Problem Sets

Knots

(K problems are from the *Knot Theory* book which is scanned an on our course website - some images must be found there.)

K1.1. If at a crossing point in a knot diagram the crossing is changed so that the section that appeared to go over the other instead passes under, an apparently new knot is created. Demonstrate that if the marked crossing in Figure 1.8 is changed, the resulting knot is trivial (draw convincing pictures - not necessarily Reidemeister moves). What is the effect of changing some other crossing instead?

K1.2. Figure 1.9 illustrates a knot in the family of 3-stranded *pretzel knots*; this particular example is the $(5, -3, 7)$ pretzel knot. Can you show that the (p, q, r)-pretzel knot is equivalent to both the (q, r, p)-pretzel knot and the (p, r, q)-pretzel knot (draw convincing pictures - not necessarily Reidemeister moves)?

K1.3 The subject of knot theory has grown to encompass the study of links, formed as the union of disjoint knots. Figure 1.10 illustrates what is called the *Whitehead link*. Find a deformation of the Whitehead link that interchanges the two components (draw convincing pictures - not necessarily Reidemeister moves). (It will be proved later that no deformation can separate the two components.)

K1.4 For what values of (p, q, r) will the corresponding pretzel knot actually be a knot, and when will it be a link? For instance, if $p = q = r = 2$, then the resulting diagram describe a simple link of three components, “chained" together.

K1.5 Describe the general procedure for drawing the (p, q)-torus knot. What happens if p and q are not relatively prime?

K1.6 The link in Figure 1.11 is called the *Borrromean rings*. It can be proved that no deformation will separate the components. Note, however that if one of the two components is removed, the remaining two can be split apart. Such a link is called *Brunnian*. Can you find an example of a Brunnian link with more than 3 components?

K1.7 The knots illustrated in Figure 1.12 were, until recently (1970, I believe) assumed to be distinct, and both appeared in many knot tables. However, a lawyer, Ken Perko, discovered a deformation that turns one into the other. As a challenging exercise, try to find it (draw convincing pictures - not necessarily Reidemeister moves).

K2.3.4 Is every polygonal knot with exactly 4 vertices unkotted?

K2.4.4 Show that the trefoil knot can be deformed so that its (nonregular) projection has exactly one multiple point.

K2.5.2 Any oriented knot, or link, determines an unoriented link. Simply ignore the orientation. Given a knot, there are at most two equivalence classes of oriented knots that determine its equivalence class, ignoring orientations. (Why? In particular why could there only be one?)

(a) What is the largest possible number of distinct oriented n component links which can determine the same unoriented link, up to equivalence? Try to construct an example in which this maximum is achieved. (Do not attempt to prove that the oriented links are actually inequivalent.)

(b) Show that any two oriented links which determine the unlink as an unoriented link are oriented equivalent.

A knot is called *reversible* if the two choices of orientation are equivalent.

K2.5.4 Show that the (p, p, q)-pretzel knot is reversible.

K2.5.5 The knot 8_{17} is the first knot in the table that is not reversible, a difficult fact to prove. Find reversing deformations for some of the knots that precede it. Several are not obvious. Include some that are not obvious.

K2.5.6 Classically, what has been defined here as the reverse of a knot was called the inverse. The inverse is now defined as follows. Given an oriented knot, multiplying the z-coordinates of its vertices by -1 yields a new knot, K^m, called the *mirror image* of the first (this can be easily drawn by merely changing each of the crossings). The *inverse* of K is defined to be K^{mr} (i.e. the reverse of the mirror image).

(a) How are the diagrams of a knot and its mirror image and inverse related?
(b) Given a knot diagram it is possible to form a new knot diagram by reflecting the diagram through a vertical line in the plane, as illustrated in Figure 2.8. To what operation on knots in 3-space does this correspond?
(c) Show that the operation described in part (b) yields a knot equivalent to the mirror image of the original knot.

K3.1.1 Show that the change illustrated in Figure 3.2 can be achieved by a sequence of two Reidemeister moves.

K3.1.2 Find a sequence of Reidemeister moves that transforms the diagram of the unknot drawn in Figure 3.3 into the diagram without crossings. Here is a harder exercise: What is the least number of Reidemeister moves needed for such a sequence? Can you prove that this is the least number that suffices?

K3.2.1 Which of the knot diagrams with seven or fewer crossing, as illustrated in a knot table, are colorable?

K3.2.2 For which integers n is the $(2, n)$-torus knot in Figure 3.6a colorable? The knot illustrated in Figure 3.6b is called the *n-twisted double of the unknot*, where $2n$ is the number of crossings in the vertical band. The trefoil results when $n = -1$. What if $n = 0$ or 1? For which values of n is the n-twisted double of the unknot colorable?

K3.2.3 Discuss the colorability of the (p, q, r)- pretzel knots.

K3.2.4 (a) Prove the coloring theorem for Reidemeister move 1a.
(b) How many cases need to be considered in proving Theorem 2 for Reidemeister move 3a?
(c) Check each of these cases.
(d) Complete the proof of Theorem 2.

K3.2.5 Given an oriented link of two components, J and K, it is possible to define the *linking number* of the components as follows. Each crossing point in the diagram is assigned a sign, $+1$ if the crossing is right-handed and -1 if it is left-handed. The linking number of K and J, $\text{lk}(K, J)$, is defined to be the sum of the signs of the crossing points where J and K meet, divided by 2.

(a) Use the Reidemeister moves to prove that the linking number depends only on the oriented link, and not on the diagram used to compute it.
(b) Figure 3.8 illustrates an oriented *Whitehead link*. Check that it has linking number 0.
(c) Construct examples of links with different linking numbers.

