CHAPTER 5

VECTOR FIELDS ON SURFACES

Henri Poincaré (1854–1912), one of the founders of topology, was a mathematician and physicist, particularly interested in the qualitative theory of differential equations. In this chapter, we explore some of the connections between differential equations and topology. References for more complete coverage of this material include Godbillion [Go] and Hartman [Hrt].

The link is the notion of a vector field. Suppose at each point of the plane or of a surface M a tangent vector to M is determined in such a way that its length and direction vary continuously with the point; that is, nearby points have nearly the same vectors assigned. More formally, a tangent **vector field** on a surface $M \subseteq \mathbb{R}^N$ is a continuous function $v : M \to \mathbb{R}^N$ so that $v(p)$ (drawn based at p) is a tangent vector to M at p. In other words, $v(p)$ lies in the tangent plane to M at p (Figure 5.1). Of course, if M is the plane, that simply means that the vectors also lie in the plane.

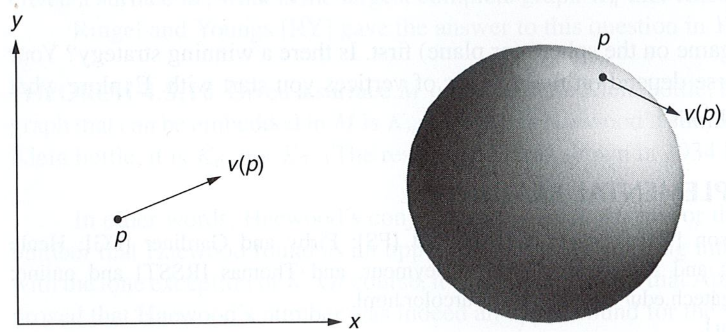

FIGURE 5.1
Tangent vectors in the plane and sphere

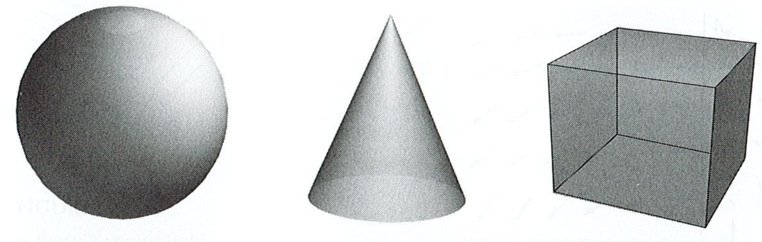

FIGURE 5.2
The surface on the left is smooth; those on the right are not smooth.

We are sneaking in some calculus here (to be expected with *differential* equations). We are asking that our surface M be smooth—that is, have a smoothly varying tangent plane at each point (Figure 5.2). We don't want M to have corners or edges. In fact, to be precise about doing differential equations on surfaces, we would need to introduce calculus on surfaces. A wonderful introduction to this subject is Spivak's *Calculus on Manifolds* [Sp]. We will rely on some intuitive notions, avoiding the technical details. The topology of the surface has some fascinating interactions with qualitative properties of vector fields on a surface. This is the direction we will go. Let's first look at vector fields in a simple setting—the plane.

5.1 VECTOR FIELDS IN THE PLANE

The relationship between vector fields and differential equations can be illustrated in the plane. A vector field in the plane is determined by a system of first-order (autonomous) differential equations:

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

The vector determined at each point (x, y) in the plane is simply $v(x, y) = (F(x, y), G(x, y))$, so we have a vector field in the plane. It is continuous since the coordinate functions F and G are continuous. In fact, *we will assume that F and G are differentiable functions.*

Example 5.1.1 The system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

determines the vector field given by $v(x, y) = (y, -x)$. This means that, at each point (x, y) in the plane we draw the vector that has coordinates $(y, -x)$. In Figure 5.3, we draw a sampling of the vectors, giving us a fairly good picture of

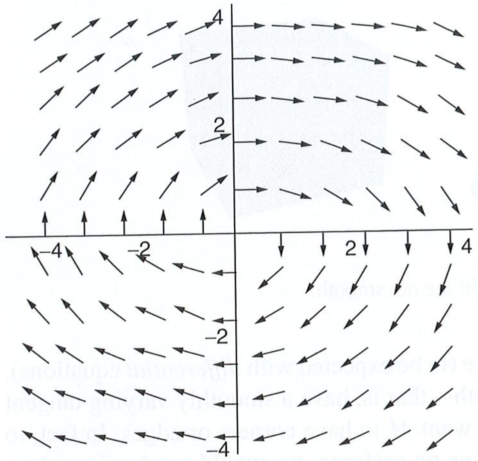

FIGURE 5.3
The vector field $v(x, y) = (y, -x)$

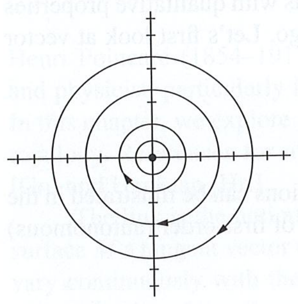

FIGURE 5.4
Some integral curves of the vector field $v(x, y) = (y, -x)$

the vector field. We have rescaled the vectors to unit length for a clearer diagram, as we will do throughout the chapter.

The origin of the system in Example 5.1.1 has the zero vector assigned to it. A point with the zero vector assigned is called a **critical point** (or, in various books, a *singular point* or *singularity* or *zero*) of the vector field.

We can draw a cleaner and more revealing picture of the system by using integral curves. The basic existence and uniqueness theorem of differential equations says that, through each point of the plane, locally there exists a unique solution curve [a curve whose tangent vector at each point equals the vector $v(x, y)$ assigned by the vector field]. We call a solution curve an **integral curve,** or **orbit.** We think of the vector field as describing the motion of points in the plane, with the integral curves being the paths of the points. If we draw some of the integral curves of our system of differential equations in Example 5.1.1, instead of all the little vectors, we get a simpler picture as shown in Figure 5.4. The uniqueness part of the theorem ensures that no two integral curves can cross.

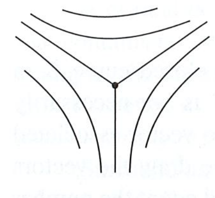

FIGURE 5.5
Is there a vector field in the plane whose phase portrait locally looks like this?

Notice that the integral "curve" assigned to the critical point (at the origin) is simply a point—that is, a curve that stays fixed at the critical point for all time. The other curves simply follow the flow of the field of vectors. This picture of the integral curves is called a **phase portrait** of the system of differential equations. The phase portraits are what we are interested in throughout this chapter—in the plane for now and later on surfaces.

The integral curves are parametrized curves, although in general we will only care about the orientation of the curve. In Section 5.3, however, it will help to work with the parametrized orbit. If x is any point, the orbit of x can be written $\{\phi_t(x)\}$, where the subscript t is the parameter, which we think of as time. When $t = 0$, we are at the point x (having traveled along the orbit for time 0). If t is positive, we travel along the orbit for time t in the direction that preserves the orientation. If t is negative, we travel from x for time t in the reverse direction.

EXERCISE 5.1.2 Draw phase portraits in the plane for each of the following systems of differential equations.

a. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = y$ b. $\dfrac{dx}{dt} = -y$, $\dfrac{dy}{dt} = 1$ c. $\dfrac{dx}{dt} = y$, $\dfrac{dy}{dt} = x$

EXERCISE 5.1.3 Does there exist a vector field in the plane whose phase portrait locally looks like that in Figure 5.5? Explain.

A simple closed curve that is an integral curve for a system is called a **periodic orbit.** Which of the systems that we've seen have periodic orbits?

As you have probably seen in other classes, vector fields occur in many appplications. For example:

- Gravitational or electromagnetic force fields
- Velocity vector fields of fluids in motion, like air or water
- Gradient fields, like a topographical map

The existence of a critical point or periodic orbit has physical and topological importance. Critical points play a key role in connecting vector fields with topology.

5.2 INDEX OF A CRITICAL POINT

Suppose we have a vector field in the plane. Consider any simple closed curve C in the plane that does not pass through a critical point. (Note that C is *not* necessarily an integral curve of the vector field.) Each point of C has a nonzero vector associated with it. As we traverse C once in a counterclockwise direction, we draw the vectors emanating from points along C, basing them at some fixed point, and count the number of counterclockwise rotations made by the vectors. This integer is called the **index (or winding number) of C with respect to the vector field v**, denoted $I(v, C)$. An alternative way of thinking of the index is the total angle variation of the vectors as we traverse C once divided by 2π (the angle variation of one revolution of a circle). As we rescale the vectors to unit length, the index is *the number of times the vectors on C wind around the unit circle*. Figure 5.6 shows several examples.

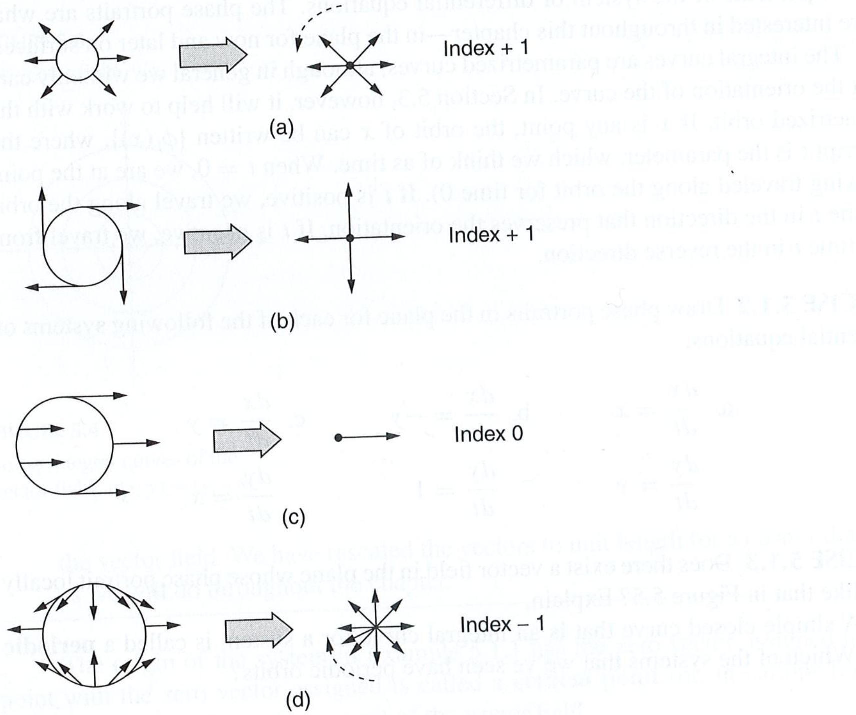

FIGURE 5.6
On the left, we draw the vectors around curve C. On the right, the same vectors are translated to the origin. To compute the index, we count the number of counterclockwise rotations of the translated vectors as one traverses C in the counterclockwise direction. Notice than in (d), the translated vectors make one *clockwise* rotation of the origin as C is transversed in the counterclockwise direction to give an index of -1.

