Lab Activity Manual

MATH 112-113: Math for Elementary Teachers

Walla Walla University

SEVENTH-DAY
ADVENTIST
HIGHER
EDUCATION

Dr. Jonathan Duncan

2014-2015

Contents

Chapter 1

Logic and Foundations

*Logic is the hygiene the mathematician practices to keep his ideas healthy
and strong.*

– Hermann Klaus Hugo Weyl

*Contradiction is not a sign of falsity, nor the lack of contradiction a sign
of truth.*

– Blaise Pascal

Contents

1.1 What Makes a Good Math Teacher?

In this activity we will examine several aspects of good mathematics teaching. After discussing attitudes and reflecting on previous experiences with math teachers, students' will work as a group to create a job description for their ideal math teacher.

Objectives:

- To understand how one's own attitude towards mathematics can affect a classroom.
- To determine the characteristics and behaviors that typify effective math teachers.

I. **Class Discussion**

What is your attitude towards math and how did it develop? In what ways did your teachers' attitudes influence your own? Participate in the class discussion of this question

II. **Individual Reflection**

Now take a few minutes to think about your math teachers from elementary or high school. Try to recall at least one particularly good teacher and at least one teacher who influenced you negatively. Reflect on what these teachers said or did and how it affected you.

III. **Group Brainstorming**

When all members of your group have had adequate time for reflection, share your stories with your peers. Then, work together as a group to compile a list of qualities, skills, and behaviors of both good and poor math teachers.

(a) Good Math Teachers:

(b) Poor Math Teachers:

IV. **Group Project: Create a Mathematics Teacher Job Description**

Reference the lists you made above and class discussions to create a one-page type-written job description for the ideal elementary school mathematics teacher. Include the most desirable qualities in a mathematics teacher and exclude those qualities which are least desirable. Do, however, be realistic.

Adapted from Frank Lester, Jr. *Mathematics for Elementary Teachers via Problem Solving*, 5.

1.2 Patterns in Pascal's Triangle

The triangle of numbers that we know today as Pascal's Triangle was well known in the ancient world. It can be found in Indian mathematics as early as the 2nd century BC, in Persian writings from the 10th century AD, and in 11th century Chinese manuscripts. However Blaise Pascal's 15th century treatise was one of the most complete study of the patters and properties of this interesting array of numbers.

Objectives:

- To introduce students to the prevasiveness of patterns.
- To identify patterns in large sets of numbers.

I. **Pascal's Triangle**

We start this activity by filling in the numbers for the first thirteen rows of Pascal's triangle. Begin and end each row with a one. Any other entries in the triangle should be filled in with the sum of the two entries directly above it. The first few numbers are done for you.
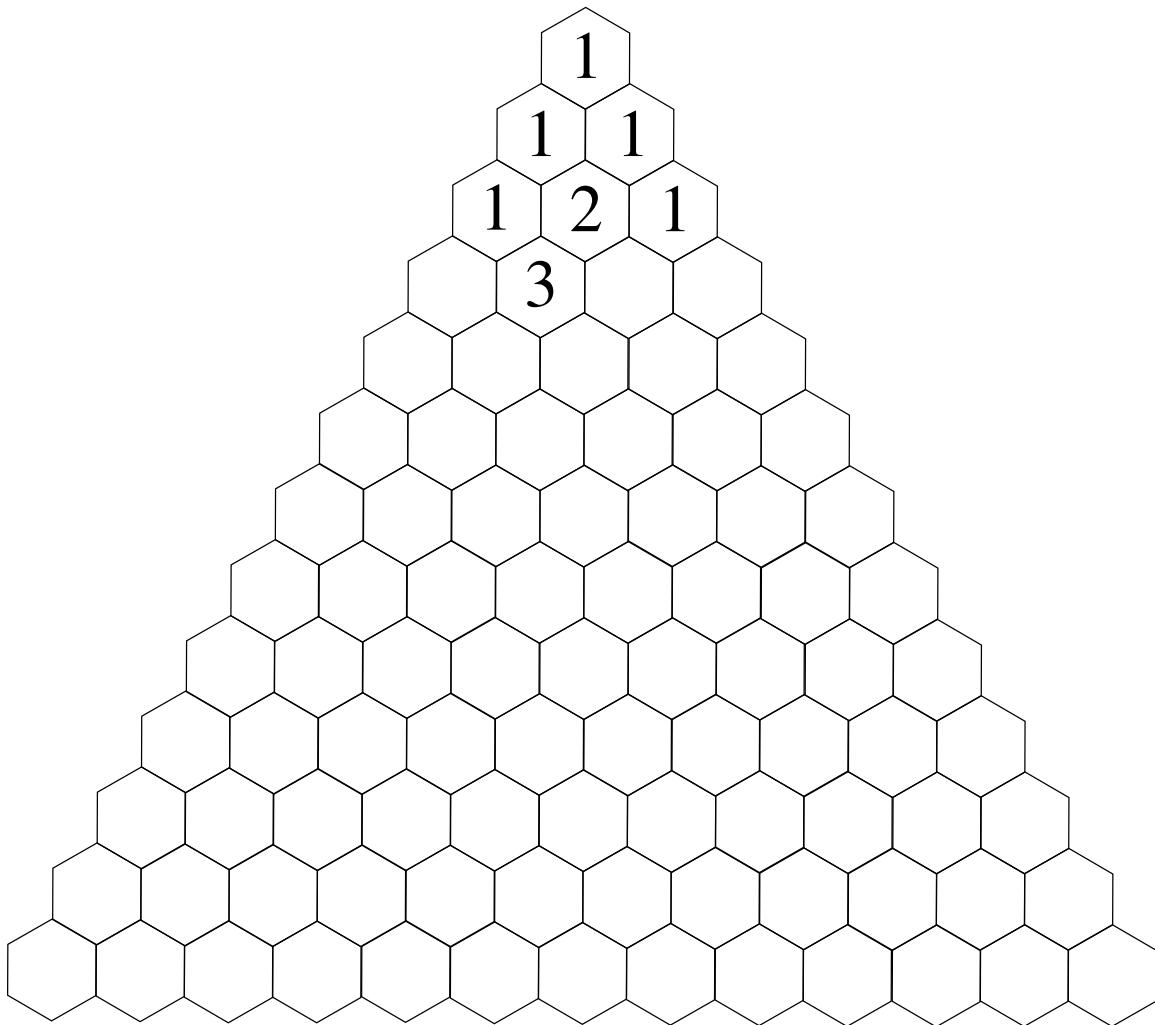
II. **Finding Patterns**