K3.2.6 This exercise demonstrates that the linking number is always an integer. First note that the sum used to compute linking numbers can be split into the sum of the signs of the crossing where K passes over J, and the sum of the crossing where J passes over K.

(a) Use Reidemeister moves to prove that each sum is unchanged by a definition.
(b) Show that the difference of the two sums is unchanged if a crossing is changed in the diagram.
(c) Show that if the crossings are changed so that K always passes over J, the difference of the sums is 0. (The link produced by changing these crossings can be deformed so that K and J have disjoint projections.)
(d) Argue that the linking number is always an integer, given by either of the two sums. (This is the usual definition of linking number. The definition in Exercise 2.5 makes it clear that $\mathrm{lk}(K, J) = \mathrm{lk}(J, K)$.)

K3.2.7 The definition of colorability is often stated slightly differently. The requirement that at least two colors are used is replaced with the condition that all three colors appear.

(a) Show that the unlink of two components has a diagram which is colorable using all three colors and another diagram which colorable with exactly two colors.
(b) Why is it true that for a knot, once two colors appear all three must be used, whereas the same statement fails for links?
(c) Explain why the proof of Theorem 2 applies to links as well as to knots.

K3.2.8 Prove that the Whitehead link illustrated in Figure 3.8 is nontrivial, by arguing that it is not colorable.

K3.2.9 In this exercise you will prove the existence of an infinite number of distinct knots by counting the number of colorings a knot has. If a knot is colorable there are many different ways to color it. For instance, arcs that were colored red can be changed to yellow, yellow arcs changed to blue, and blue arcs to red. The requirements of the definition of colorability will still hold. There are six permutations of the set of three colors, so any coloring yields a total of six colorings. For some knots there are more possibilities.

(a) Show that the standard diagram for the trefoil knot has exactly six colorings.
(b) How many colorings does the *square knot* shown in Figure 3.9 have?
(c) The number of colorings of a knot projection depends only on the knot; that is, all diagrams of a knot will have the same number of colorings. Outline a proof of this.
(d) Use the connected sum of n trefoils, illustrated in Figure 3.10, to show that there are an infinite number of distinct knots.

K3.3.1 Determine which knots with 6 or fewer crossings can be labeled mod 5.

K3.3.2 For what primes p can the trefoil knot diagram be labeled mod p?

K3.3.3 Prove Theorem 3 by showing that if any Reidemeister move is performed on a labeled diagram, the resulting diagram can again be labeled.

K3.3.4 Show that if all the labels of a knot that is labeled mod 3 are multiplied by 5, the resulting labeling is a labeling mod 15. This gives some indication as to why p is restricted to primes.

K3.3.5 If p is 2, other difficulties arise. Explain why no knot can be labeled mod 2. (Modulo 2, what does the crossing relationship say?)

K3.3.6 Check that the theory of labelings applies to links of many components.

K3.3.7 Show that the knots $4_1, 7_1$, and 8_{16} are distinct by using mod 5 and mod 7 labelings. (Find mod 5 and mod 7 labelings of 8_{16}.)

K3.4.1 For each knot with 6 or fewer crossings find the associated matrix, and its determinant. In each case, for what p is there a mod p labeling?

K3.4.2 The knots 8_{18} and 9_{24} both have determinant 45. Check that one has a mod 3 rank of 1, while the other has a mode 3 rank of 2. The knots 8_8 and 9_{49} both have determinant 25. Compute their mod 5 ranks.

K3.4.3 Prove the linear algebra results stated in the proof of Theorem 5.

K3.4.4 Because the unknot has some particularly simple diagrams, the arguments given above really need to be modified slightly. The two diagrams for the unknot that cause difficulties are the diagram with no crossings, and the diagram with exactly one crossing. What goes wrong in these cases? Why don't these problems occur in other situations? How would you correct for these minor problems? (Define the determinant and nullity of a 0×0 matrix to be 1.)

K3.4.5 Prove the determinant of a knot is always odd. (See Exercise 5 of the previous section, relating to mod 2 labelings. Also, this result does not apply for links of more than one component.)

K3.4.6 Show that if a knot has mod p rank n, then the number of mod p labelings is $p(p^n - 1)$.

K3.5.1 Compute the Alexander polynomial for several knots in the tables.

K3.5.2 Relate the value of the Alexander polynomial of a knot evaluated at -1 to the determinant of the knot, defined in the previous section.

K3.5.3 Check that Reidemeister move 1a does not change the Alexander polynomial.

K3.5.4 It is possible to construct knots with the same polynomial, but which can be distinguished by their mod p ranks for some p. Compute the polynomials of 8_{18} and 9_{24} to check that they are identical. In Exercies 4.2 of this chapter these knots were distinguished using the mod 3 ranks.

K3.5.5 Show that the knot in Figure 3.18 has Alexander polynomial 1. (This is one of only two knots with 11 or fewer crossings that has trivial polynomial, other than the unknot.) Use Exercise 5.2 to argue that the knot cannot be distinguished from the unknot using labelings. Stronger tools can be used to do this.

K3.5.6 Prove that a knot and its mirror image, as illustrated in Figure 3.19, have the same polynomial. (Hint: Label the mirror image in the obvious way, but reverse its orientation.)

K3.5.7 Show that the Alexander polynomial of K with its orientation reversed is obtained from the polynomial of K by substituting t^{-1} for t, and multiplying by the appropriate power of t, and perhaps changing sign.

O1. Find or draw a 2-component link that is not interchangable (recall that we saw that the Whitehead link is interchangable). Find or draw a noninterchangable 2-component link with the property that both components can be straightened into circles. In both of these you do not have the tools to prove your results (in fact, such tools are not well known among professionals), but be fairly certain that your results are true from experiment.

