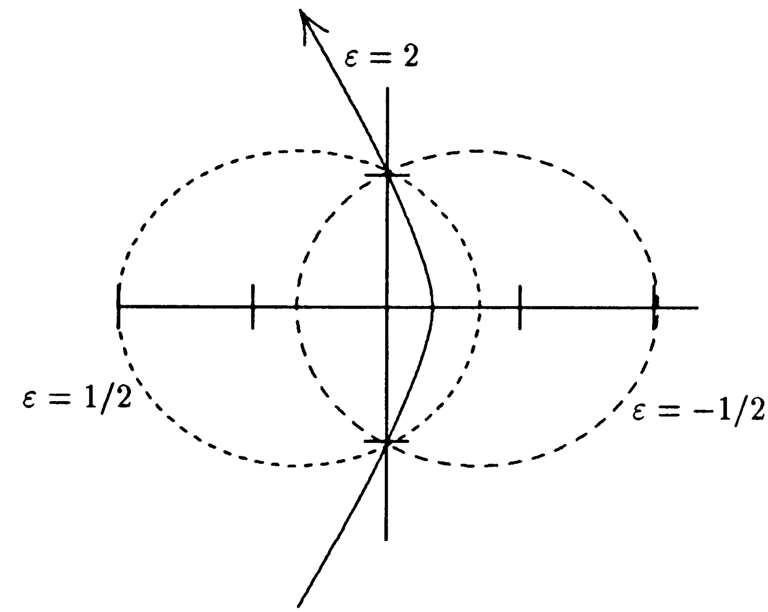

FIGURE 3.6. Orbits for $\varepsilon = -\frac{1}{2}$, $\frac{1}{2}$, and 2.

that achieves a parabolic or hyperbolic orbit is said to reach *escape velocity.* Assuming it is on the outbound arm (and so is in no danger of crashing into the earth), the earth's gravity cannot hold it back.

Eccentricity and Escape Velocity

Let us narrow our focus to satellites orbiting the earth where the value of GM is approximately

$$GM = 4 \times 10^{14} \quad \text{m}^3/\text{s}^2. \tag{3.47}$$

To simplify matters, we shall ignore the effects of the moon and other bodies. If we know the position, $\vec{r}$, and velocity, $\vec{v}$, of our satellite at any given time, we can find K and ε and thus compute the orbit. Our first problem will be to find the escape velocity from the earth.

The constant K is easily computed from Equation (3.29): $K = rv\sin\phi$, where ϕ is the angle between $\vec{r}$ and $\vec{v}$. To find ε from r, v, and ϕ is a little trickier. From Equations (3.39) and (3.27), we see that

$$\vec{u}_r + \vec{\varepsilon} = \frac{\vec{v} \times \vec{K}}{GM} = \frac{\vec{v} \times (\vec{r} \times \vec{v})}{GM}. \tag{3.48}$$

We can solve this for $\vec{\varepsilon}$ and then get

$$\varepsilon^2 = \vec{\varepsilon} \cdot \vec{\varepsilon}$$

$$= \left(\frac{\vec{v}\times(\vec{r}\times\vec{v})}{GM} - \vec{u}_r\right)\cdot\left(\frac{\vec{v}\times(\vec{r}\times\vec{v})}{GM} - \vec{u}_r\right)$$

$$= \frac{1}{G^2M^2}|\vec{v}\times(\vec{r}\times\vec{v})|^2 - \frac{2}{GM}[\vec{v}\times(\vec{r}\times\vec{v})\cdot\vec{u}_r] + 1. \tag{3.49}$$

Now, $\vec{v}$ is perpendicular to $\vec{r}\times\vec{v}$, and therefore,

$$|\vec{v}\times(\vec{r}\times\vec{v})| = rv^2|\sin\phi|.$$

We can rearrange our scalar triple product $\vec{v}\times(\vec{r}\times\vec{v})\cdot\vec{u}_r$ to $[(\vec{u}_r\times\vec{v})\cdot(\vec{r}\times\vec{v})]$. Since $\vec{u}_r\times\vec{v}$ is parallel to $\vec{r}\times\vec{v}$, the dot product of these vectors is simply the product of their magnitudes:

$$\vec{v}\times(\vec{r}\times\vec{v})\cdot\vec{u}_r = (v\sin\phi)(rv\sin\phi) = rv^2\sin^2\phi.$$

