371 Problem Set 2 Comments

1. It says to interpret the formula using Figure 2.1. This is not optional. And when doing so, your vectors need to make sense. In particular, on the second part your vector sum needs to look possible. This does imply there best be pictures. Furthermore, a picture alone is not sufficient for geometric interpretation unless it is extensively labeled. There should be some geometric reasoning also.
2. Since $|z|=|\bar{z}|$, the bound for $S^{*}$ is the same as the bound for $S$. This can also be seen visually, that reflecting keeps within the circle of boundedness. Finally remember in $\mathbb{C}$ boundedness is always centred at the origin.
3. Interpret geometrically again is not optional - the best interpretation is that the sum of squares of the diagonals equals the sum of squares of the four sides. Also this is better done with conjugates than reducing to rectangular coordinates.
4. This was done in class.
5. Probably the intent here was to use rectangular coordinates and reduce to a standard form of the equation of a line. I find it interesting to also start with the $\operatorname{Re}(z)=\frac{c}{2}$ line and then compose that set with the effects of multiplying by $a$ (which is premultiplied, so the effect is to scale distance from the origin by the reciprocal of the magnitude, and to rotate the negative of the angle of $a$ ). In either case, each of these equations represents a line and any line can be expressed in this form. We will see more about lines in Chapter 3.
6. Through either rectangular or conjugate means, this can be reduced to an equation of a circle, in particular the circle is given by the equation $|z-(-\bar{a})|=\sqrt{|a|^{2}-c}$.
7. It says to use the imaginary part of something. The simple key here is that the imaginary part of a real number is $0 . a$ is parallel to $b$ if $a=r b$ for a real $r$. Solving for $r=\frac{a}{b}$ and since it is real $\operatorname{Im}\left(\frac{a}{b}\right)=0$. Since $\frac{a}{b}=\frac{a \bar{b}}{|b|^{2}}$ and $|b|^{2}$ is real, this is equivalent to saying $\operatorname{Im}(a \bar{b})=0$.
8. Note that multiplication by $i$ rotates by a right angle. But, $a$ and $b$ might be different lengths. Including that $a$ is parallel to $b$ if $a=r b i$ for a real $r$. Solving for $r i=\frac{a}{b}$ and since this is purely imaginary $\operatorname{Re}\left(\frac{a}{b}\right)=0$ or $\operatorname{Re}(a \bar{b})=0$. Notice the last versions of both of these imply that 0 is perpendicular and parallel to all vectors, which seems reasonable.
9. Presented in class.
10. This is tedious, but not terribly interesting. It's less work if you use conjugates than if you descend down to rectangular. In the end it is merely checking the properties from the definition of a metric.
10.5. Both + or both - works fine, but one of each does not check with the original square roots. This is rather direct algebra.
10.6. No one did, sadly, but it does work out.
11. No credit offered.
12. Letting $z=w-\frac{A}{3}$ eliminates the quadratic term.
13. Presented in class.
14. Sadly, I think the bottom line is that this works well in theory, but produces not-very-helpful answers in practice. Because of the method, not the execution.
15.     - 29. We skipped sections 3,4 and 5 . We'll come back to them after the exam before we do chapter 4. If you have worked on any of these problems, please keep them for the Chapter 4 problem set.
1. Don't ignore the conjugation question. This is tricky. It's in the limit. What limit? The limit that is hiding as part of the definition of the infinite series of $e^{z}$. Make sure to include the steps for both the sum and the product as well.
2. This is just expanding the definitions in terms of $e^{i z}$. Nothing sophisticated.
3. Use complex somehow expand from Euler's Theorem. Do not give a HS trigonometry proof.
4. Use the exponential definitions. The answers are $\cos (z+w)+\cos (z-w)=$ $2 \cos (z) \cos (w)$ and $\cos (z+w)-\cos (z-w)=-2 \sin (z) \sin (w)$.
5. Again use complex somehow. These are the familiar formulas used for trig. integration in Calc. II. For reference, $\cos ^{2}(z)=\frac{1+\cos (2 z)}{2}$ and $\sin ^{2}(z)=\frac{1-\cos (2 z)}{2}$.
6. presented in class
7. Merely check that the magnitude of this point is 1 . Then use either tangent or sine or cosine to find the angle. Substitute the angle in to $e^{i \theta}$. Don't skip this step. Not very interesting.