Locate each of the following sets of numbers in Pascal's triangle. Describe the pattern in which these numbers appear. Some patters may involve adding several numbers together, such as the square numbers 1, 4, 9, 16, ... which appear in the third diagonal as 1, 1+3, 3+6, 6+10, ...

(a) Counting Numbers: 1, 2, 3, 4, ...

(b) Triangular Numbers: 1, 3, 6, 10, ...

(c) Fibonacci Numbers: 1, 1, 2, 3, 5, 8, ...

(d) Powers of 2: 2, 4, 8, 16, ...

III. **Visual Patterns**

The arrangement of Pascal's triangles leads to some interesting visual patterns. Use a highlighter or pencil to shade in the cells which would contain the indicated types of numbers. Below these triangles, write a sentence describing the patters you see.
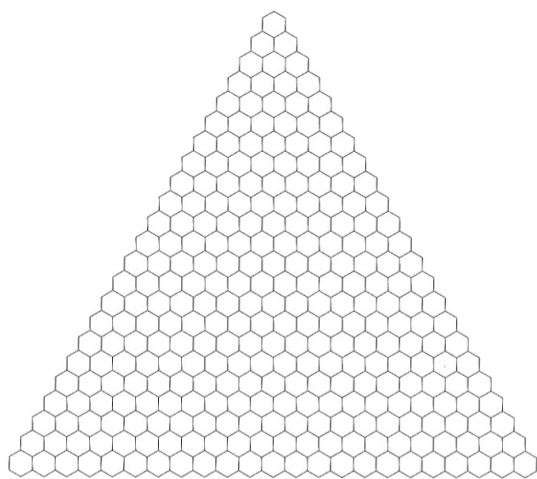
Multiples of 2   Multiples of 3

Adapted from Bassarear *Mathematics for Elementary Teachers Explorations*, 2.
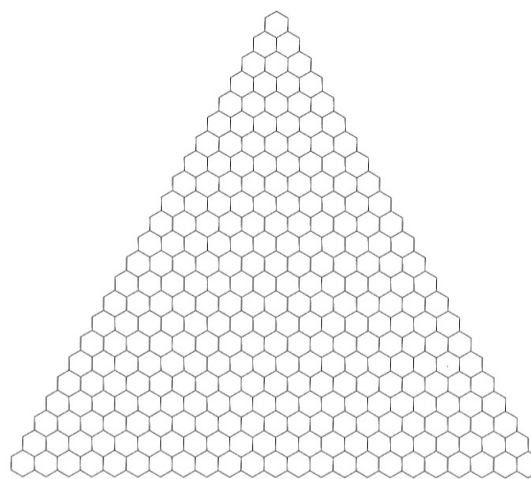
1.3 Stacking Cereal Boxes

Inductive reasoning is a useful problem solving tool. In inductive reasoning, you make generalizations and solve complicated problems by looking at smaller examples of the same problem. This activity asks you to make use of inductive reasoning as well as several other problem solving strategies.