Making these substitutions into the last line of Equation (3.49), we obtain

$$\begin{aligned}\varepsilon^2 &= \frac{r^2v^4\sin^2\phi}{G^2M^2} - \frac{2rv^2\sin^2\phi}{GM} + 1 \\ &= 1 + \frac{rv^2\sin^2\phi}{G^2M^2}\left(rv^2 - 2GM\right).\end{aligned} \tag{3.50}$$

If, instead of factoring out rv^2/G^2M^2, we replace the 1 in the first line by $\cos^2\phi + \sin^2\phi$, we see that we can also write ε^2 as

$$\varepsilon^2 = \sin^2\phi\left(1 - \frac{rv^2}{GM}\right)^2 + \cos^2\phi. \tag{3.51}$$

If $\phi = 0$, then our satellite is moving vertically. It either keeps going forever in a straight line or slows down, reverses direction, and crashes back into the earth. Since neither of these cases is particularly interesting, we shall assume that ϕ is not zero.

Because r and v are positive, we have a nonperiodic orbit ($|\varepsilon| \geq 1$) if and only if

$$rv^2 \geq 2GM = 8\times10^{14} \quad \text{m}^3/\text{s}^2,$$

$$v \geq \sqrt{\frac{8\times10^{14}}{r}} \quad \text{m/s}. \tag{3.52}$$

As we get further from the earth, the escape velocity decreases. At the surface of the earth where r is roughly 6.4×10^6 meters, the escape velocity is about 11 200 m/s or 24 800 miles/h. Note that escape velocity does *not* depend on the angle between r and v. In practice, you must check that you are not on a trajectory that will collide with the earth. As long as this will not happen, heading in *any* direction at 25 000 miles/h will launch you toward the ends of the universe.

A circular orbit has an eccentricity of $\varepsilon = 0$, which is achieved if and only if

$$rv^2 = GM \quad \textbf{AND} \quad \phi = \pi/2.$$

In a properly elliptic orbit ($0 < |\varepsilon| < 1$), the angle between $\vec{r}$ and $\vec{v}$ is $\pi/2$ at precisely two points: the apogee and the perigee. At these points, we have the relationship

$$\varepsilon^2 = \left(1 - \frac{rv^2}{GM}\right)^2,$$

or

$$\varepsilon = \pm\left(1 - \frac{rv^2}{GM}\right). \tag{3.53}$$

The choice of sign is determined by whether the perigee occurs on the positive or negative x axis. The eccentricity is positive when $rv^2 > GM$ and negative when $rv^2 < GM$, and so

$$\varepsilon = \frac{rv^2}{GM} - 1. \tag{3.54}$$

Kepler's Third Law and Geosynchronous Orbit

Kepler's third law now comes for free. The area inside an elliptic orbit is π times the product of the semimajor and semiminor axes, which is

$$\pi \frac{\gamma}{1-\varepsilon^2} \frac{\gamma}{\sqrt{1-\varepsilon^2}} = \frac{\pi K^4}{G^2 M^2 (1-\varepsilon^2)^{3/2}}.$$

Since the area is swept out at the constant rate $K/2$, the *period* of the orbit (the time needed to complete one orbit) is the area divided by the rate:

$$\begin{aligned} \text{period} &= \frac{2\pi K^3}{G^2 M^2 (1-\varepsilon^2)^{3/2}} \\ &= \frac{2\pi}{\sqrt{GM}} \frac{\gamma^{3/2}}{(1-\varepsilon^2)^{3/2}} \\ &= \frac{2\pi}{\sqrt{GM}} a^{3/2}, &(3.55) \\ \text{period}^2 &= \frac{4\pi^2}{GM} a^3, &(3.56) \end{aligned}$$

where a is the semimajor axis. Equation (3.56) is Kepler's third law.

An orbit is said to be *geosynchronous* if its period is the same as the time it takes the earth to complete one rotation, that is, 1 day or 86 400 s. Inserting this and the value of GM into Equation (3.55), we get an a value of about 42 000 km, or roughly 35 600 km above the surface of the earth.

Accelerating while in Orbit

A curious phenomenon happens to a vehicle in orbit that fires its rockets to achieve acceleration in a nonradial direction. If two vehicles are traveling in tandem in a circular orbit and one of them produces a brief acceleration in the direction of its velocity, then instead of pulling ahead of its companion, it will swing out into an orbit of greater eccentricity and actually fall behind. The mathematics behind this comes out of our equation for eccentricity [Equation (3.51)].