8. The work of manipulating formulas is somewhat dull, going between the hyperbolic and circular trigonometric definitions in terms of exponentials, but the results are neat: $\cosh (z)=\cos (z i)$ and $\sinh (z)=-i \sin (z i)$.
9. This is just applying formulas. The cube roots of unity are 1 , and $-\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$. The eighth roots of unity are $\pm 1, \pm i, \pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}}$.
10. We've gotten rather dull in these questions. Because rotating by $2 \pi$ doesn't change the complex number, rotating by $\pi$ gives another square root. We have established long ago in Lemma 2.1 that there are at most two square roots. The important detail here is that for nonzero complex numbers two are distinct.
11. Presented in class.
12. Even more dull, $-\sqrt{3}+i=2 e^{i 5 \pi / 6}$.
13. Discussed in class
14. Because $\omega^{m}=1, \omega$ is on the unit circle. So, multiplying is just rotating it. Because $\omega^{m}=1, \omega^{m-1}$ is rotated backwards once from 1. In other words it is rotated in the opposite direction. This can be seen to be the same as reflecting $w$ over the real axis, or taking the conjugate. Therefore $\omega^{m-1}=\bar{\omega}=a-i b$. No computation done! Oh, and for those who computed, you needed to use that $\omega$ was on the unit circle twice to get at the end that $a^{2}+b^{2}=1$ and find this answer.
15. Because of the cubing function dividing angles by three, this produces all complex numbers with argument between 0 and $\pi / 3$, not including end points, or $z=0$. The restriction that you end in the first quadrant rules out other values, e.g. $e^{i 5 \pi / 6}$ which cube to have positive imaginary values.
16. Direct from polar. $z=|z| e^{i \theta}$, so $z^{m}=|z|^{m} e^{i m \theta}$. So what happens? The magnitude is raised to the $m$ th power, and the angle is multiplied by $m$. It is a common mistake to say that $z$ is rotated by an angle of $m \theta$. This is not true. If it were it would end at $\theta+m \theta$. Be careful with your words.
17. One correct solution. Think of factoring the polynomial and consider $n$th roots of 1 .
18. Answers: $\log (e)=1+i 2 n \pi, \log (i)=i \pi(1 / 2+2 n)$, and $\log (-1)=i \pi(1+2 n)$. Remember logarithms always have infinitely many values.
19. Clearly this is solved by logarithms. Taking $\log$ of both sides we get $2 z=i \pi(1+2 n)$, so $z=i \pi / 2(1+2 n)$ Very dull.
20. Makes 48 more interesting. The first answer is the same as 48: $z=i \pi / 2(1+2 n)$ and the second is merely $z=2 n i \pi$.
21. Standard uniqueness proof. Suppose you have two of them equal. Reach a contradiction. Don't use logarithms, but instead think of their position on a circle.
22. Didn't we do these already? Maybe better as answers just $e^{i \pi / 4}, e^{3 i \pi / 4}, e^{5 i \pi / 4}, e^{7 i \pi / 4}$.
23. One unsuccessful attempt - not really complex related anyway.
24. These kinda things are getting old. Multiple valued functions have multiple values if you just try picking one it'll not work out. Let's see here:

$$
2 n i \pi=\log (1)=\log ((-i) i)=\log (-i)+\log (i)=\frac{3 \pi i}{2}+2 a i \pi+\frac{\pi i}{2}+2 b i \pi=2(a+b+1) i \pi
$$

This is the same set of values. The point that the problem is with multiple values needed to me made in your answer. In fact, restricting the range does not solve this problem. This is the problem that arises from that restricting.
54. Presented in class
55. The answer lies in multiple values. They are equal if you think of them as sets.
56. Discussed in class.
57. It's wordy, but mostly just do what it says. There was one tripping point. To show a function equals a particular constant using derivatives, e.g. $f(x)=4$, you need to show two things. You need to show 1. that $f^{\prime}(x)=0$. Almost everyone did this. This shows that $f(x)=$ a constant, but it doesn't tell you which particular constant. Then you also need to show for some particular $a, f(a)=4$. This part was missing. Everyone who attempted this problem proved that $c^{2}+s^{2}=$ a constant, but no one proved that it equaled 1. Alas.
58. Not attempted. One integral is by converting $\cos (b x)$ into exponentials. The other is Calc II.
59. not attempted