A critical point is said to be **isolated** if there is an open set containing the point that contains no other critical points. In general, critical points need not be isolated. For example, the vector field given by $v(x, y) = (x, 0)$ has critical points all along the y axis. However, we will focus on those systems with isolated critical points. *So let's now assume for the rest of the chapter that all critical points are isolated.*

Surround an isolated critical point p by a simple closed curve C so that no other critical point lies inside or on C. We define **the index of p with respect to C** to be the index of C; that is, $I(p, C, v) \equiv I(C, v)$. We will show that this index is in fact independent of the curve C we take (as long as p is the only critical point inside C). Figure 5.7 shows some examples of critical points and their indices.

EXERCISE 5.2.1 Determine the index of each of the critical points shown in Figure 5.8.

EXERCISE 5.2.2 Draw critical points of index 4, 0, and -3, respectively. In general, what values do you think are possible indices for critical points? Experiment with finding a procedure to create an example of whatever index you think possible. We'll return to this topic later.

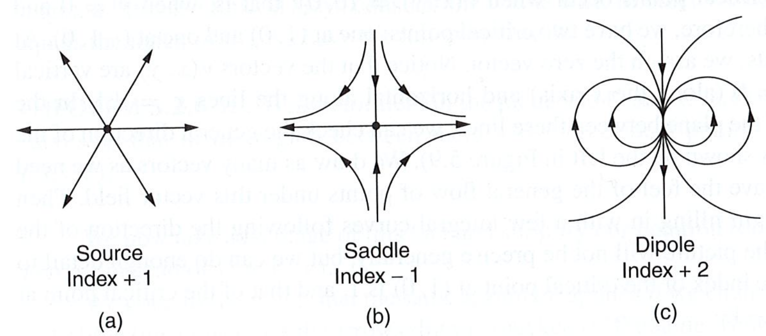

FIGURE 5.7
Examples of some critical points and their indices: (a) a source, (b) a saddle, and (c) a dipole

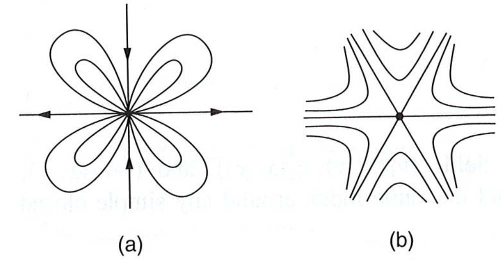

FIGURE 5.8
Find the index of each of these critical points.

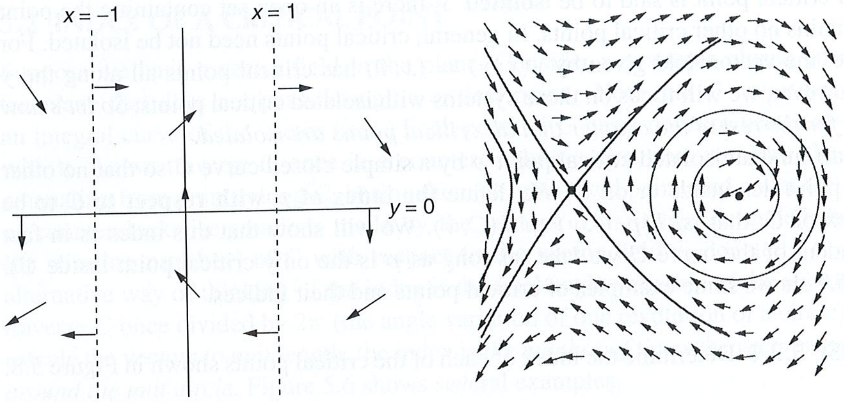

FIGURE 5.9
The vector field in the plane given by $v(x, y) = (y, 1 - x^2)$

Example 5.2.3 Consider the vector field in the plane given by $v(x, y) = (y, 1 - x^2)$. The critical points occur when $v(x, y) = (0, 0)$; that is, when $y = 0$ and $1 = x^2$. Therefore, we have two critical points: one at $(1, 0)$ and one at $(-1, 0)$. At these points, we assign the zero vector. Notice that the vectors $v(x, y)$ are vertical when $y = 0$ (along the x axis) and horizontal along the lines $x = \pm 1$. In the regions of the plane between these lines, we can check the general direction of the vectors (as shown on the left in Figure 5.9). We draw as many vectors as we need until we have the feel of the general flow of points under this vector field. Then we can begin filling in with a few integral curves following the direction of the vectors. The picture will not be precise generally, but we can do enough detail to see that the index of the critical point at $(1, 0)$ is 1, and that of the critical point at $(-1, 0)$ is -1.

EXERCISE 5.2.4 Roughly sketch phase portraits of the following vector fields in the plane. Find each critical point and determine its index.

a. $v(x, y) = (-y, x)$
b. $v(x, y) = (x, x^2 - y)$
c. $v(x, y) = (x - y, x + y)$
d. $v(x, y) = (xy, -y + 1)$

EXERCISE 5.2.5 Show that the vector fields $(v_1(x, y), v_2(x, y))$, and $(-v_2(x, y), v_1(x, y))$ have the same critical points and the same index around any simple closed curve C not meeting a critical point.

Here is a more formal definition of the index, as a line integral. If you've not seen line integrals before, you may skip it. Let $C : [0, 1] \to \mathbb{R}^2$ be a parametrized simple

closed curve and $v(C(s))$ a nonzero vector field on C with $v(C(s)) = (v_1(C(s)), v_2(C(s)))$. The index $I(C, v)$ is the total angle change around C divided by 2π (one revolution); that is, if $\theta(s)$ is an angle function, then the total angle variation around C divided by 2π is $\int_0^1 \theta'(s)\, ds = \oint_C d\theta = \oint_C d(\arctan \frac{v_2}{v_1})$. So the index of C is

$$I(C, v) = \frac{1}{2\pi} \oint_C d\theta = \frac{1}{2\pi} \oint_C d\left(\arctan \frac{v_2}{v_1}\right)$$

$$= \frac{1}{2\pi} \oint_C \frac{1}{1 + \left(\frac{v_2}{v_1}\right)^2} \cdot \frac{v_1 dv_2 - v_2 dv_1}{v_1^2} = \frac{1}{2\pi} \oint_C \frac{v_1 dv_2 - v_2 dv_1}{v_1^2 + v_2^2}.$$

Let's now see that the index of a critical point is independent of the choice of C. We will give a good intuitive argument here, though a careful one depends upon knowledge of line integrals and Green's theorem. A formal proof, using the line integral definition of index, is given at the end of this section for those who have the appropriate background. An alternative approach will be available in the next chapter.

For the intuitive proof, first notice that if we only move the curve a little then we only move the vectors a little, by continuity; so the index changes only a little. However, the index is always an integer and hence must stay constant. As we move from one circle C to another C' (without critical points in between so that the index is defined at each stage), a little at a time, it stays constant the whole way. In other words, the index of C equals the index of C'. We have the following result.

THEOREM 5.2.6 If v is a vector field in the plane and C and C' are two simple closed curves so that in the region between them v has no critical points, then $I(C, v) = I(C', v)$.

We now take advantage of this fact and simplify our notation for the index of p to $I(p, v)$ instead of $I(p, C, v)$.

We note, for later use, that the same argument applies if we change the vector field just a little (meaning move the arrows slightly) but keep C the same. Then the index should change just a little, but since it can only change in "big" shifts of 1 or more, it apparently stays constant. In other words, $I(C, v) = I(C, w)$ for vector fields v close to w.

Let's put this fact to good use. Suppose a simple closed curve C encloses several critical points. Then C can be continuously deformed into a curve C', as shown in Figure 5.10. Since there are no critical points between C and C', their indices are the same.

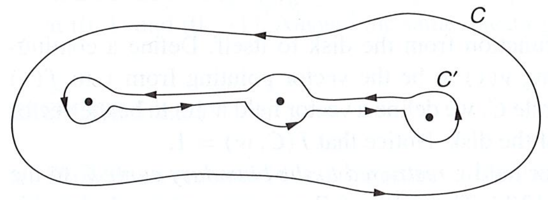

FIGURE 5.10
The indices of C and C' are the same.

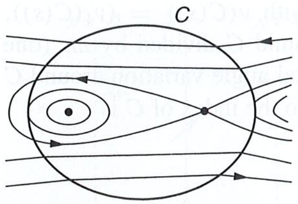

FIGURE 5.11
Check that the index of C is 0.

Notice that when we move along the arc joining the circle about one critical point with the circle about another critical point, we later traverse that same arc but in the opposite direction. So the rotation of vectors contributed by that arc will be canceled the next time through (since traversing it in the opposite direction reverses the direction of rotation). Thus, the contribution to total angle rotation of the vectors is by the circles surrounding the critical points. In other words, $I(C, v) = \sum I(p_i, v)$, where the p_i's are the critical points encompassed by C. Hence, we have the following.

THEOREM 5.2.7 The index of any simple closed curve C is the sum of the indices of the critical points that it encloses.

Example 5.2.8 The curve C in Figure 5.11 encircles two critical points: of index 1 and -1, respectively. So the index of C should be $1 - 1 = 0$. Count the number of revolutions made by the vectors based at points of C to check that this is correct.

COROLLARY 5.2.9 If the index of a simple closed curve is nonzero, then the curve encircles at least one critical point.

This corollary is quite useful. We look at one application now and another a little later.