Objectives:

- To use various problem solving strategies possibly including: inductive reasoning, using manipulatives, drawing a picture, making a table, and looking for a pattern.
- To solve a real-world problem without direct guidance.

I. **Stacking A Given Number of Boxes**

A store clerk was told that she had to stack cereal boxes in a display window. She must create a complete triangle as shown below out of the given number of boxes. Use the provided tiles to help you visualize these problems. Record your answers below, drawing pictures when helpful.

(a) How many boxes should the clerk place on the bottom row in order to stack 45 cereal boxes?
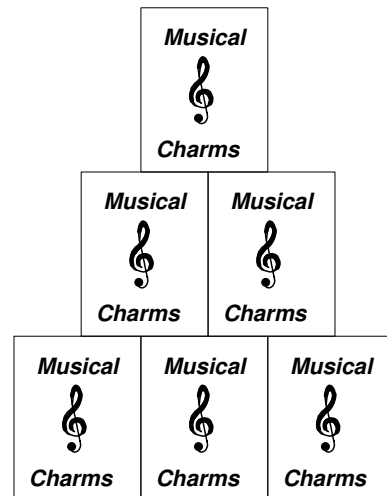
(b) Suppose that the clerk needs to stack 210 boxes. How many should be put on the bottom row?

(c) What if the clerk is given 301 boxes?

II. **Starting with A Certain Number of Boxes**

Suppose that instead of being given a total number of boxes, the clerk was instead told how many boxes to put on the bottom row. How would she decide how many boxes she would need for the triangle?

(a) What if the clerk used 30 boxes in the bottom row?

(b) How many boxes are needed if the bottom row is to have n boxes? Your answer should be a formula in terms of n.

1.4   Scoring Darts

How do we prove something mathematical? Certainly finding and analyzing patterns is an important part of this process, but how do we convince others that our solution to a problem is in fact correct? In this activity you will work with these concepts in the context of scoring a dart game.

Objectives:

- To understand the concept of a mathematical proof.
- To see a model of the reasoning process used in proofs.

I. **Finding and Testing Patterns**

You have four darts and a dart board as shown below. Your score for a game is calculated by adding the values of the regions you hit with each of your four darts. Answer the following questions, making sure to explain your reasoning.

(a) Which of the following game scores are possible?

$$6 \qquad 10 \qquad 13 \qquad 15 \qquad 20 \qquad 28$$
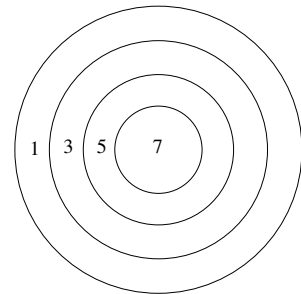
(b) Predict the kinds of scores that are possible and not possible. Justify your prediction.

(c) Systematically list all of the possible scoring outcomes (e.g. 3, 5, 5, 1). Do the total scores match your prediction?

II. **Proving an Assertion**

Many students make use of the following assertion when answering part (b) of the previous section. Prove that this assertion is true.

The sum of two odd numbers is always an even number.

Hint: You may use algebra or geometry to prove this assertion. Think about how to represent odd and even numbers using one of these methods.

1.5   A Logic Puzzle

Many people buy books full of logic puzzles and solve them for fun. In case you are not one of them, your instructor will work through an example logic puzzle with the class to help you become familiar with how to solve these puzzles. As for why we solve them, consider the objectives below.

Objectives:

- To develop a logical representation of given information.
- To use the process of elimination and deductive reasoning.

I. **A Sample Puzzle**

Janet Davis, Sally Adams, Collete Eaton, and Jeff Clark have the following occupations: architect, carpenter, diver, and engineer. Find the occupation of each using the following clues.

- The first letter of each person's last name and occupation are different.
- Jeff and the engineer are going sailing together.
- Janet lives in the same neighborhood as the carpenter and the engineer.

II. **Your Group Problem**

A game warden in charge of a bear population in a certain park is troubled. One of a family of 4 bears is stealing potato salad from picnickers. Based on the warden's observations, can you find the name and color of the culprit?