O2. The Borromean rings have the property that if you erase any component, they fall apart into the unlink (disjoint circles that are not connected). Any link with this property is called Brunnian. Find or draw a 4-component brunnian link. Can you find one with 5-components? n-components? Research why these are called Brunnian links.

O3. Seifert surfaces are not unique. Draw two nonhomeomorphic Seifert surfaces for the same knot diagram. This question is not particularly difficult, please do not make it so.

O4. Draw a non-split link (i.e. one that cannot be pulled apart into two pieces) for which there are Seifert surfaces for each component that are disjoint. Draw the surfaces, then connect them with a tube to create a Seifert surface for entire link.

O5. Draw two nonhomeomorphic Seifert surfaces for the Borromean rings both produced via Seifert's algorithm.

Topological Spaces

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

2. Let X be a set; let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X \backslash U$ either is countable or is all of X; also let $\mathcal{T}_\infty = \{U | X \backslash U$ is infinite or empty or all of $X\}$. Which of these are a topology on X?

3. If $\{\mathcal{T}_\alpha\}$ is a collection of topologies on X, which of $\cap \mathcal{T}_\alpha$ or $\cup \mathcal{T}_\alpha$ is a topology on X?

4. Give an example of a family $(U_i)_{i \in I}$ of open sets of $\mathbb{R}$ such that $\cap_{i \in I} U_i$ is not open.

5. Give an example of a topology $\mathcal{T}$ on $\mathbb{R}$ different from the ones we have studied (I think those are the usual, discrete, indiscrete, and finite complement, but please don't give an example we have studied that I merely have forgotten to include). Prove it is a topology.

6. Let X be a set and $\leq$ an order relation on X. Show that $\mathcal{T} = \{U | U \in 2^X, x \in U$ and $y \leq x \implies y \in U\}$ is a topology on X.

7. List all possible topologies on the three-element set $\{a, b, c\}$.

Closed, boundary

1. Famous result of Kuratowski: If S is a subset of a topological space X, then there are at most 14 subsets of X that can be obtained from S by successively taking either complements or closures. Find a subset S of $\mathbb{R}$ such that exactly fourteen subsets of $\mathbb{R}$ can be obtained from S in this manner. (Many hints - ignore them if you're too cool to need help: Prove these things: $S'' = S$, $\overline{\overline{S}} = \overline{S}$, $\overline{(\overline{((\overline{S})')})}' = \overline{S}$. For the example, use a subset of $\mathbb{R}$ with different types of pieces separated from each other.)

2. Let S be a subset of a topological space X. A point $x \in X$ is a *limit point* of S if every open set containing x contains a point of S other than x itself. A point $s \in S$ is an *isolated point* of S if there is an open set U containing s such that $U \cap S = \{s\}$. Show that the set of limit points of S is closed. Show that $\overline{S}$ is the disjoint union of the set of limit points of S and the isolated points of S.

3. Let S be a subset of a set X. Describe the closure of S when (i) X has the discrete topology, (ii) X has the indiscrete topology, and (iii) X has the finite complement topology (all of these are examples on page 72, numbers 5, 3 and 6, respectively).

4. Give an example of a family $(F_i)_{i \in I}$ of closed sets of $\mathbb{R}$ such that $\cup_{i \in I} F_i$ is not closed.

5. Prove that a set A is closed if and only if $\partial A \subset A$.

6. Prove that for any set A, ∂A is closed.

7. Let $\mathcal{T} = \{S \subset \mathbb{R} : 0 \in \mathbb{R} \backslash S\} \cup \{\mathbb{R}\}$. Show that $\mathcal{T}$ is a topology on $\mathbb{R}$ and find the closure of the interval $A = (1, 2)$ and of the interval $B = (-1, 1)$.

8. For each $n \in \mathbb{N}$, let $S_n = \{k \in X : k \geq n\}$. Show that $\mathcal{T} = \{S_n : n \in \mathbb{N}\} \cup \{\emptyset\}$ is a topology for $\mathbb{N}$, and find the closure of the set of even naturals. Find the closure of the singleton set $A = \{100\}$.

9. Let $\mathcal{S}$ and $\mathcal{T}$ be topologies for a set X. Prove that $\mathcal{S} \subset \mathcal{T}$ if and only if for every set $A \subset X$, it is true that $\overline{A}_{\mathcal{T}} \subset \overline{A}_{\mathcal{S}}$.

10. Show that if U is open in X and A is closed in X, then $U \backslash A$ is open in X, and $A \backslash U$ is closed in X.

Separation

1. A property of a topological space is *hereditary* if, whenever a topological space X has that property, then every subspace of X has the property. Show that the properties of being a T_1-space, Huasdorff space, and regular space are hereditary.

2. Let X have the finite complement topology. When is X a T_1-space? When is X a Hausdorff space?

3. Determine which separation axioms each of the topologies on a three-element set satisfy.

4. Find examples of topological spaces with at least three elements that demonstrate the separation axioms are distinct. That is, as we had begun, find an example that satisfies none of them, an example that is only $T_{1/2}$, an example that is $T_{1/2}$ and not T_1, and so forth.

5. Prove: A topological space satisfies T_1 if and only if every one point set is closed.

6. Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x | x \in X\}$ is closed in $X \times X$.

7. Show that the T_1 axiom is equivalent to the condition that for each pair of points of X, each has an open set not containing the other.

Continuity

1. Prove the following statements about continuous functions and discrete and indiscrete topological spaces.

(a) If X is discrete, then every function f from X to a topological space Y is continuous.

(b) If X is not discrete, then there is a topological space Y and a function $f : X \to Y$ that is not continuous. *Hint:* Let Y be the set X with the discrete topology.

(c) If Y is an indiscrete topological space, then every function f from a topological space X to Y is continuous.

