371 Problem Set 5 Comments

1. It's important to understand why $\lim _{\zeta \rightarrow 0} \frac{\bar{\zeta}}{\zeta}$ does not exist. Aside from that, although other ways are definitely possible, it is acceptable (and we proved as much) to compute $\frac{\partial}{\partial \bar{z}}$ just like you would compute $\frac{\partial}{\partial x}$.
2. Discussed in class.
3. Start with $\Phi=A|z|^{2}+\beta z+\overline{\beta z}+C$. Compute $\frac{\partial}{\partial \bar{z}}$. Force it to always be zero to conclude that $A=\beta=0$, and thus $\Phi=C$.
4. Presentation.
5. I would start with the left-hand-side, but there were many successful solutions.
6. Compute step by step. One technical detail - although the rules are the same, you're not multiplying differential operators. Be careful about that detail. The answer, on p. 91, is $4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}$.
7. 

$$
\bar{\partial} \omega=\frac{\partial \omega}{\partial \bar{z}} d \bar{z}=\frac{\partial(A d z+B d \bar{z})}{\partial \bar{z}} d \bar{z}=\frac{\partial A}{\partial \bar{z}} d \bar{z} \wedge d z+\frac{\partial B}{\partial \bar{z}} d \bar{z} \wedge d \bar{z}=\frac{\partial A}{\partial \bar{z}} d \bar{z} \wedge d z
$$

since $d \bar{z} \wedge d \bar{z}=0$. There are many different ways to do this. This way is pretty short, so worth looking at.
8. Presentation.
9. $F(z)=z^{3}$.
10. $F(x, y)=e^{x}(\cos y+i \sin y)=e^{x} e^{i y}=e^{x+i y}=e^{z}$.
11. $F(r, \theta)=(\log (r), \theta)=\log (z)$
12. $\frac{\partial g}{\partial \bar{z}}=\frac{2 \bar{z}}{z}=0$ at 0 . But the limit has more occurences of $\frac{\bar{\zeta}}{\zeta}$ so does not exist.
13. This is a real analysis question, but a cute one. If it did satisfy this, the limit definition of derivative would produce 0 always, so it would be a constant. OTOH, letting $z=0$ produces $f(h)=h^{2}$, which is not a constant.
14. This is either real analysis or calc. I; either way, not our main business. I was very happy no one did it.
15. Presentation.
16. Make sure the level curves are indicated sufficiently to indicate how they cover the entire plane. Since the function is defined for all complex numbers, the each must go somewhere.
17. Same as above. In particular, be sure to include the left half of the plane. $z=-1$ does go somewhere.
18. similar - two important points - only first quadrant is specified. The level sets of the logarithm function are real part concentric circles around the origin, and imaginary part, radial rays from the origin. The origin itself is not included, neither are the real or imaginary axes.
18.5 no one attempted to fix d'Angelo's problems with 5.1
19. After all d'Angelo's problems with Theorem 5.1, I'm not surprised no one attempted this.
20. Presentation.
21. Answers: yes, the sum of harmonics is harmonic; for the product to be harmonic, the gradients of the two functions need to be perpendicular; the $x$-partial of a harmonic is always harmonic, if it is sufficiently differentiable.
22. Using $|f|^{2}=f \bar{f}$ works well, as does the expression for $\operatorname{Re}(f)$. Application of the indicated derivatives is key. But it hinges on what $\frac{\partial \bar{f}}{\partial z}$ is. I think if it is the conjugate of $\frac{\partial f}{\partial \bar{z}}$ then this might be not that bad. You might benefit to think of this question under that assumption.
23. Answers appear elsewhere.
24. The standard method is differentiation/integration as discussed in the preceding paragraph about "many texts". $f(z)=z^{2}$.
25. $f(z)=z^{3}$
26. $f(z)=e^{z}$
27. $f(z)=\log \left(z^{2}\right)=2 \log (z)$. New method needs revision, not abandonment. Here letting $\bar{z}=1$ works well.
28. $f(z)=\frac{i}{z}$. Again, new method needs revision. Here letting $\bar{z}=i$ works well.
29. Please don't forget this is presentation in chapter 6.
30. Suppose it did achieve its maximum, then it would be higher than the points around it, but the mean value property says it equals the average of the points around it. That's the entire idea.
31. The Hessian matrix is $\left(\begin{array}{cc}\frac{\partial^{2} u}{\partial x^{2}} & \frac{\partial^{2} u}{\partial x \partial y} \\ \frac{\partial^{2} u}{\partial x \partial y} & \frac{\partial^{2} u}{\partial y^{2}}\end{array}\right)$. The determinant is therefore $\frac{\partial^{2} u}{\partial x^{2}} \frac{\partial^{2} u}{\partial y^{2}}-$ $\left(\frac{\partial^{2} u}{\partial x \partial y}\right)^{2}$. Using $\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}=0$ the determinant equals $-\left(\frac{\partial^{2} u}{\partial x^{2}}\right)^{2}-\left(\frac{\partial^{2} u}{\partial x \partial y}\right)^{2}$ which is nonpositive.

32-3. Unattempted.
34. Unattempted. I might put this on the final as an extra credit question, just to see what happens. I might.