Brouwer fixed-point theorem *Let f be a continuous map of the closed unit disk to itself. Then $f(c) = c$ for some point c in the disk. In other words, the closed disk has the fixed-point property.*

Proof: Suppose f is a continuous function from the disk to itself. Define a continuous vector field on the disk by taking $v(x)$ to be the vector pointing from x to $f(x)$ [Figure 5.12(a)]. On the boundary circle C, we define a vector field $w(x)$ to be the vector pointing from x on C to the center of the disk. Notice that $I(C, w) = 1$.

Continuously deform the vector field v *restricted to the boundary curve* C to the vector field w, as shown in Figure 5.12(b). The index of C remains constant during this deformation since, as we noted after Theorem 5.2.6, the vectors change continuously, and the index is always integer valued. So $I(C, v) = I(C, w) = 1$. Corollary 5.2.9

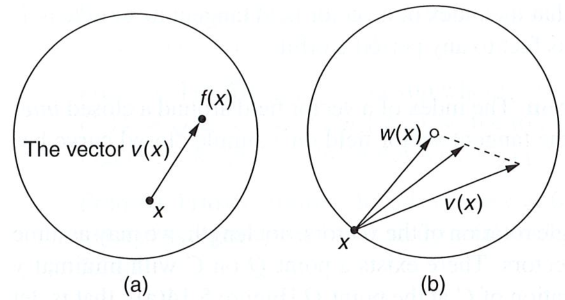

FIGURE 5.12
Deforming v to a vector field with index 1 around the boundary

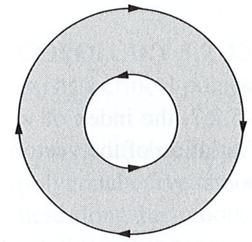

FIGURE 5.13
Can you fill in a vector field without critical points on the interior of this annulus?

implies that a critical point of v is encompassed by C. In other words, there is a point c in the disk such that $v(c) = 0$ that in turn means $f(c) = c$. ■

EXERCISE 5.2.10

a. Suppose we have a vector field on the plane with exactly two critical points: a source at the origin and a saddle at (0, 1) (see Figure 5.7). Can there be a periodic orbit along $x^2 + y^2 = 4$? Draw an example or explain why it cannot happen.
b. In the same setting, suppose the only critical points are a saddle at the origin and sinks at (0, 1) and (0, -1). Answer the same question as in part (a).

EXERCISE 5.2.11

a. Show that a vector field on an annulus can be without critical points only if the indices of the boundary circles are equal.
b. Draw a vector field on the annulus without critical points given that the boundary circles are integral curves with opposite orientations, as shown in Figure 5.13.

From Figure 5.6(b), we saw that the index of a vector field tangent to a circle is 1. The following result generalizes this fact to any periodic orbit.

THEOREM 5.2.12 Hopf's theorem The index of a vector field around a closed *integral* curve C is 1. In other words, the tangent vector field on a simple closed curve has index 1.

Proof: Since we only care about angle rotation of the vectors, not length, we may assume that all the vectors on C are unit vectors. There exists a point Q on C with minimal y coordinate. We start our parametrization of C at the point Q [Figure 5.14(a)]; that is, let $Q = C(0) = C(1)$. Define a continuous vector field w on the triangle in the plane given by $0 \le x \le y \le 1$ as follows:

$$w(x, y) = \begin{cases} \text{unit vector from } C(x) \text{ to } C(y), \text{ except for the point } (0, 1) \\ v(C(y)) \text{ along } x = y \\ -v(C(0)) \text{ when } x = 0,\ y = 1 \end{cases}$$

See Figure 5.14(b). You should verify that w is indeed continuous.

Clearly, $w \neq 0$ anywhere in the triangle, so by Theorem 5.2.7, the index of w around the boundary of the triangle is 0. We compare the angle variation of the vector fields w and v on C:

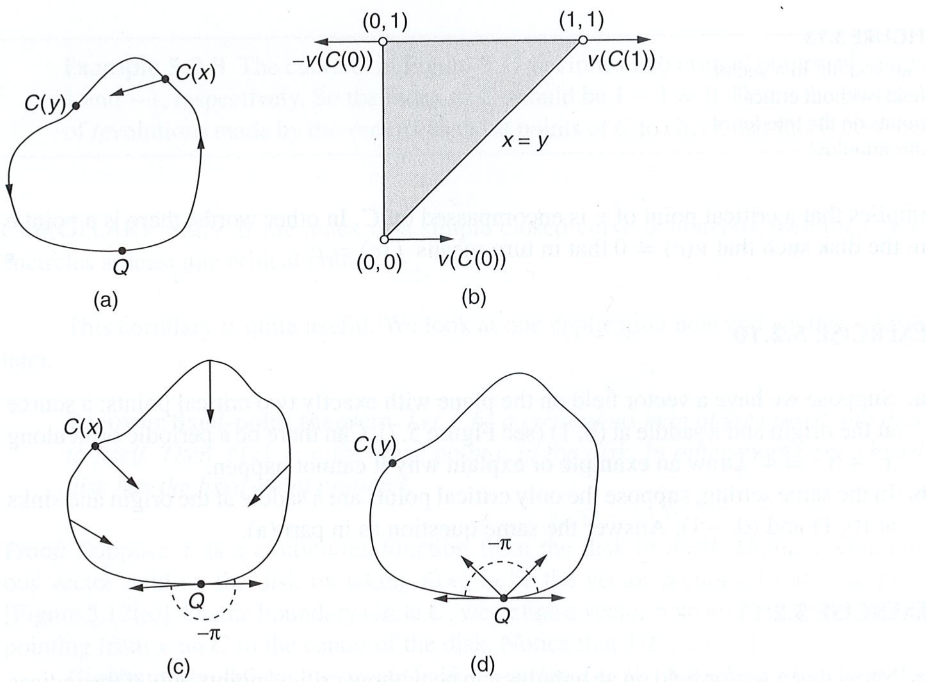

FIGURE 5.14
Proving that the index of a vector field around a closed *integral* curve is always 1 (Hopf's theorem)

From $(0, 0)$ to $(1, 1)$ along the hypotenuse, $w(x, y) = v(C(y))$, so the angle variation of w is $2\pi I(v, C)$).

From $(1, 1)$ to $(0, 1)$ along the top where $y = 1$, w is the unit vector from $C(x)$ to $C(1) = Q$ as x goes *from* 1 *to* 0, so the total angle variation is $-\pi$, as shown in Figure 5.14c. [*Note:* This follows from $y(Q)$ being minimal.]

From $(0, 1)$ to $(0, 0)$ along the side where $x = 0$, w is the unit vector from $C(0) = Q$ to $C(y)$ as y goes *from* 1 *to* 0, so the total angle variation is $-\pi$, as shown in Figure 5.14d.

Hence, $0 = \text{index}(w) = 2\pi I(C, v) - \pi - \pi$; that is, $I(C, v) = 1$, completing the proof. [*Note:* We have illustrated the proof with a counterclockwise vector field on C. You should check that the proof works with a clockwise vector field. Remember that the index is still measured in a counterclockwise fashion.] ∎

COROLLARY 5.2.13 If a vector field in the plane has a closed integral curve, then there exists a critical point inside.

Note that Hopf's theorem can be generalized to the case where the curve C is a polygonal curve, smooth except at a finite number of corners and tangent to the vector field along the smooth regions, as shown in Figure 5.15. However, in this case, in counting rotations of the tangent vectors, we must add in the angles at the corners of C.

Example 5.2.14 A simple example of a polygonal curve C (where the edges are straight so that there is no rotation *along* them) is drawn in Figure 5.16. If the interior angles of C are α_1, α_2, and α_3, then the total variation of the vector field is $2\pi I(C, v) = (\pi - \alpha_1) + (\pi - \alpha_2) + (\pi - \alpha_3) = 3\pi - (\alpha_1 + \alpha_2 + \alpha_3) = 2\pi$. So the index is 1.

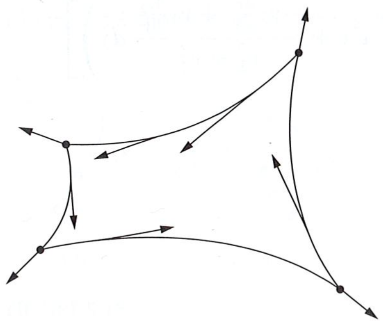

FIGURE 5.15
Hopf's theorem holds for a polygonal curve too, but we must add in the rotation at the corners.

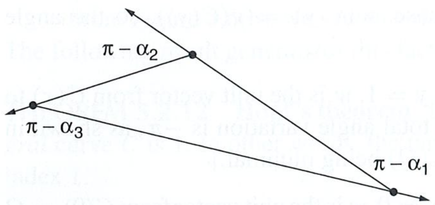

FIGURE 5.16
Compute the index of the tangent vector field around this polygonal curve, making sure to include the angle rotation at the corners.

5.2.1 Appendix to Section 5.2 (Optional)

Here is a proof of Theorem 5.2.6, that the index is independent of the choice of C, using line integrals and Green's theorem. We saw an intuitive argument earlier. We will use the more formal definition of index that we gave following Exercise 5.2.5.

Proof: Let $v(x, y) = (v_1(x, y), v_2(x, y))$ be a vector field in the plane with v_1 and v_2 differentiable functions. We want to show that the value of the index is independent of the choice of enclosing curve, as long as p is the only critical point enclosed in either curve (that is, no critical points lie between the two curves). What if we take two curves, C and C^*, surrounding p with no other critical point inside either curve? Let C' be a curve surrounding p and interior to both C and C^*, as shown in Figure 5.17.