(d) If Y is not indiscrete, then there is a topological space X and a function $f : X \to Y$ that is not continuous.

2. Prove that a function $f : X \to Y$ is continuous if and only if for each closed set $C \subset Y, f^{-1}(C)$ is closed in X.

3. Prove that all open intervals in $\mathbb{R}$ (finite, semi-infinite, or infinite) are homeomorphic. Prove that all half-open intervals in $\mathbb{R}$ are homeomorphic.

4. Show that the punctured plane $\mathbb{R}^2 \backslash \{0, 0\}$ is homeomorphic to the exterior of the closed unit ball $\mathbb{R}^2 \backslash \{(x, y) : x^2 + y^2 \leq 1\}$. State and prove an analogous result for $\mathbb{R}^n$.

5. Let X be a topological space and let X_0 be the topological space that is the set X with the finite complement topology. Show that the identity map of X to X_0 is continuous if and only if X is a T_1-space.

6. Let $(X, \mathcal{T})$ be a topological space. Let $C_X(X)$ be the set of all continuous functions from X into X. Prove that if $f \in C_X(X)$ and $g \in C_X(X)$ then $g \circ f \in C_X(X)$ (this is stated as $C_X(X)$

is *stable*). Let $\mathcal{H}(X)$ be the set of all homeomorphisms of X. Show that $\mathcal{H}(X)$ is a group under the operation of composition.

7. *The Pasting Lemma* Let X be a topological space with closed subsets A and B such that $X = A \cup B$. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions such that for each $x \in A \cap B$, $f(x) = g(x)$. Define a new function $f \cup g : X \to Y$ by

$$f \cup g(x) = \begin{cases} f(x) & \text{for } x \in A \\ g(x) & \text{for } x \in B \end{cases}$$

(1) Prove that $f \cup g$ is continuous.
(2) Give an example to show that the condition that A and B must be closed is necessary.
(3) Why is this called the pasting lemma?

8. Is the continuous image of a Huasdorff space still Hausdorff? Is Hausdorff a topological property?

9. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Show that f is continuous.

11. Suppose $i : (X, \mathcal{T}) \to (X, \mathcal{S})$ is the identity function. Prove that i is continuous if and only if $\mathcal{S} \subset \mathcal{T}$.

12. Suppose $f : X \to Y$ and $x \in X$. The f is said to be **continuous at** x if the inverse image of every open set containing $f(x)$ is an open set containing x. Prove that f is continuous if and only if f is continuous at every $x \in X$.

13. Let $\{f_n : n \in F\}$ be a finite collection of continuous functions from X into the space of real numbers with the usual topology. Let $f : X \to \mathbb{R}$ be defined by setting $f(x) = \min\{f_n(x) : n \in F\}$. Prove that f is continuous.

14. Suppose $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous functions.

(a) Prove that the function $(fg) : X \to \mathbb{R}$ defined by $(fg)(x) = f(x)g(x)$ is continuous.
(b) Suppose that $g(x) \neq 0$ for all $x \in X$. Prove that the function $(f/g) : X \to \mathbb{R}$ defined by $(f/g(x) = f(x)/g(x)$ is continuous.

15. An *open* function takes open sets to open sets. A *closed* function takes closed sets to closed sets. Prove or disprove: if $f : X \to Y$ is one-to-one, onto, and continuous, then $f^{-1} : Y \to X$ is a closed function. Are open functions closed, or vice versa?

16. Prove or disprove: If X and Y are homeomorphic and Y and Z are homeomorphic, then X and Z are homeomorphic.

17. Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

Subspaces

1. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$? Justify.

$$A = \left\{ x : \frac{1}{2} < |x| < 1 \right\}.$$

$$B = \left\{ x : \frac{1}{2} < |x| \leq 1 \right\},$$

$$C = \left\{ x : \frac{1}{2} \leq |x| < 1 \right\},$$

$$D = \left\{ x : \frac{1}{2} \leq |x| \leq 1 \right\},$$

$$E = \left\{ x : 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+ \right\}.$$

2. Let X be a topological space, let S be a subspace of X, and let E be a subset of S. Show that the subspace topology that E inherits from S coincides with the subspace topology that E inherits from X.

3. Prove that if A and S are subsets of a topological space X, then the closure of $A \cap S$ in S in the subspace topology for S is a subset of the intersection $\overline{A} \cap S$, where $\overline{A}$ is the closure of A in X. Give an example where the subspace closure of $A \cap S$ is a proper subset of $\overline{A} \cap S$.

4. Prove that if $f : X \to Y$ is continuous and if S is a subspace of X, then the restriction $f|_S : S \to Y$ s continuous.

5. Let X and Y be sets such that $X \subset Y$. Suppose $\mathcal{T}$ is a topology on X. Show that $\mathcal{W} = \mathcal{T} \cup \{Y\}$is a topology on Y and the subspace topology on X as a suspace of $(Y, \mathcal{W})$ is $(X, \mathcal{T})$.

6. Show that any set A is both open and closed relative to itself, and that $\emptyset$ is both open and closed relative to A.

7. Give an example of sets $B \subset A \subset \mathbb{R}^3$ where B is open relative to A but not open in $\mathbb{R}^3$.

8. Suppose A is a subspace of X. Show that C is closed in A if and only if $C = A \cap D$ for a closed set D in X.

9. Consider $\mathbb{Q} \subset \mathbb{R}$ with the usual topology. For each of the following sets, tell whether in the subspace topology the set is open, closed, neither or both: $A = \{r \in \mathbb{Q} : 0 < r < 1\}, B = \{r \in \mathbb{Q} : 0 < r \leq \sqrt{2}\}, C = \{r \in \mathbb{Q} : -\sqrt{2} \leq r \leq \sqrt{2}\}$.