Initially, since we are in a circular orbit, we have

$$rv^2 = GM,$$

and the semimajor axis is r, so the period is

$$\text{period} = \frac{2\pi}{\sqrt{GM}} r^{3/2}.$$

We let our first vehicle continue in this orbit, but we briefly fire the rockets on the second vehicle. In view of the distances involved, a "burn," or rocket firing, of a few seconds can be viewed as an instantaneous increase in velocity, so that at the moment of the burn the position of our second vehicle, $\vec{r}_2(t)$, is still the same as that of the first vehicle, but the velocity has changed from $\vec{v}$ to

$$\vec{v}_2 = c\vec{v},$$

for some positive constant c. We have assumed that our instantaneous acceleration is parallel to $\vec{v}$, so that initially the angle between $\vec{r}_2$ and $\vec{v}_2$ is still $\pi/2$. It is convenient to define the positive x axis so that the burn occurs as we cross it.

The new value of K is

$$K_2 = |\vec{r}_2 \times \vec{v}_2| = |\vec{r} \times c\vec{v}| = cK. \tag{3.57}$$

Since the angle between $\vec{r}_2$ and $\vec{v}_2$ is $\pi/2$ and $rv^2 = GM$, the new value of ε satisfies

$$\varepsilon_2 = \frac{r_2 v_2^2}{GM} - 1 = c^2 \frac{rv^2}{GM} - 1 = c^2 - 1. \tag{3.58}$$

Since c is assumed to be positive, we have a noncircular orbit unless $c = 1$. Note that we stay in a periodic orbit if and only if $c < \sqrt{2}$. If c is less than 1 (we have decelerated), then we are at our apogee and ε_2 is negative. If c is larger than 1 and less than $\sqrt{2}$, then we are at our perigee and ε_2 is positive.

We have demonstrated that if the second vehicle fires its afterburners, it will swing out into a wider orbit. But as long as c is less than $\sqrt{2}$, it will continue to return to its perigee on each orbit. Which vehicle gets back first? The semimajor axis for the second vehicle is

$$a_2 = \frac{K_2^2}{GM(1-\varepsilon_2^2)} = \frac{c^2K^2}{GM(2c^2-c^4)} = \frac{K^2}{GM(2-c^2)} = \frac{r}{2-c^2},$$

and so its period is

$$\frac{2\pi}{\sqrt{GM}}\frac{r^{3/2}}{(2-c^2)^{3/2}},$$

as opposed to the period of the first vehicle,

$$\frac{2\pi}{\sqrt{GM}}r^{3/2}.$$

If the second vehicle has accelerated, $1 < c < \sqrt{2}$, then it will take it longer to return to the point of the burn. To beat the first rocket back to that point, it must *decelerate*, $0 < c < 1$. Care is required, however, as deceleration puts you into an eccentric orbit passing closer to the earth, and it is desirable to avoid colliding with it.

Caveat

Before using the mathematics of this chapter to send a satellite into orbit, be aware that in practice we cannot ignore the moon's influence. For a few orbits staying relatively close to the earth, the moon will not have much effect, but over time it will modify the orbit considerably. In fact, in time, the sun and each of the planets, even each of the asteroids, will exert a measurable sway over the satellite. The mathematics we have developed is incomplete as an exact model of our universe because our universe consists of more than two objects.

Our model is a good approximation, and the influence of the other bodies can be calculated to almost any degree of accuracy. But, we are placed in a position uncomfortably close to that of pre-Keplerian astronomers: we possess a beautiful and simple theory that is only a first approximation. To make it agree with observational accuracy, we need to complicate it considerably.

This is not to suggest that we are no better off than our medieval predecessors. It is Newton's laws that tell us how to make most of the corrections. There is no need to resort to convoluted inventions to account for them. Yet, there is something basically dissatisfying about the present state of affairs. One wishes for a model that combines elegance, utility, and simplicity, probably in forms we would not yet recognize, in an explanation of the intricate dance of many bodies under gravitational attraction. There is some indication that the collection of results now being grouped under the heading of chaos theory is groping in this direction.