- The bears are named Wilbur, Bob, Sally, and Jane. They are brown, black, cinnamon, and gray, not necessarily in that order. The bears also have strange dietary habits. One of them likes only fish, another eats only berries and nuts, the third is a vegetarian (no meat or fish), while the last one eats anything.
- The warden has seen the black and brown bears by the river eating fish. The gray bear will eat apples out of the warden's hand.
- Wilbur and Sally do not like nuts or berries.
- The gray and brown bears are female.
- Jane eats fish, but it is not her favorite dinner.
- A bear who would eat potato salad would have to be willing to eat absolutely anything!

9
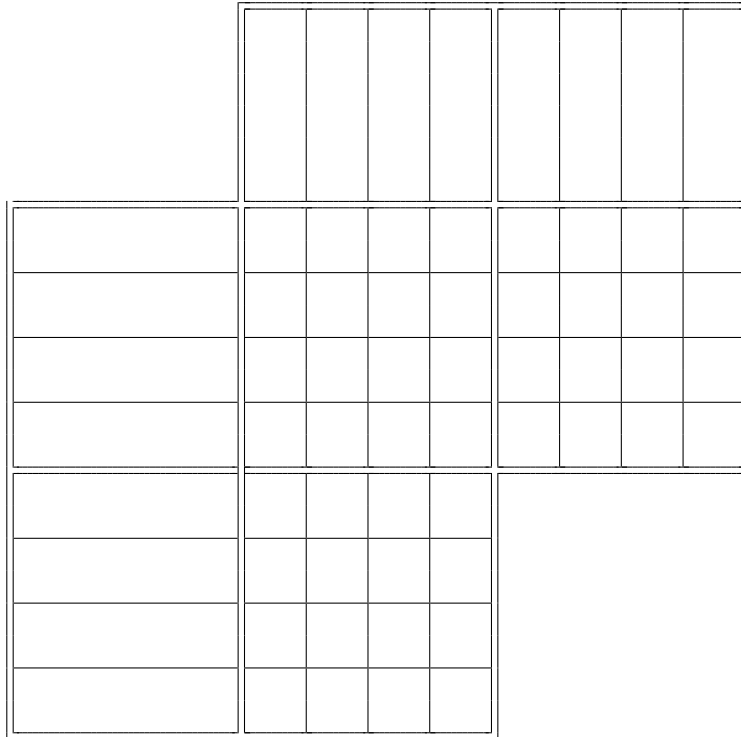
1.6   Glicks and Glocks

In each of the problems below, a statement is given which you may accept as true. Based on that statement, decide if each of the lettered statements is "true", "false", or if the truth value "can not be determined". Work with your group to reach a consensus on each question. Record your answers and a short explanation.

Objectives:

- To practice deductive reasoning
- To learn to make and recognize valid arguments

I. **All Glick numbers that are greater than 20 are even.**

(a) No odd number greater than 20 is a Glick.

(b) If an even number is greater than 20, then it is a Glick.

(c) No odd number is a Glick.

(d) No number that is less than 20 can be a Glick.

(e) All even numbers are Glicks.

(f) 33 is not a Glick.

II. **If a number is a multiple of 6 or is divisible by 7, then it is a Gluck.**

(a) 17 is a Gluck.

(b) 42 is a Gluck.

(c) 38 is not a Gluck.

(d) No even number is a Gluck.

(e) No positive number less than 5 is a Gluck.

(f) All Glucks are divisible by 42.

III. **All odd numbers that are less than 29 and all divisors of 48 are Glacks.**

    (a) All Glack numbers less than 15 are odd.

    (b) 14 is a Glack.

    (c) All odd numbers less than 21 are Glacks.

    (d) 16 is a Glack.

    (e) No odd number greater than 27 is a Glack.

    (f) Exactly Nine Glacks are prime.

IV. **All Glock numbers are divisible by 2 and are multiples of 13.**

    (a) 169 is a Glock.

    (b) No prime number is a Glock.

    (c) 52 is a Glock.

    (d) All Glocks are even.

    (e) All Glock numbers are divisible by 26.

    (f) All numbers divisible by 26 are Glocks.

1.7   Using Venn Diagrams

Venn diagrams are an extremely useful tool for organizing and visualizing sets. In this exercise, you will use given Venn diagrams to identify sets.

Objectives:

- To determine if a number is an element of a given set.
- To represent set membership using Venn Diagrams.