Bases, Products

1. Let X be a topological space with the discrete topology. Find a base $\mathcal{B}$ of open sets for X such that $\mathcal{B}$ is included in any other base of open sets for X.

2. Let X and Y be topological spaces and let $\mathcal{B}$ be a base of open sets for Y. Show that a function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is an open subset of X for every $U \in \mathcal{B}$.

3. Let $\mathcal{B}$ be the family of subsets of $\mathbb{R}$ of the form $[a, b)$, where $-\infty < a < b < \infty$.

(a) Show that $\mathcal{B}$ is a base of open sets for a topology $\mathcal{T}$ of $\mathbb{R}$. The topology determined by $\mathcal{B}$ is the *half-open interval topology.*

(b) Show that every open subset of $\mathbb{R}$ (in the standard topology) is $\mathcal{T}$-open.

(c) Show that each interval $[a, b)$ is $\mathcal{T}$-closed.

4. Show that if E_j is a closed subset of $X_j, 1 \leq j \leq n$, then $E_1 \times \cdots \times E_n$ is a closed subset of $X_1 \times \cdots \times X_n$.

5. Suppose $X = X_1 \times \cdots \times X_n$, where each X_j is nonempty. Prove that if X is Hausdorff, then each X_j is Hausdorff.

6. Let X be the place, and let $\mathcal{B}$ be the collection of all circles centred at the origin, including the origin itself. Show that $\mathcal{B}$ is a base for a topology on X, and find the closure of the square $S = \{(x, y) : -1 \leq x \leq 1, and -1 \leq y \leq 1\}$.

7. Suppose $\mathcal{U}$ is the usual topology on $\mathbb{R}$. Let $\mathcal{I}$ be the collection of all subsets of the irrational numbers. Let $\mathcal{T}$ be the topology generated by $\mathcal{U} \cap \mathcal{I}$. In the topological space $(\mathbb{R}, \mathcal{T})$, find the closure of the interval $(0, \sqrt{2})$.

8. Let X be the place, and let $\mathcal{T}$ be the topology generated by the set of all straight lines through the origin. In the topological space $(X, \mathcal{T})$ find the closure of each of the following sets: $A = \{(0, 0)\}, B = \{(1, 1)\}, C = \{(x, 1) : 0 < x < 1\}$.

9. Let $Z = \Pi\{X_a : a \in A\}$ be the product of finitely many topological spaces. Prove or disprove:

> (a) If each U_a is an open subset of X_a, then the product $\Pi\{U_a : a \in A\}$ is an open subset of Z.
> (b) If each F_a is a closed subset of X_a, then the product $\Pi\{F_a : a \in A\}$ is a closed subset of Z.

10. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

11. When we defined it in class, what we called the product topology is actually the *box topology*. They are identical for finite products, but different for infinite products. Here is the actual product topology:

The product topology on ΠX_α has as a basis all sets of the form ΠU_α, where U_α is open in X_α for each α *and* U_α *equals* X_α *except for finitely many values of* α.

Here is another view of the same topology, using another new concept. Any collection of sets whose union equals the whole space can be a *subbasis*. From a subbasis we get a basis by taking finite intersections of subbasis elements. Then we get open sets by taking unions of basis elements, as always. For the product topology the subbasis is the set of all $\pi_\alpha^{-1}(U_\alpha)$, where U_α is open in X_α.

Why would we want that? Here is a key theorem. Prove it is true in the product topology but false in the box topology:

> Let $f : A \to \Pi X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))$ where $f_\alpha : A \to X_\alpha$ for each α, then the function f is continuous if and only if each function f_α is continuous.

Quotients

1. Let $\sim$ be an equivalence relation on a topological space X. Prove that $X/\sim$ is a T_1-space if and only if each equivalence class is closed. Give an example of a T_1-space X and an equivalence relation $\sim$ such that $X/\sim$ is not a T_1-space.

2. Let B^n be the closed unit ball in $\mathbb{R}^n$. Prove that the quotient space obtained from B^n by identifying its boundary S^{n-1} to a point is homeomorphic to the n-sphere S^n. Prove this is also homeomorphic to the identification of two copies of B^n along their boundaries.

3. For $n \geq 1$, define $P^n = S^n/\sim$, where the equivalence relation is defined by declaring $x \sim y$ if and only if $x = y$ or $x = -y$. In other words, P^n is obtained from S^n by identifying pairs of antipodal points. The space P^n is called *real projective space* of dimension n, and it can be regarded as the set of liens in $\mathbb{R}^{n+1}$ which pass through the origin. Establish the following assertions:

(a) P^n is a Hausdorff space.
(b) The projection $\pi : S^n \to P^n$ is a local homeomorphism, that is each $x \in S^n$ is contained in an open set that is mapped homeomorphically by π onto an open set containing $\pi(x)$.
(c) P^1 is homeomorphic to the circle S^1.
(d) P^n is homeomorphic to the quotient space obtained from the closed unit ball B^n in $\mathbb{R}^n$ by identifying antipodal points of its boundary S^{n-1}.

4. Describe the space $X/\sim$ for the following spaces and equivalence relations:

(1) Find $I/\sim$ for $X = I = [0,1]$ and the equivalence classes defined by

$$[x] = \begin{cases} \{x\} & \text{if } 0 < x < 1 \\ 0 \sim 1 & \text{if } x = 0, 1 \end{cases}$$

(2) X is the unit square $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ with the equivalence classes:

$$[(x, y)] = \begin{cases} \{(x, y)\} & \text{if } x \neq 0, 1 \text{ and } 0 \leq y \leq 1 \\ (0, y) \sim (1, 1 - y) & \text{if } x = 0, 1 \text{ and } 0 \leq y \leq 1 \end{cases}$$

(3) X is the disc $D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$. Let S^1 denote the boundary circle: $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Define equivalence classes of a point (x, y) by

$$[(x, y)] = \begin{cases} \{(x, y)\} & \text{if } (x, y) \notin S^1 \\ (x, y) \sim (1, 0) & \text{if } (x, y) \in S^1 \end{cases}$$

(4) $X = S^1$ and $(x, y) \sim (-x, -y)$ for each $(x, y) \in S^1$.