We claim that $I(C, v) = I(C', v) = I(C^*, v)$. Since $dv_i = \frac{\partial v_i}{\partial x}\,dx + \frac{\partial v_i}{\partial y}\,dy$,

$$I(C, v) - I(C', v) = \frac{1}{2\pi}\left[\oint_C \frac{-v_2 dv_1 + v_1 dv_2}{v_1^2 + v_2^2} - \oint_{C'} \frac{-v_2 dv_1 + v_1 dv_2}{v_1^2 + v_2^2}\right]$$

$$= \frac{1}{2\pi}\left[\oint_C \left(\frac{-v_2\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_2}{\partial x}}{v_1^2 + v_2^2}\,dx + \frac{-v_2\frac{\partial v_1}{\partial y} + v_1\frac{\partial v_2}{\partial y}}{v_1^2 + v_2^2}\,dy\right)\right.$$

$$\left. - \oint_{C'} \left(\frac{-v_2\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_2}{\partial x}}{v_1^2 + v_2^2}\,dx + \frac{-v_2\frac{\partial v_1}{\partial y} + v_1\frac{\partial v_2}{\partial y}}{v_1^2 + v_2^2}\,dy\right)\right]$$

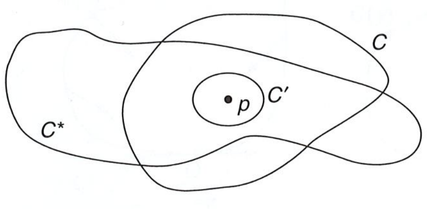

FIGURE 5.17
We use Green's theorem to show that the indices of C and C' are the same, as are the indices of C^* and C'. Therefore, the index around C is the same as that around C^*.

Let D be the annular region between the curves C and C'. Notice that $C - C'$ is the oriented boundary of D, with D to the left of the boundary as we travel it in a positively oriented fashion. (We traverse C in the counterclockwise direction and use $-C'$ to indicate traversing C' in the clockwise direction.) Hence, by Green's theorem with the above integrand being $M\,dx + N\,dy$, we have

$$\oint_{C-C'} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \frac{1}{2\pi}\iint_D \frac{\partial}{\partial x}\left(\frac{-v_2\frac{\partial v_1}{\partial y} + v_1\frac{\partial v_2}{\partial y}}{v_1^2 + v_2^2}\right) - \frac{\partial}{\partial y}\left(\frac{-v_2\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_2}{\partial x}}{v_1^2 + v_2^2}\right) dx\,dy$$

$$= 0$$

since mixed partials are equal and therefore cancel. We therefore have that $I(C, v) = I(C', v)$. Similarly, we can show $I(C', v) = I(C^*, v)$, giving that the indices of C and C^* are the same. ∎

5.3 LIMIT SETS IN THE PLANE

Suppose an orbit enters a bounded region of the plane and never leaves. What can its limit set look like? A few examples are shown in Figure 5.18. Experimentation will convince you that the orbit must either spiral in around a periodic orbit or have a critical point in its limit set. The proof of that fact is the focus of this section. Although this material is not essential for the rest of the chapter (with one exception in Section 5.4), it fits beautifully with our primary interests because it is a strictly two-dimensional result. In fact the main theorem only works for the plane and the sphere. We take the approach of Palis and de Melo [PdM].

Let's introduce a little terminology first. Recall from Section 5.1 that we can indicate a parametrized orbit by $\{\phi_t(x)\}$, where the subscript t is the time parameter. If $t = 0$, $\phi_0(x) = x$ since we travel for zero time. The **positive semiorbit** of a point x, denoted $\mathcal{O}^+(x)$, is $\{\phi_t(x) \mid t > 0\}$. The negative semiorbit is defined similarly for $t < 0$.

The **ω-limit set** of a point x, $L_\omega(x)$, is the set of points y so that a sequence of points $\{\phi_{t_n}(x)\}$ on the orbit of x, with $t_n \to \infty$, converges to the point y $(\phi_{t_n}(x) \to y)$. Figure 5.18 shows some ω-limit sets.

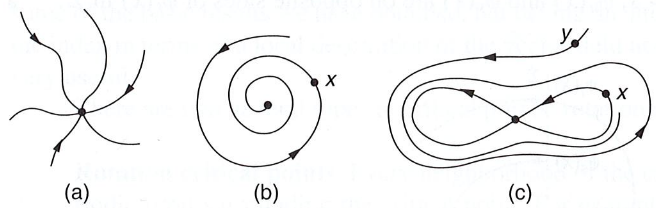

FIGURE 5.18
(a) The ω-limit set of any point in a small neighborhood of a sink is the critical point. (b) The ω-limit set of x is the periodic orbit toward which it spirals. (c) The ω-limit set of x is the critical point. The ω-limit set of y is the union of three orbits: the critical point plus the two orbits adjoining it.

NOTE: The **α-limit set** is defined similarly but with $t_n \to -\infty$. The results we show for ω-limit sets translate naturally to similar results for α-limit sets simply by reversing the orientation of the integral curves.

EXERCISE 5.3.1 Find the α-limit sets of the point x in the bottom two examples in Figure 5.18.

EXERCISE 5.3.2 In Figure 5.7, which points have the critical point as their ω-limit set?

EXERCISE 5.3.3 If the positive semiorbit of a point x is contained in a compact subset of the plane, show that $L_\omega(x)$ is nonempty. [*Hint:* Recall some basic properties of compact subsets of $\mathbb{R}^n$ in Section 1.3.4.]

PROPOSITION 5.3.4 Any ω-limit set is a closed union of orbits; that is, $L_\omega(x)$ contains all of its limit points and, if it contains some point y, then it contains the entire orbit of y.

Proof: We first show $L_\omega(x)$ is closed. Suppose z is not in $L_\omega(x)$. There is a neighborhood V of z such that V is disjoint from $\{\phi_t(x) \mid t \geq T\}$ for some time T. Then points of V are not in $L_\omega(x)$, meaning that the complement of $L_\omega(x)$ is open and $L_\omega(x)$ is closed.

Showing that $L_\omega(x)$ is a union of orbits is just as easy. Suppose z is in $L_\omega(x)$ and $\tilde{z} = \phi_{t_0}(z)$ is some point on the orbit of z. We know we have a sequence of points, $\{\phi_{t_n}(x)\}$, with $t_n \to \infty$, such that $\phi_{t_n}(x) \to z$. If we let the sequence flow forward for time t_0, we get $\phi_{t_0+t_n}(x) = \phi_{t_0}(\phi_{t_n}(x)) \to \phi_{t_0}(z) = \tilde{z}$, giving us that $\tilde{z}$ is in $L_\omega(x)$. ■

PROPOSITION 5.3.5 Suppose Σ is an arc that is never tangent to an integral curve. Then if a positive semiorbit of a point x, $\mathcal{O}^+(x)$, meets Σ, it does so monotonically. That is, if $\phi_s(x)$, $\phi_t(x)$, and $\phi_u(x)$ meet Σ, with $s < t < u$, then $\phi_t(x)$ lies between $\phi_s(x)$ and $\phi_u(x)$ along Σ.

Proof: The piece of the orbit between $\phi_s(x)$ and $\phi_t(x)$ and the piece of Σ between these same two points form a simple closed curve C. (See Figure 5.19.) By the Jordan separation theorem, C disconnects the plane with $\mathcal{O}^+(\phi_t(x))$ contained entirely in one component. Since $u > t > s$, $\phi_u(x)$ and $\phi_s(x)$ are on opposite sides of $\phi_t(x)$ in Σ. ■

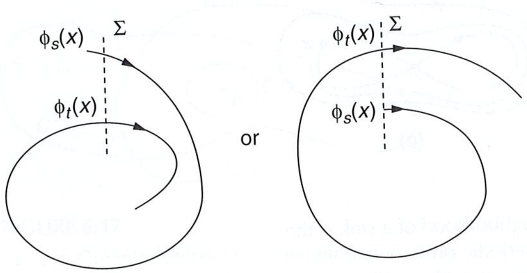

FIGURE 5.19
A semiorbit must meet an arc Σ monotonically.

PROPOSITION 5.3.6 $L_\omega(x)$ meets Σ in at most one point.

EXERCISE 5.3.7 Prove Proposition 5.3.6. Suppose $L_\omega(x)$ meets Σ in two points and use Proposition 5.3.5 to reach a contradiction.

> **Poincaré–Bendixson theorem** *Any positive semiorbit contained in a compact subset of the plane must have either a critical point or a periodic orbit in its closure.*

Proof: Let x be a point whose positive semiorbit stays within a compact set. Suppose there is no critical point in the closure of $\mathcal{O}^+(x)$. We know from Exercise 5.3.3 that there is a point y in $L_\omega(x)$ (not a critical point by our preceding assumption). We will show that the orbit of y is periodic. Let z be a point in $L_\omega(y)$. Take an arc Σ through z that is nowhere tangent to the integral curves of ϕ. By Proposition 5.3.5, the positive semiorbit of y meets Σ in a monotone sequence $\phi_{t_n}(y)$. However, by Proposition 5.3.4, $\phi_{t_n}(y) \in L_\omega(x)$; hence, Proposition 5.3.6 implies that $\phi_{t_n}(y)$ is a single point; that is, the orbit of y is periodic. ■

EXERCISE 5.3.8 Can you find an example in the plane so that a nonempty ω-limit set contains neither a critical point nor a periodic orbit? You will need to look at a noncompact ω-limit set, of course.

EXERCISE 5.3.9 Let A be an annulus in the plane. Suppose we have a vector field with no critical points on A and that the vector field is never tangent to the boundary of A, always pointing in toward the interior of A. Show that there is a periodic orbit inside A.

EXERCISE 5.3.10 Suppose v and w are two vector fields in the plane that are always perpendicular to each other. Suppose v has a periodic orbit. Show that w must have a critical point.

5.4 A LOCAL DESCRIPTION OF A CRITICAL POINT

The definition of the index of a critical point works quite well for simple examples and for some of the basic results we have obtained, but having an alternative way of computing the index in terms of a local description of the vector field near the critical point will be very useful.

There are two general types of critical points: rotation and nonrotation.

> **Rotation critical points** Every neighborhood of the critical point contains a periodic orbit surrounding the critical point. For example, we could have a center (which we saw in $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x$), where every nearby orbit is periodic, or a sequence of periodic orbits nearing the critical point with spirals in-between. Figure 5.20 gives examples of each. From Hopf's theorem (Theorem 5.2.12), we see that the index of a rotation critical point is always 1, since we can choose C to be a periodic orbit.