3.4 Exercises

1. Prove that $\vec{u}_r \times (\vec{u}_r \times \vec{u}_\theta) = -\vec{u}_\theta$.

2. Prove that in a properly elliptic orbit, the angle between $\vec{r}$ and $\vec{v}$ is $\pi/2$ only at the apogee and the perigee.

3. Mars has a radius of approximately 3300 km and a mass 0.15 times that of Earth. Find the escape velocity on the surface of Mars.

   For Exercises 4 through 7, we are considering a rocket that is fired to 300 km above the surface of the earth, 6.7×10^6 m from the center of the earth. At this point, the engines are cut off and the rocket enters orbit. The angle between $\vec{r}$ and $\vec{v}$ is denoted by ϕ.

4. What velocity must it have attained if it is to remain in a circular orbit at this height? What is the period of this orbit?

5. If its speed is 9000 m/s and $\phi = \pi/2$, what are the values of the apogee and perigee of the resulting orbit? What is the period of this orbit?

6. If its speed is 9000 m/s and $\phi = \pi/3$, what are the values of the apogee and perigee of the resulting orbit? What is the period of this orbit?

7. If its speed is 9000 m/s, find the angle ϕ that will result in an orbit whose perigee is 6.5×10^6 m. What are the values of the eccentricity, apogee, and period of this orbit?

8. A rocket has attained a circular orbit around the earth at 6.6×10^6 m from the center of the earth. It is traveling at a speed of 7785 m/s. We want to move it out to a circular orbit of $r = 7.0 \times 10^6$ m by executing a burn, increasing its speed to v_1 so that it enters an eccentric orbit whose apogee is 7.0×10^6 m. When it reaches this apogee, we perform a second burn to increase its speed from v_2, the speed of the eccentric orbit at the apogee, to 7560 m/s, the speed needed to maintain a circular orbit at this height. Find v_1 and v_2.

9. Show that the absolute value of the eccentricity is the difference between the apogee and the perigee divided by their sum. Find the absolute value of the eccentricity of an orbit whose apogee is 12×10^6 meters and whose perigee is 8×10^6 m.

   For Exercises 10 through 14, we consider the New York to Tokyo space shuttle now being planned. Our shuttle accelerates until it is 160 km above New York (6.56×10^6 m from the center of the earth). At that point, the engines are cut, and

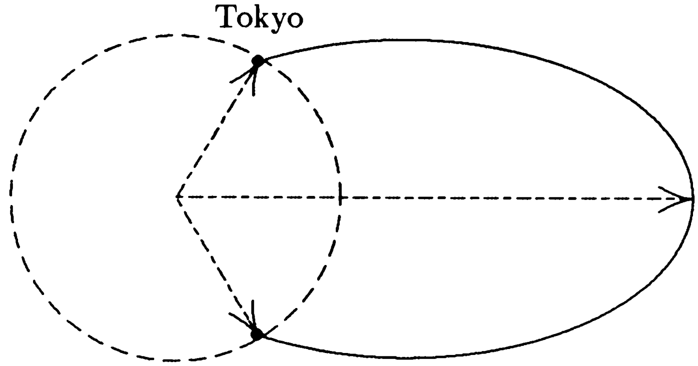

FIGURE 3.7. The New York to Tokyo shuttle, Exercises 10 to 14.

the shuttle enters an orbital glide until it is 160 km above Tokyo, at which time it decelerates for the landing. For the purposes of simplification, we shall ignore the rotation of the earth until the last problem in this set.

10. New York is at 70° W, 41° N; Tokyo is at 140° E, 36° N. Find the angle between the lines connecting the center of the earth, O, to New York and Tokyo, respectively. All of our calculations are on the plane defined by these three points, and we take the bisector of this angle to be the positive x axis (Figure 3.7).

11. Find the speed needed to achieve a circular orbit at $r = 6.56 \times 10^6$. How many minutes will it take for the orbital glide between New York and Tokyo?

12. What speed must it reach if instead of a circular orbit it is to enter an elliptic orbit with apogee at

    (a) 7.0×10^6 m,
    (b) 7.5×10^6 m,
    (c) 8.0×10^6 m,
    (d) 9.0×10^6 m?

    In each of these cases, how many minutes will it take for the orbital glide?

13. Find the value of the apogee that *minimizes* the speed we need when we enter the glide. What is this minimal speed, and how long will the glide last?

14. Redo Exercise 11 taking into consideration the fact that the rotation of the earth is moving Tokyo eastward at the rate of 15°/h.