5. Let X be a topological space with an equivalence relation $\sim$ and let $X/\sim$ be the quotient space. Show that the quotient map $p : X \to X/\sim$ defined by $x \mapsto [x]$ is continuous.

6. Define an equivalence relation on the plane $\mathbb{R}^2$ as follows: $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_1^2 = x_2 + y_2^2$. Let $\mathbb{R}^2*$ be the identification space resulting from the quotient topology. This is homeomorphic to a familiar space; what is it? Repeat the previous question for the relation given by $x_1^2 + y_1^2 = x_2^2 + y_2^2$.

7. §3.8 3 - do not do this technically, but decipher the question enough to figure out why it is intuitively believable. Give me a proof by pictures. (Hint: X/Y is the space obtained by starting with X and identifying all of Y to a point, that is gluing all of Y together into one point.)

8. (a) Let $p : X \to Y$ be a continuous map. Show that if there is a conitnuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

(b) If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. show that a retraction is a quotient map.

Orientability / Bundles

1. Cut a Möbius strip in third lengthwise. What do you get? What if it is cut in fourths or fifths? Justify a general rule.
2. Think up fun things to do in a nonorientable universe. Write them in a story.
3. Is a projective plane orientable?
4. Draw some pictures of square bundles over circles.
5. Describe two *different* circle bundles over circles.
6. A solid doughnut is topologically a disk cross a circle ($D^2 \times S^1$). Describe a disk bundle over a circle which is not a product.
7. Describe two ways in which $S^2 \times I$ can be glued up to make an S^2 bundle over S^1.

Connectedness

1. Prove in detail that connectedness is a topological property.
2. Prove that every connected subset of $\mathbb{R}$ is an interval.
3. A point p of a topological space X is a cut point if $X \backslash p$ is disconnected. Show that the property of having a cut point is a topological property. Use this to prove that no two of the intervals [0, 1], (0,1), and [0, 1) are homeomorphic.
4. Prove that each connected component of a topological space is closed. Show by counterexample that a connected component of a topological space is not necessarily open.
5. Suppose $X = X_1 \times \cdots \times X_n$, where each X_j is nonempty. Prove that if X is connected, then each X_j is connected.
6. Let $X/\sim$ be the quotient space determined by an equivalence relation $\sim$ on a topological space X. Prove that if X is connected, then $X/\sim$ is connected.
7. Give examples of sets A and B in $\mathbb{R}^2$ which satisfy:

   (a) A and B are connected, but $A \cap B$ is not connected.
   (b) A and B are connected, but $A \backslash B$ is not connected.
   (c) Neither A nor B are connected, but $A \cup B$ is connected.

8. Show that if X is a non-empty topological space with the discrete topology, then the only connected sets are the singletons. If X is has the indiscrete topology, show that any subset of X is connected.
9. Show that $\mathbb{N}$ is connected in the finite complement topology.
10. State whether each of the following sets is connected; if not connected, find a separation.

    (a) A circle with one point deleted; with two points deleted.
    (b) An arc of a circle; an arc with its midpoint deleted.
    (c) A finite set of points (in the standard topology of $\mathbb{R}$); the singleton set consisting of a single point; the empty set.

11. Show by an example that the inverse image of a connected set is not necessarily connected.
12. Give an example of two connected sets whose intersection is not connected.

13. Explain whether the points in the plane having at least one rational coordinate form a connected set; those having exactly one rational coordinate; those having two rational coordinates. If not connected, show a separation.

14. We have proven if $f : [0,1] \to [0,1]$ is continuous, then there is a fixed point $f(x) = x$. Is this still true for $f : (0,1] \to f : (0,1]$? Discuss and explain.

15. Consider $\mathbb{R}$ with the topology $\mathcal{T} = \{\mathbb{R}, \emptyset, [0,1]\}$. Is $(\mathbb{R}, \mathcal{T})$ connected? Explain.

16. Prove or disprove: A function $f : [a,b] \to \mathbb{R}$ with a connected graph is continuous.

17. Consider $\mathbb{R}$ with the topology $\mathcal{T} = \{U \subset \mathbb{R} : 0 \in U\} \cup \{\emptyset\}$. Is the space $(X, \mathcal{T})$ connected? How about the subspace $X \backslash \{\emptyset\}$? Explain.

18. Let $S = \{1/n : n \in \mathbb{Z}_+\} \cap \{0\}$. Which components of S are open? Explain.

19. Give an example of a space having no open components, or prove there is no such space.

20. Are $\mathbb{R}$ and $\mathbb{R}^2$ homeomorphic?

Path-connectedness

1. Prove that any interval of $\mathbb{R}$ is path-connected.

2. Prove that path connectedness is a topological property.

3. Prove that if X is path-connected and $f : X \to Y$ is a continuous function, then $f(X)$ is path-connected.

4. Suppose $X = X_1 \times \cdots \times X_n$, where each X_j is nonempty. Prove that if X is path-connected, then each X_j is path-connected.

5. Let $X/\sim$ be the quotient space determined by an equivalence relation $\sim$ on a topological space X. Prove that if X is path-connected, then $X/\sim$ is path-connected.

6. Prove: The cartesian product of finitely many path-connected spaces is path-connected.

7. Prove or disprove: If X is a path-connected space and $f : X \to Y$ is a continuous function from X onto a topological space Y, then Y is path-connected.