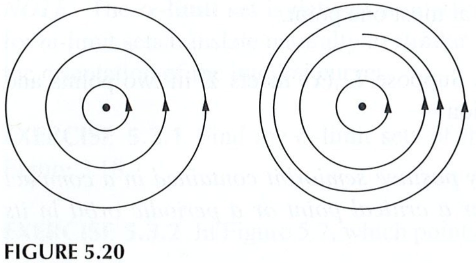

FIGURE 5.20
Examples of rotation critical points

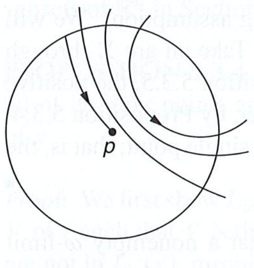

FIGURE 5.21
A nonrotational critical point has an orbit limiting on it in positive or negative time.

Nonrotation critical points We claim that, if p is an isolated critical point that is not a rotational critical point, then some integral curve approaches p as a limit in positive or negative time. To see this, take a small disk D surrounding p with no periodic orbits in D. Every orbit that enters D also leaves D in finite time *or* approaches p as a limit, as we want. (Apply the Poincaré–Bendixson theorem.) If we consider a sequence of orbits that get arbitrarily close to p but enter and leave in finite time and we focus on their points of entry, we see that they must limit on a point whose orbit must approach p in positive time (Figure 5.21).

An integral curve that meets the boundary of D and then stays inside D forever, approaching p in positive time or in negative time, is called a **separatrix** of D. A **sector** (relative to D) is a subregion bounded by separatrices, and is one of three possible types, which we now study more closely.

1. **Elliptic sector** The two bounding separatrices of the sector are portions of a single solution curve which stays entirely in D, with one separatrix approaching p in positive time and the other approaching p in negative time.
2. **Parabolic sector** The separatrices both approach p in positive time or both in nagative time, and no other separatrix in the sector has the opposite orientation.

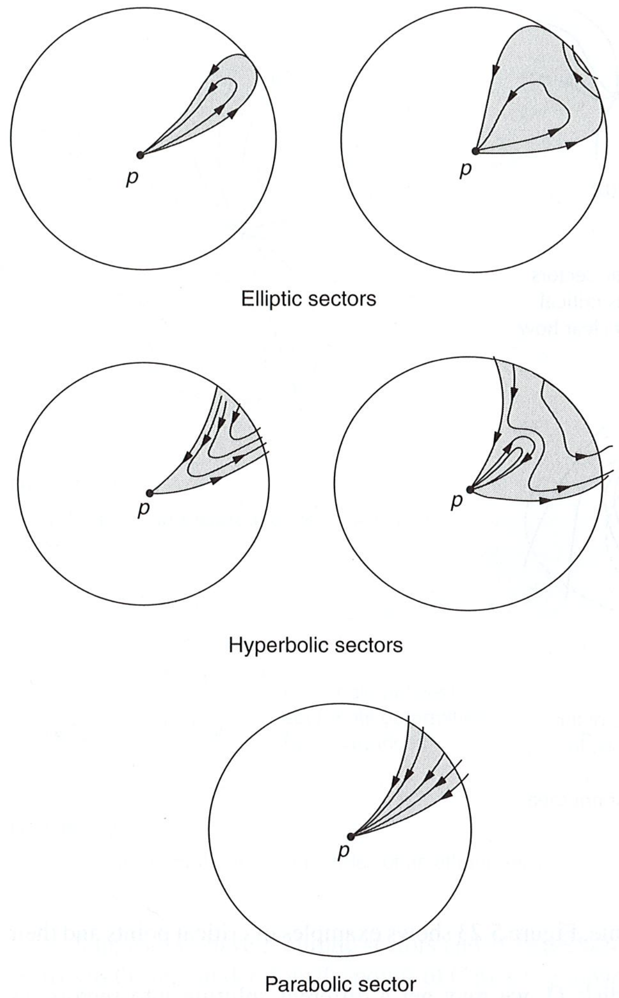

FIGURE 5.22
The three types of sectors: elliptic, parabolic, and hyperbolic

3. **Hyperbolic sector** All solution curves entering the sector along the boundary of D also exit the sector in finite time. One of the separatrices of a hyperbolic sector approaches p in positive time, the other in negative time.

Figure 5.22 illustrates the three types of sectors.

Notice that the subdivision of D into sectors is not uniquely determined, since a parabolic sector is a union of parabolic sectors. However, this ambiguity will not cause a problem. Also notice that the number of hyperbolic and elliptic sectors of a region D is finite.

If the critical point is surrounded entirely by parabolic sectors, it is called a **focus.** More specifically, it is a **sink** if all orbits limit on p in positive time or a **source** if all

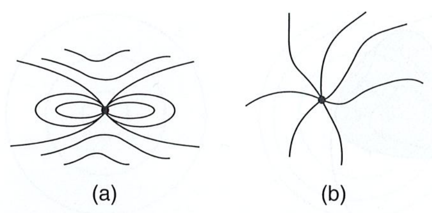

FIGURE 5.23
(a) This critical point has two elliptic sectors and two hyperbolic sectors. (b) This critical point has all parabolic sectors. Is it clear how many?

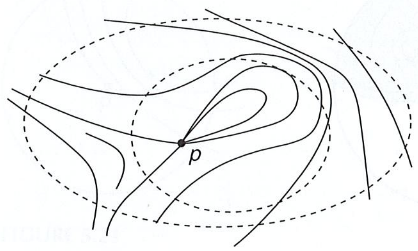

FIGURE 5.24
Looking at different neighborhoods of the critical point may change the sectors. In particular, notice that the smaller neighborhood has an elliptic sector not seen along the boundary of the larger neighborhood.

orbits limit on p in negative time. Figure 5.23 shows examples of critical points and their sectors.

If we take a different disk D, we may get a different splitting into sectors, as illustrated by Figure 5.24. However, for any disk, the number of hyperbolic and elliptic sectors is finite.

EXERCISE 5.4.1 Why must the number of hyperbolic sectors plus the number of elliptic sectors around a critical point be even?

Next we want to find a formula for finding the index of a nonrotational critical point in terms of the sectors surrounding it. (Recall that we know the index of a rotational critical point is always 1.) We will do so by computing the contribution to the angle variation made by each type of sector.

Draw a small curve C around a nonrotational critical point p so that C encloses no other critical points. At points where the separatrices meet C, we will assume that they do so at right angles. Although this may not quite be true, any error is made up in adjacent sectors.

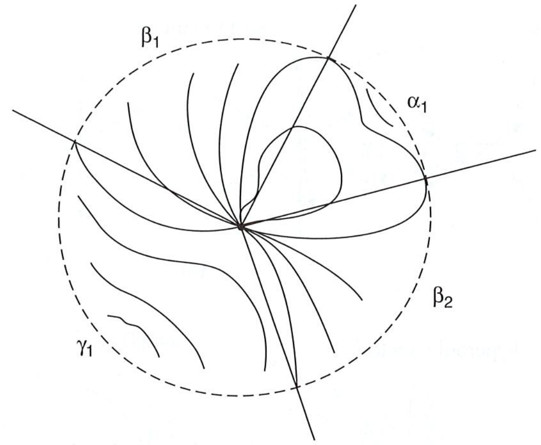

FIGURE 5.25
A subdivision into sectors and their associated angles

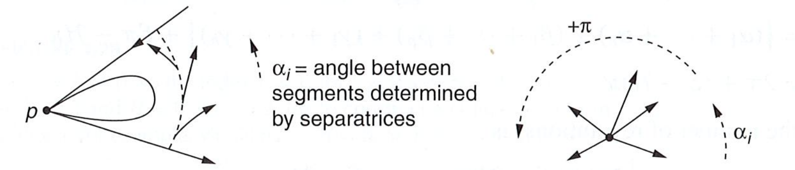

FIGURE 5.26
Computing the contribution to the index of an elliptic sector

Suppose we have $\mathcal{E}$ elliptic sectors and $\mathcal{H}$ hyperbolic sectors, leaving $\mathcal{P}$ parabolic sectors as the remainder. Join the points of C meeting separatrices to p along straight-line segments. Let $\alpha_1, \ldots, \alpha_e$ be the angles made by these segments for the associated elliptic sectors. Similarly, let $\gamma_1, \ldots, \gamma_h$ be the angles of the hyperbolic sectors and $\beta_1, \ldots, \beta_p$ the remaining parabolic sectors. Figure 5.25 illustrates the subdivision into sectors and the associated angles for a critical point with sectors of each type.

Let's now compute the contribution to the index made by each type of sector.

- *Contribution of an elliptic sector* We must calculate the amount of turning of the vectors as we move through the sector along C. By studying Figure 5.26, we see that for an elliptic sector the vectors rotate through angle of $\alpha_i + \pi$. There is a rotation of α_i from one straight-line segment to the other, plus an additional rotation of π to reverse the direction from pointing away from p to pointing in toward p.
- *Contribution of a hyperbolic sector* We do a similar analysis. The total rotation in the sector is through angle of $\pi - \gamma_i$, but it is in the *clockwise* direction, so we count it as $\gamma_i - \pi$ (Figure 5.27).
- *Contribution of a parabolic sector* See Exercise 5.4.2.

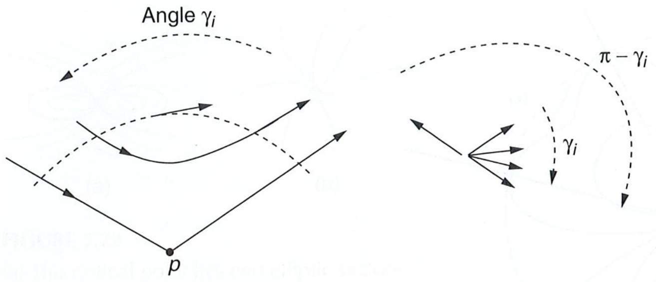

FIGURE 5.27
Computing the contribution to the index of a hyperbolic sector

EXERCISE 5.4.2 Show that the angle rotation through a parabolic sector is simply β_i.

We now put this information together. Traveling once, counterclockwise, around C, the vectors turn counterclockwise through angle

$$\begin{aligned}(\alpha_1 + \pi) + \cdots + (\alpha_e + \pi) + \beta_1 + \cdots + \beta_p + (\gamma_1 - p) + \cdots + (\gamma_h - \pi) \\ = \{(\alpha_1 + \cdots + \alpha_e) + (\beta_1 + \cdots + \beta_p) + (\gamma_1 + \cdots + \gamma_h)\} + \mathcal{E}\pi - \mathcal{H}\pi \\ = 2\pi + (\mathcal{E} - \mathcal{H})\pi\end{aligned}$$

The index (the number of revolutions) is

$$\frac{2\pi + (\mathcal{E} - \mathcal{H})\pi}{2\pi} = 1 + \frac{\mathcal{E} - \mathcal{H}}{2}$$

We now have the following.

THEOREM 5.4.3 The index of an isolated critical point is $1 + \dfrac{\mathcal{E} - \mathcal{H}}{2}$, where $\mathcal{E}$ = the number of elliptic sectors and $\mathcal{H}$ = the number of hyperbolic sectors.

NOTE: The parabolic sectors contribute nothing. Also, a rotation critical point has index 1, so the formula for the index still holds with $\mathcal{E} = \mathcal{H} = 0$.

Figure 5.28 illustrates a few examples that show we get the same results as with using the angle-rotation definition but with less work.

EXERCISE 5.4.4 Find the index of each of the critical points shown in Figure 5.29.

EXERCISE 5.4.5 Draw a phase portrait for a critical point of index 5 with $\mathcal{E} = 10$ and $\mathcal{H} = 2$.

EXERCISE 5.4.6 Draw a phase portrait for a critical point of index −3.

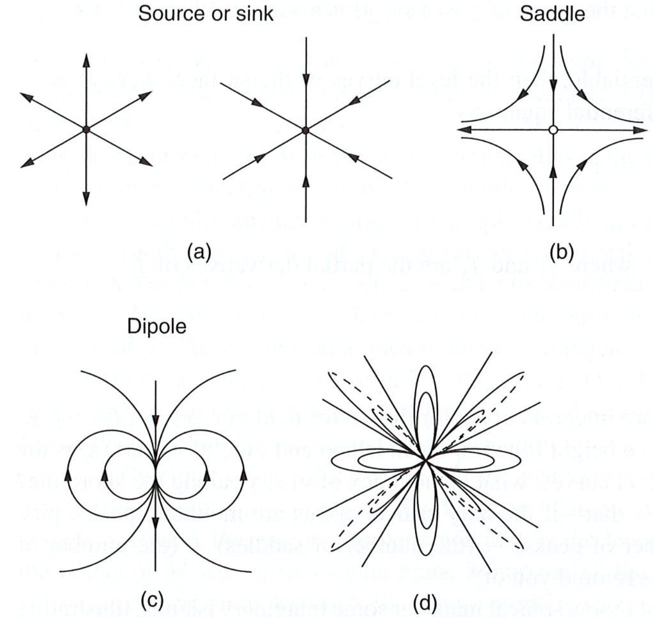

FIGURE 5.28
(a) $\mathcal{E} = \mathcal{H} = 0$, so the index $I = 1$. (b) $\mathcal{E} = 0$ and $\mathcal{H} = 4$, so $I = -1$. (c) $\mathcal{E} = 2$ and $\mathcal{H} = 0$, so $I = 2$. (d) To construct a critical point with index 5, for example, we take $\mathcal{E} - \mathcal{H} = 8$. So if $\mathcal{E} = 8$ then $\mathcal{H} = 0$.

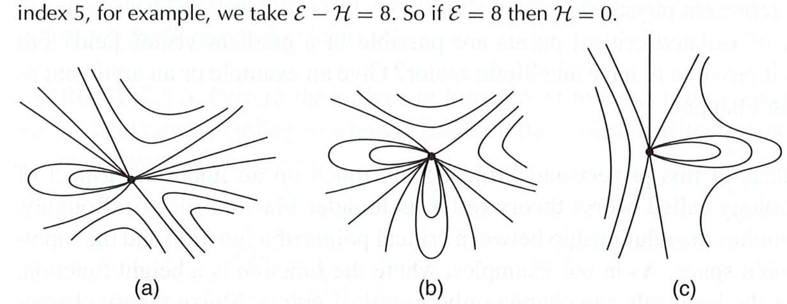

FIGURE 5.29
Use the formula $1 + \dfrac{\mathcal{E} - \mathcal{H}}{2}$ to compute the index of each of these critical points.

EXERCISE 5.4.7 Among all the critical points with no more than four sectors, find and sketch the distinct ones with index 1, 2, or 3.

EXERCISE 5.4.8 Show that any critical point with an odd number of sectors has at least one parabolic sector.

EXERCISE 5.4.9 Draw all critical points of index 3 that have no more than six sectors. Explain why you have described all such.

PROJECT 5.4.10 Recall that the graph of $z = f(x, y)$ is a surface.

a. Show that if f is differentiable, then the level curves of the surface, $f(x, y) = c$, satisfy the system of differential equations

$$\left.\begin{aligned} \frac{dx}{dt} &= f_y \\ \frac{dy}{dt} &= -f_x \end{aligned}\right\} \quad \text{where } f_x \text{ and } f_y \text{ are the partial derivatives of } f$$

Hence, the level curves are integral curves for the vector field $v(x, y) = (f_y, -f_x)$.

b. If the function $f(x, y)$ is a height function on an island and $v(x, y)$ in part (a) is the vector field giving the level curves, what is the index of $v(x, y)$ along the shoreline? Use your answer to show that—if the only critical points are mountain peaks, pits, and saddles—(the number of peaks) − (the number of saddles) + (the number of pits) = 1. What does this remind you of?

c. Draw several examples of topographical maps for some imaginary islands, illustrating your findings.

d. The gradient vector field associated to the function f is (f_x, f_y). What is the geometric relationship between the gradient vector field and $v(x, y)$? What does the gradient vector field represent physically?

e. What types of isolated critical points are possible in a gradient vector field? For example, is it possible to have an elliptic sector? Give an example or an argument as to why it can't happen.

NOTE: The ideas in this project and Project 5.5.4 touch on an important aspect of differential topology called Morse theory (after its founder Marston Morse). Roughly, Morse theory studies the relationship between critical points of a function and the topology of the domain space. As in our examples, where the function is a height function, the topology of the level sets can change only at critical points. Morse theory characterizes the type of changes that can occur. We can begin to reconstruct the topology of the domain surface by knowing these changes. The short introductory chapter in Milnor [Mi] gives an elegant and informal description for surfaces. There is also a classic video, distributed through the Mathematical Association of America, featuring Morse as he discusses the above topics with models and animation. It is called *Pits, Peaks and Passes*.

Morse theory and its generalizations have been used in various geometric modeling applications in computer graphics. For example, how can the shape of a three-dimensional object be recovered from a two-dimensional image of the object? To be useful, one must factor in the variation that can occur in the two-dimensional image when viewing the object from a different perspective. Such questions become critical in interpreting medical images—for example, in radiotherapy, biopsies, and neurosurgery. Koenderink's *Solid Shape* [Koe] gives an excellent introduction to these applications.

5.5 VECTOR FIELDS ON SURFACES

So far, we have only looked at vector fields on the plane. Recall that, in the introduction to this chapter, we defined a vector field on any surface. A (tangent) **vector field v on a surface** $M \subseteq \mathbb{R}^N$ is a continuous function $v : M \to \mathbb{R}^N$ so that $v(p)$ (drawn based at p) is a tangent vector to M at p. The vector $v(p)$ lies in the tangent plane to M at p. We will restrict our attention to compact *orientable* surfaces.

Let's examine an important set of examples: gradient vector fields. In Project 5.4.10, we looked at the special case of a planar surface with a real-valued function $z = f(x, y)$ (interpreted as height). The associated gradient vector field (f_x, f_y) is a vector field on the plane, perpendicular to the level sets of f and pointing in the direction of greatest increase of f. We use the same idea to create a (tangent) vector field on any surface. Take M to be a compact surface in $\mathbb{R}^3$. We assume that M is a smooth surface, with smoothly varying tangent planes. The level curves of a height function for M (with the z coordinate as height) are obtained by intersecting the surface with horizontal planes $z = k$ for various values of k. We define the **gradient vector field** on M to be tangent at every point and pointing in the direction of steepest ascent (with respect to the height z). Hence, the vectors are perpendicular to the level curves of M. Precisely at the points of M where the tangent plane is horizontal, the gradient vector is the zero vector—a critical point. Figure 5.30 shows a gradient vector field and its level curves on the sphere.

In general, the critical points of a gradient vector field on M may not be isolated, as Figure 5.31 illustrates. However, it can be proved (not here) that a slight perturbation of the surface can make the critical points isolated. *We will assume that isolated critical points are in all our vector fields.*

EXERCISE 5.5.1 Perturb the surface in Figure 5.31 to make it into one whose gradient vector field corresponding to a height function has isolated critical points. (You should see many ways of doing so.)

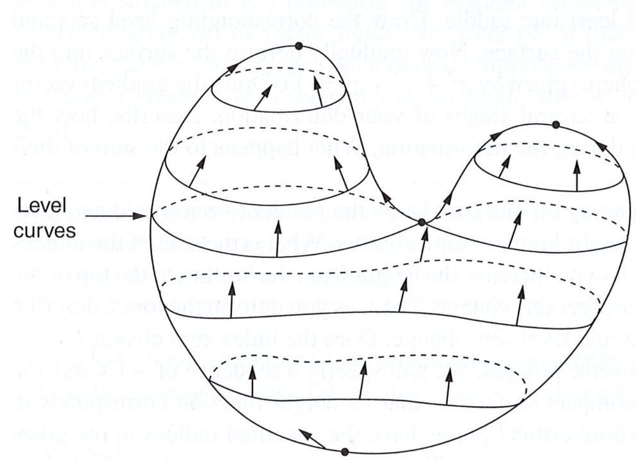

FIGURE 5.30
The gradient vector field of a deformed sphere

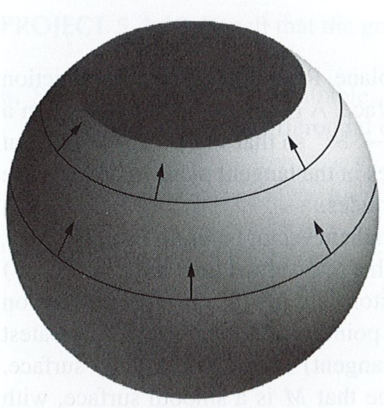

FIGURE 5.31
The gradient vector field on a flat-topped sphere has an entire disk of critical points.