8. Prove or disprove: If S is a path-connected subset of a space X, and $S \subset K \subset \overline{S}$, then K is path-connected.

9. Prove or disprove: If $\mathcal{C}$ is a collection of path-connected subsets of a space, and if there is a C^* in $\mathcal{C}$ that meets each C in $\mathcal{C}$, then $\cap\mathcal{C}$ is path-connected.

Fundamental group

1. Suppose that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for any three paths in X for which the product is defined. Show that each path component of X consists of a single point.

2. Let X be path-connected and let $b \in X$. Show that every path in X is homotopic with endpoints fixed to a path passing through b.

3. Prove that if there are simply connected open subsets U and V of X such that $U \cup V = X$ and $U \cap V$ is nonempty and path-connected, then X is simply connected.

4. Prove that the product of simply connected spaces is simply connected.

5. Prove that if $n \geq 3$, then $\mathbb{R}^n/\{0\}$ is simply connected.

6. Let X be the *comb space*, that is, the subset of $\mathbb{R}^2$ consisting of the horizontal interval $\{(x,0) : 0 \leq x \leq 1\}$ and the closed vertical intervals of unit length with lower endpoints at $(0,0)$ and at $(0, \frac{1}{n})$, $1 \leq n < \infty$. Show that X is contractible to $(0,0)$ with $(0,0)$ held fixed. Show that X is not contractible to $(0,1)$ with $(0,1)$ held fixed.

7. Let (X, x_0) and (Y, y_0) be pointed spaces. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the direct product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

8. Classify up to homotopy type: ABCDEFGHIJKLMNOPQRSTUVWXYZ1234567890. Also classify up to homeomorphism.

9. Choose a point $x \in T^2$. Show that the punctured torus $T^2 \setminus \{x\}$ deformation retracts to the figure-eight space and use this fact to compute $\pi_1(T^2 \setminus \{x\})$.

10. Prove that homotopy of paths is reflexive and symmetric.

11. Prove that concatenation of homotopy classes of paths is associative. Do so differently from the book. At least upside down.

12. Let $p, q \in X$ and $\gamma : [0,1] \to X$ a path from p to q.

a. For a loop α in X based at p, show that $\gamma^{-1}\alpha\gamma$ is a loop based at q.
b. Show that the map $[\alpha] \to [\gamma^{-1}\alpha\gamma]$ is a group isomorphism from $\pi_1(X,p)$ to $\pi_1(X,q)$.

13. If a space X is simply connected and α and β are any two paths from x_0 to x_1 in X, show that α and β are path homotopic. [Hint: Look at $[\alpha\beta^{-1}] \star [\beta]$.]

14. Let A be a convex subset of $\mathbb{R}^n$; that is, assume that for any two points in A the line segment joining those points is also in A. Show that A is simply connected and, in particular, then $\mathbb{R}^n$ is simply connected.

15. Consider the following loops in S^1. For each, find the endpoint of a path $\tilde{\alpha}$ in $\mathbb{R}^1$ that begins at 0 and gets mapped to the given loop by $p : \mathbb{R} \to S^1, p(x) = (\cos 2\pi x, \sin 2\pi x)$.

a. $\alpha(t) = \cos 2\pi nt, \sin 2\pi nt)$ for $0 \le t \le 1$, n a positive integer.
b. $\alpha(t) = \cos 2\pi nt, \sin 2\pi nt)$ for $0 \le t \le 1$, n a negative integer.
c. $\alpha(t) = \begin{cases} (\cos 4\pi t, \sin 4\pi t) & \text{for } 0 \le t \le \frac{1}{2} \\ (\cos 4\pi t, -\sin 4\pi t) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$

(This is part of proving $\pi_1(S^1) = \mathbb{Z}$)

16. Let $\alpha(t) = \cos 2\pi nt, \sin 2\pi nt)$ and $\beta(t) = (\cos 4\pi t, \sin 4\pi t)$, $0 \le t \le 1$, be loops in S^1. Let $\tilde{\alpha}$ be the path for 15a for $n = 1$. Let $\tilde{\tilde{\beta}}$ be a path in $\mathbb{R}^1$ that begins at the point where $\tilde{\alpha}$ ends and that gets mapped to β by p. What is the endpoint of $\tilde{\tilde{\beta}}$? If you were to lift the product path $\alpha\beta$ to a path in $\mathbb{R}^1$ beginning at 0 and getting mapped by p to $\alpha\beta$, how does its endpoint compare with that of $\tilde{\tilde{\beta}}$? (This is part of proving $\pi_1(S^1) = \mathbb{Z}$)

17. a. Show that the circle is a deformation retract of the annulus *and* of the Möbius band. (Pictures suffice for this part.) Notice that therefore fundamental group doesn't detect orientability
b. For the annulus $\{(x,y) | 1 \le x^2 + y^2 \le 4\}$, write out an explicit formula for a homotopy H taking the annulus to the unit circle.

18. For each of the following spaces, which has a deformation retract of (i) a point, (ii) a circle, (iii) a figure eight, or (iv) none of these?

a. $\mathbb{R}^3$ minus the nonnegative x, y, and z axes
b. $\mathbb{R}^2$ minus the positive x axis
c. $S^1 \cup \{(x,0) | -1 < x < 1\}$, wher S^1 is the unit circle in the plane
d. $\mathbb{R}^3$
e. S^2 minus two points
f. $\mathbb{R}^2$ minus three points
g. S^2 minus three points
h. T^2 minus a nonseparating simple closed curve

19. Suppose S^1 is the unit circle $\{(x, y, 0)|x^2 + y^2 = 1\}$ in $\mathbb{R}^{\nvDash}$, M is the z axis, and N is the vertical line given by $x = 2$ and $y = 1$.

a. Find $\pi_1(\mathbb{R}^3\ M)$.
b. It is intuitively clear that $S^1 \cup M$ and $S^1 \cup N$ are different as subsets of $\mathbb{R}^3$. Make this as precise as you can by showing that there does *not* exist a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(S^1 \cup M) = S^1 \cup N)$.