EXERCISE 5.5.2 Draw a deformation of the sphere so that it has a gradient vector field associated with a height function with four critical points of index 1 and two saddles. [*Note:* Your sphere might be quite distorted, but it will still be a *topological* sphere.]

EXERCISE 5.5.3 Do you think one could have a sphere deformed such that its gradient vector field has no critical points of index 1? Why or why not? What about for $2\mathbb{T}$ or other compact surfaces? Explain your reasoning. You need not give a formal proof.

PROJECT 5.5.4

1. **a.** Draw a (topological) sphere shaped so that its height function gives a gradient vector field having at least one saddle. Draw the corresponding level sets and gradient vector field on the surface. Now gradually deform the surface into the standard geometric sphere (given by $x^2 + y^2 + z^2 = 1$). Draw the gradient vector fields of the surfaces at several stages of your deformation. Describe how the critical points change during the deformation. What happens to the sum of their indices?
   b. Begin with a torus standing on one end. Draw the gradient vector field and level sets associated to the height function for this torus. What is the sum of the indices of the critical points? As you increase the height from the bottom to the top of the torus, describe how the level sets change. Then, as you deform the torus, describe how the critical points and level sets change. Does the index sum change?
2. Let's now examine the reverse process. We will specify a sequence of -1's and 1's. Your goal is to build a compact surface so that its height function corresponds to a gradient vector field whose critical points have the specified indices in the given order. Identify the surface.
   a. $1, -1, 1, 1$
   b. $1, -1, -1, -1, -1, 1$

   c. $1, 1, -1, -1, 1, 1$
   d. $1, -1, -1, -1, 1, 1, -1, 1$
   e. Were you able to identify the surface before building it? Can you think of any restrictions on the achievable types of sequence of 1's and -1's? Explain.

It is important to note that the local existence and uniqueness theorem for differential equations extends to surfaces since the theorem is local and surfaces are locally homeomorphic to a plane. Hence, we have integral curves on M, at least locally. We do not know, however, that an integral curve is globally defined—that is, defined for all time t.

Example 5.5.5 The differential equation in $\mathbb{R}^1$ given by $\frac{dx}{dt} = 1 + x^2$ has integral curves given by $x = \tan(t - c)$. These integral curves cannot be extended beyond the interval $(c - \frac{\pi}{2}, c + \frac{\pi}{2})$.

However, there is a completeness theorem in the theory of differential equations stating that integral curves on a compact surface M can be extended indefinitely (that is, can continue for all time t). In particular, we have a phase portrait on all of M, with integral curves defined for all t in $\mathbb{R}^1$.

EXERCISE 5.5.6 Is Corollary 5.2.13 true on $\mathbb{S}^2$? On $\mathbb{T}$? Explain.

EXERCISE 5.5.7 Explain why the Poincaré–Bendixson theorem (see Section 5.3) is true on the sphere. For other surfaces, it is not necessarily true. Explain why the proof does not work for other surfaces. Can you find an example showing it is not true on the torus.

The gradient vector fields above illustrate how the geometry of a surface (the way it is situated in $\mathbb{R}^3$) influences the gradient vector field. Does the *topology* of M influence what sort of vector fields can occur on M? Of course! That is why we are studying them in this course. For example, suppose we ask, "On which surfaces M can we have a nowhere zero (tangent) vector field—that is, a vector field without critical points?"

Example 5.5.8 On a torus, an example of a nowhere zero vector field is easy; cover the torus by longitudinal circles as the integral curves and take unit tangent vectors, as shown in Figure 5.32.

How about the sphere $\mathbb{S}^2$? If we draw some examples of vector fields on the sphere, we might guess not. Look at those in Figure 5.33, for example.

EXERCISE 5.5.9 Draw two or three other vector fields tangent to the sphere. Do they all have critical points? Can you find one with exactly one critical point? If so, what is the index of the critical point?

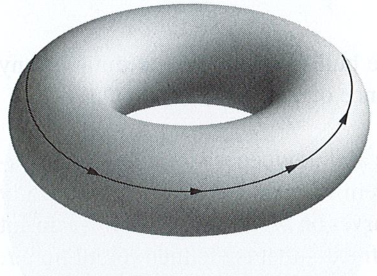

FIGURE 5.32
A nowhere zero vector field on a torus: Take vectors tangent to the longitudinal circles.

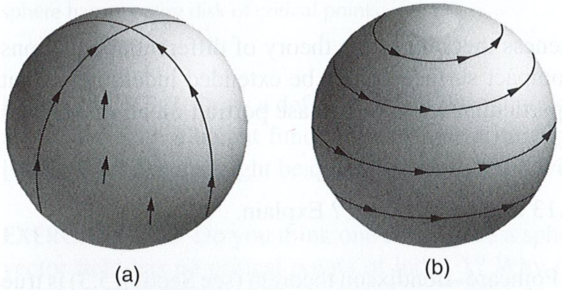

FIGURE 5.33
(a) The gradient vector field along longitudes produces a sink at the north pole and a source at the south pole. (b) The tangent vector field along latitudes produces two centers: one at the north pole and one at the south. Could one construct a nowhere zero vector field on the sphere?

PROJECT 5.5.10 Draw some examples of vector fields on various surfaces. Conjecture a relationship between the indices of the critical points and the topology of the surface. You do not need to prove your conjecture. We will return to this topic shortly.

We will now *prove* that a vector field tangent to the sphere always has a critical point. Our results on the index of critical points still apply in the context of surfaces because index was defined locally. One can simply project a small neighborhood of a critical point p and the nearby vectors onto the tangent plane at p and compute the index there, as illustrated in Figure 5.34.

Here is a beautiful proof from Chinn and Steenrod [CS] that vector fields tangent to the sphere always have critical points. It is actually a special case of our main theorem about vector fields on surfaces and gives a partial answer to what kind of vector fields can occur on the sphere.

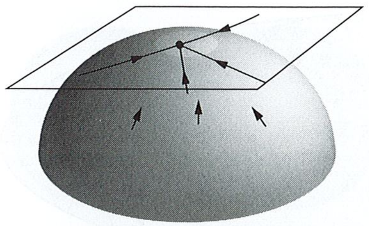

FIGURE 5.34
The sink at the north pole projects to a sink on the tangent plane at the north pole (with index 1).

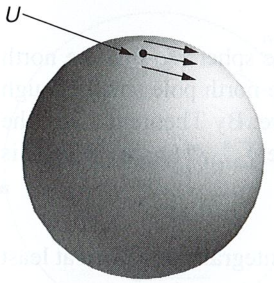

FIGURE 5.35
In a very small neighborhood of the north pole p, the vectors are nearly parallel to $v(p)$.

THEOREM 5.5.11 Every (tangent) vector field on $\mathbb{S}^2$ has at least one critical point. Further, if the vector field v has finitely many critical points $p_1, \ldots, p_n$ then $\sum_{i=1}^{n} I(p_i, v) = 2$.

Proof: We do a proof by contradiction. So suppose v is a tangent vector field on $\mathbb{S}^2$ without critical points. Take a small enough neighborhood U of the north pole p so that the vectors on the boundary are nearly parallel to $v(p)$ (Figure 5.35).

Take the stereographic projection from $(\mathbb{S}^2 - U)$ onto D, a disk in the plane tangent to the south pole of the sphere. We note that we can also project the vectors to give a nonzero vector field w on D. (Do you see that nonzero tangent vectors on $\mathbb{S}^2$ project to nonzero vectors in the plane?) To visualize the resulting vector field on D, it might help to imagine reaching inside the sphere and stretching it open to lie flat, keeping the vectors tangent to the stretching surface along the way (Figure 5.36).

Count the number of rotations of the vectors as you traverse the boundary of the disk. The index of the projected vector field w around the boundary of D is 2, not 0. Hence, by Theorem 5.2.7, there is a critical point in the disk, hence a critical point of v on $(\mathbb{S}^2 - U)$ and therefore on $\mathbb{S}^2$.

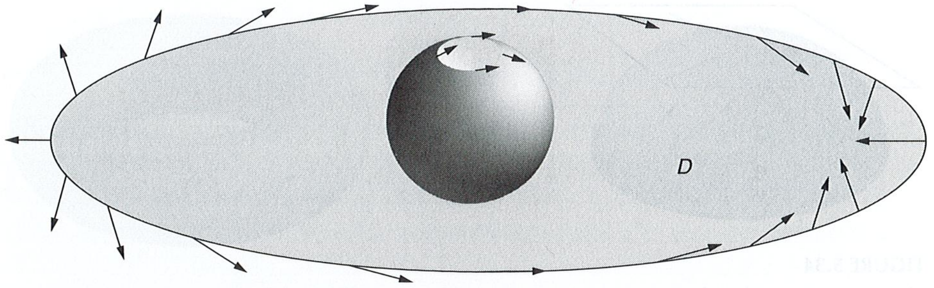

FIGURE 5.36
Stereographically project $\mathbb{S}^2 - U$ and the vector field onto the disk D. What is the index of the projected vector around the boundary of D?

For the second part of the theorem, simply position the sphere so that the north pole is not a critical point. Taking the neighborhood U of the north pole small enough to miss any critical points, apply the same argument as above. By Theorem 5.2.7, the sum of the indices of the critical points of w on D is 2. Hence $\sum_{i=1}^{n} I(p_i, v) = 2$. This completes the proof. ■

PROPOSITION 5.5.12 If a vector field on $\mathbb{S}^2$ has a closed integral curve, then at least two critical points exist.

EXERCISE 5.5.13 Prove Proposition 5.5.12.

EXERCISE 5.5.14 Why, at any instant, is there at least one point on Earth's surface where the wind is still?

EXERCISE 5.5.15 Show that any vector field on the sphere, tangent or not, either has a critical point or has a vector perpendicular to the sphere.

What about vector fields on other surfaces? The following major theorem about vector fields on surfaces connects all the ideas of this chapter with our old friend, the Euler characteristic.

Poincaré–Hopf index theorem *If w is a (tangent) vector field with isolated critical points $p_1, \ldots, p_n$ on an orientable surface M, then $\sum_{i=1}^{n} I(w, p_i) = \chi(M)$.*

Proof: We do a proof by induction on the genus g of M. If g is zero then M is a sphere and the result is simply Theorem 5.5.11. Inductively assume that the result is true for a surface of genus $g - 1$, and suppose M is homeomorphic to gT. Take a simple closed curve C on M that does not disconnect M and such that there are no critical points of w on C. Cut along C and glue in disks along each of the two resulting boundary circles to produce a surface M^* of genus $g - 1$ (Figure 5.37).

We extend the vector field w to the interior of the two disks. To do so, imagine the disks as flattened disks in the plane and simply extend the vector field radially. That is,

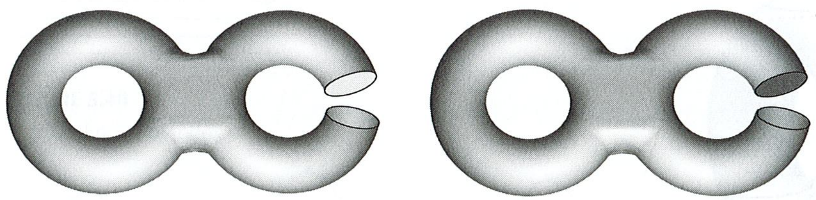

FIGURE 5.37
Cut a surface along a non-disconnecting curve and glue in disks to create a surface with lower genus.

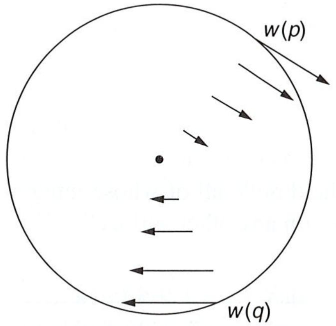

FIGURE 5.38
Extend the vector field radially from the boundary of the disk to the interior.

along each ray from the center of the disk to a point p on the boundary, assign the vector $r \cdot w(p)$ where r is the distance from the center of the disk (Figure 5.38).

Let's denote the resulting vector field on M^* by w^*. Notice that the two disks we glued in could be glued to each other to form a sphere $\mathbb{S}^2$, and since w^* restricted to the boundaries of both these disks matches w on C, w^* gives a vector field on $\mathbb{S}^2$. We know by Theorem 5.5.11 that the sum of the indices of the critical points of w^* on $\mathbb{S}^2$ is 2. By the induction hypothesis, the sum of the indices of w^* on M^* is $\chi(M^*) = 2 - 2(g - 1)$. So we have

$$\sum_{i=1}^{n} I(p_i, w) + 2 = 2 - 2(g - 1),$$

or equivalently,

$$\sum_{i=1}^{n} I(p_i, w) = 2 - 2g = \chi(M),$$

completing the proof.

[*Note:* This theorem is also true for nonorientable surfaces, but it takes a different proof.] ■

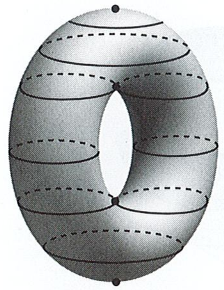

FIGURE 5.39
The level sets of the height function on this upright torus give a vector field with two saddles and two centers.

EXERCISE 5.5.16 In Example 5.5.8, we saw a vector field on $\mathbb{T}$ all of whose integral curves are simple closed curves. Are there such examples on any other surface?

Example 5.5.17 Suppose we want to draw a phase portrait on $\mathbb{T}$ with exactly two saddles and whatever other critical points are needed. We first notice that each saddle has index -1, so we need the other critical points to have indices totaling $+2$ so that the entire sum will be $0 = \chi(\mathbb{T})$. One way of doing this is to have two more critical points, each of index $+1$. Sources, sinks, and centers provide easy ways of producing critical points of index $+1$. We can, for example, stand the torus on end and take the level sets of the height function as the integral curves, as illustrated in Figure 5.39.

Example 5.5.18 Suppose we want to draw a phase portrait on $\mathbb{S}^2$ with exactly two critical points, one of index 2. We first notice that the other must be index 0 since the sum of the two indices must equal $\chi(\mathbb{S}^2) = 2$. A dipole has index 2. What is a critical point with index 0? We know the index of a critical point is given by $1 + \dfrac{\mathcal{E} - \mathcal{H}}{2}$. Setting this expression equal to 0 and solving for h yields $\mathcal{H} = \mathcal{E} + 2$. For example, if we take $\mathcal{E} = 0$ and $\mathcal{H} = 2$, we get index 0. An index 0 critical point is shown in Figure 5.40. We have simply made the flow stationary at a point along one orbit (creating a critical point). Notice the nearby orbits must slow down as they pass the critical point (to preserve continuity), but that is not obvious from the phase portrait. Now put a vector field with a dipole on $\mathbb{S}^2$, and along any integral curve insert a critical point like the one in Figure 5.40.

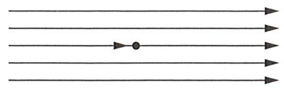

FIGURE 5.40
A critical point of index 0

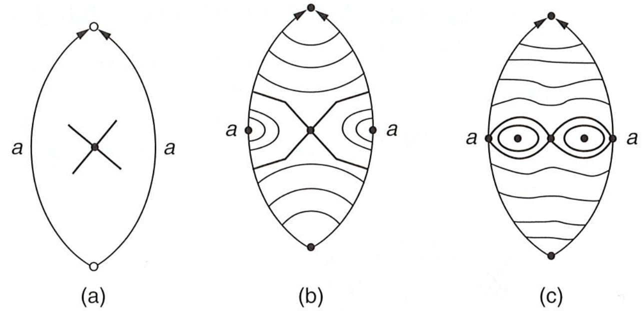

FIGURE 5.41
The task is to complete the drawing in (a) to a vector field on the sphere. Two possible solutions are given in (b) and (c). Can you find others?

Example 5.5.19 Suppose we wish to complete the sketch shown in Figure 5.41(a) to a vector field on $\mathbb{S}^2$. We know that the sum of the indices of the critical points must be 2. The saddle has index -1. We have many options; two possible solutions are shown in Figure 5.41(b) and (c).

In Figure 5.41(b), we have added three centers (each index 1). One is at the top vertex of the plane model, one is at the bottom, and the third is along the edge. Notice that the third critical point is shown as two points, with those points being identified. You should draw in the arrows along the integral curves to ensure that they can be consistently oriented.

In Figure 5.41(c), we have added five critical points to the existing saddle. What is the index of each one? Is the total index sum equal to 2 as it should be?

Can you find other ways of completing Figure 5.41(a)?

EXERCISE 5.5.20 Draw the phase portrait of a vector field on the sphere with

a. Exactly three critical points
b. One saddle and whatever else you need
c. A "monkey saddle" (six hyperbolic sectors) and whatever else you need

EXERCISE 5.5.21 Draw a phase portrait on the torus (use the plane model if it is easier) with

a. A center and a saddle as the only critical points
b. Exactly three critical points, one of which has index -1
c. A dipole and whatever else you need.

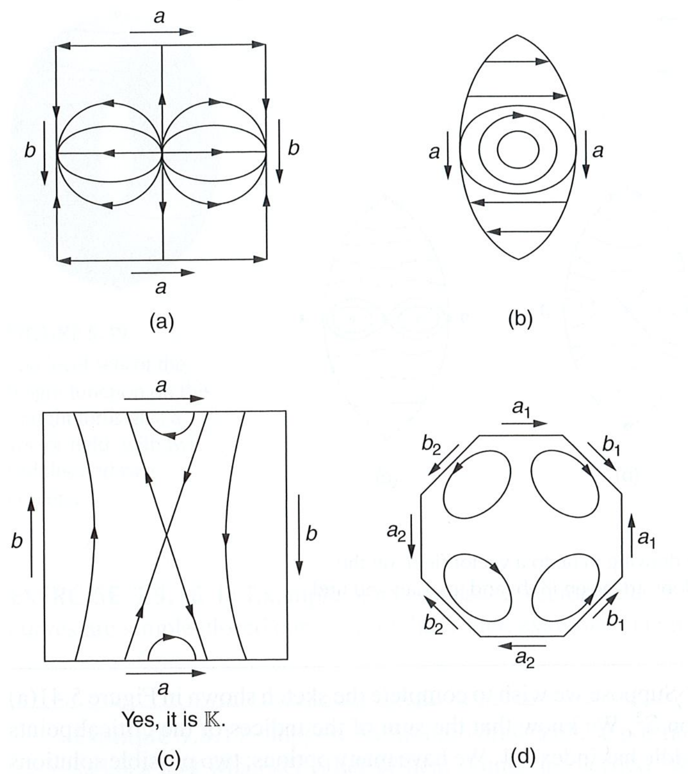

FIGURE 5.42
Complete these partial phase portraits. Make sure that the Poincaré–Hopf theorem holds in each case.

EXERCISE 5.5.22 Suppose we have a vector field on $2\mathbb{T}$ with exactly one critical point. What is its index? Draw a phase portrait.

***EXERCISE 5.5.23** Which orientable surfaces have vector fields with precisely one critical point? Describe a procedure for constructing examples. What if you want precisely two critical points?

EXERCISE 5.5.24 Figure 5.42 shows parts of phase portraits on the plane models of some surfaces. For each figure, clearly indicate the necessary critical points, finding the index of each such critical point, and verify that the Poincaré–Hopf theorem does hold in that case.

PROJECT 5.5.25 In the exercises, we have seen various examples in which certain conditions are placed upon some of the critical points of a vector field on a surface and you are asked to complete the vector field in some way. Are there any combinations

that can't be completed? That is, if given some number of critical points with indices specified, with the sum of their indices equal to the Euler characteristic of the surface, can you always construct a vector field on the entire surface with precisely these critical points? Explain your reasoning.

GUIDE TO SUPPLEMENTAL READINGS

Vector Fields on Surfaces Chinn and Steendrod [CS], Firby and Gardiner [FG], Godbillon [Go], Palis and de Melo [PdM], Prasalov [Pr], and Spivak [Sp].
Morse Theory Milnor [Mi].
Shape Theory Koenderink [Koe].