20. Use deformation retracts to give a proof of the Brouwer fixed-point theorem for the disk D^2: If f is a continuous map of the closed unit disk to itself, then $f(c) = c$ for some point c in the disk. Here's a start. Suppose f is a conitnuous map from D^2 to D^2 so that $f(x) \neq x$ for any x. Take the ray from $f(x)$ to x and define $r(x)$ to be the point where this ray hits the boundary of D^2.

21. Use 3. to show that S^2 is simply connected. Let $S^n = \{(x_1, \dots x_n) \in \mathbb{R}^{n+1}|x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$. Show that S^n is simply connected whenever $n > 1$. Use this to show that $\mathbb{R}^n\ \{0\}$ is simply connected for $n > 2$.

Compactness

1. Prove in detail that compactness is a topological property.
2. Show that any space with the finite complement topology is compact.
3. Show that a discrete topological space is compact if and only if it is finite.
4. Show that a continuous real-valued function on a compact space attains its maximum value and its minimum value. In particular, show that a continuous real-valued function on a compact space is bounded.
5. Suppose $X = X_1 \times \dots \times X_n$, where each X_j is nonempty. Prove that if X is compact, then each X_j is compact.
6. Let $X/\sim$ be the quotient space determined by an equivalence relation $\sim$ on a topological space X. Prove that if X is compact, then $X/\sim$ is compact.
7. Let E be a closed subset of a compact Hausdorff space X. Prove that the quotient space obtained from X by identifying E to a point is homeomorphic to the one-point compactification of $X \backslash E$. (See definition of one-point compactification on p. 172)
8. Show that any space X with the indiscrete topology is compact.
9. Show that the union of two compact sets is compact. Similarly, show the union of any finite number of compact sets is compact. Give an example of an infinite collection of compact sets whose union is not copmact.
10. Let X be the closed interval $[0, 10] \subset \mathbb{R}$. Show that the set C of all open intervals of $\mathbb{R}$ of length 1 is a covering of X. Find a finite subcollection of C covering X. What is the least number of such intervals in a covering of X?
11. Find an expanding sequence of open subsets $U_1, U_2, \cdots, U_k, \cdots$ of the half-open interval $X = [0, 1)$ whose union is X but no one of them is all of X.
12. A topological space X is said to be locally compact if each point x in X has a compact set N containing an open set U containing x for each x [thus $x \in$ open $U \subset$ compact N]. Prove that the real line and $\mathbb{R}^n$ are locally compact. (This is restated §5.4 2 FYI, a neighbourhood is a set containing an open set containing the point.)
13. Let X be a locally compact Hausdorff space. Take some object outside X, call it ∞. Consider $Y = X \cup \{\infty\}$. Create a topology on Y by defining the collection of open sets in Y to be all sets of the following types:

(1) U, where U is an open subset of X,
(2) $Y\backslash C$, where C is a compact subset of X.

Y is called the one-point compactification of X. Prove that this in fact defines a topology on Y. Prove that Y is compact. Prove that the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1. (This is a restatement of §5.4 3)

14. Prove or disprove:

(a) The intersection of a collection of compact subset of a space is compact.
(b) The intersection of a collection of compact subsets of a Hausdorff space is compact.

15. Suppose $f : X \to Y$ is a continuous function from one compact Hausdorff space onto another. Suppose further that for each $y \in Y$, $f^{-1}(y)$ is connected. Prove that if K is compact and connected, then $f^{-1}(K)$ is compact and connected.

16. Suppose $(X, \mathcal{T})$ is a compact Hausdorff space. Prove that if $\mathcal{S}$ is any Hausdorff topology for X such that $\mathcal{S} \subset \mathcal{T}$, then $\mathcal{S} = \mathcal{T}$.

Surfaces

1. Figure out intrinsic ways to distinguish $\mathbb{R}^2$, S^2, T^2, and the Möbius band.
2. Prove that P^2 is a sphere S^2 with a disc removed and a Möbius band glued in its place.
3. Prove that the connected sum of two surfaces is a surface, and that for any surface F, $F\#S^2 = F$.
4. What surface does one obtain from a Möbius band if one shrinks the boundary circle to a point.
5. Compute the Euler characteristics for the torus, projective plane, Klein bottle, cylinder, and Möbius band.
6. Prove $\chi(F_1\#F_2) = \chi(F_1) + \chi(F_2) - 2$ for any surfaces F_1 and F_2. Calculate the Euler characteristic of the n-handled torus and the connected sum of n projective planes.
7. What is the largest number of pairwise disjoint simple closed curves in nT^2 such tat cutting nT^2 along these curves gives a single path-connected piece?
8. §5.7 4 (finish 4 by showing that $K = P\#P$)
9. Identify the surface represented by this code: $CBB^{-1}ADA^{-1}C^{-1}D^{-1}$. Identify the surface represented by this code: $EECBB^{-1}ADA^{-1}C^{-1}D^{-1}$ (ahh, that changes everything!)
10. List all surfaces with non-negative Euler characteristic.
11. Identify the surface represented by the code: $A_1A_2 \cdots A_nA_1^{-1}A_2^{-1} \cdots A_n^{-1}$. (Distinguish between n even and n odd).
12. Identify the surface represented by the code: $A_1A_2 \cdots A_nA_1A_2 \cdots A_n$. (Distinguish between n even and n odd).