I. **Identifying Sets**
In each of the following Venn diagrams, A and B are sets selected from the following list. Use the given elements to identify the sets A and B in each diagram. Sets may be used more than once or not at all.

multiples of 2	odd numbers
prime numbers	divisors of 24
whole numbers smaller than 10	multiples of 3
whole numbers larger than 10	multiples of 6
multiples of 5	divisors of 15

$A =$

$B =$

Explanation:
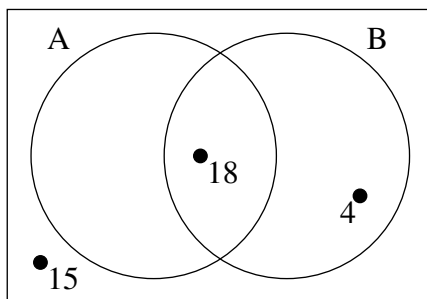
$A =$

$B =$
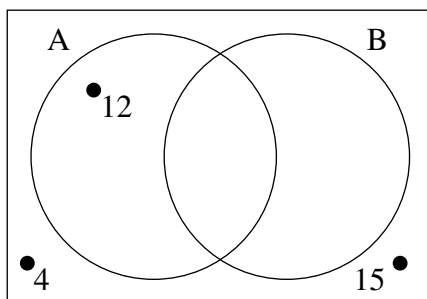
Explanation:

$A =$

$B =$

Explanation:

$A =$

$B =$

Explanation:

$A =$

$B =$

Explanation:

$A =$

$B =$

Explanation:

$A =$

$B =$
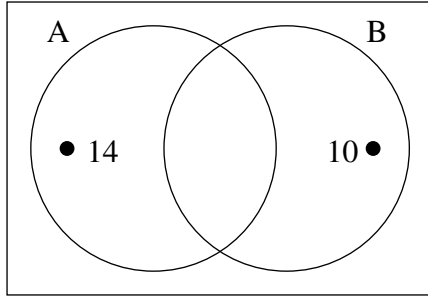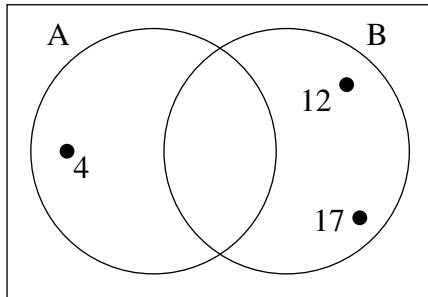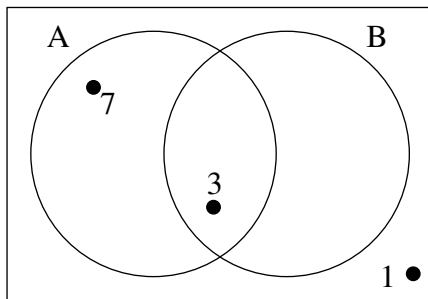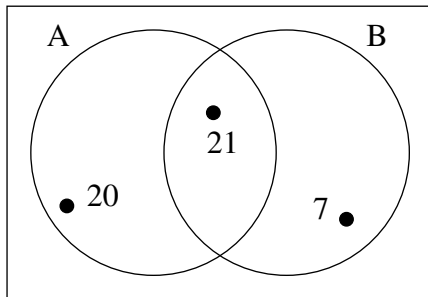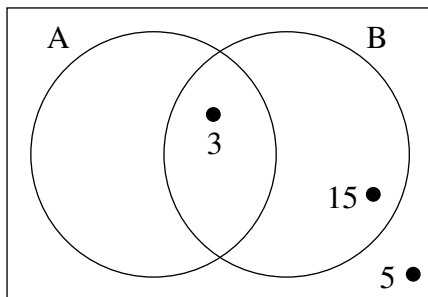
Explanation:

$A =$

$B =$

Explanation:

Adapted from McGinty, Robert & Van Beynen, John (1985). Deductive and analytic thinking. *The Mathematics Teacher*, 78(3), 189.

1.8 Counting with Venn Diagrams

Use the Venn Diagram provided below to help you answer the following question. Keep a careful record of your steps as you fill in each region of the Venn Diagram. Provide justification for each of those steps.

Objectives:

- To use visual representations to organize information.
- To follow a systematic approach in solving a problem and justify solution steps.

I. A class contains 32 students, 25 of whom are freshmen and 18 of whom are girls. Ten of the freshmen girls play sports. In total, 12 girls in the class play sports. Seven freshman boys play sports. All together, 21 students play sports. Finally, five of the freshman girls do not participate in any sport. How many boys in the class are not freshmen and do not play sports?
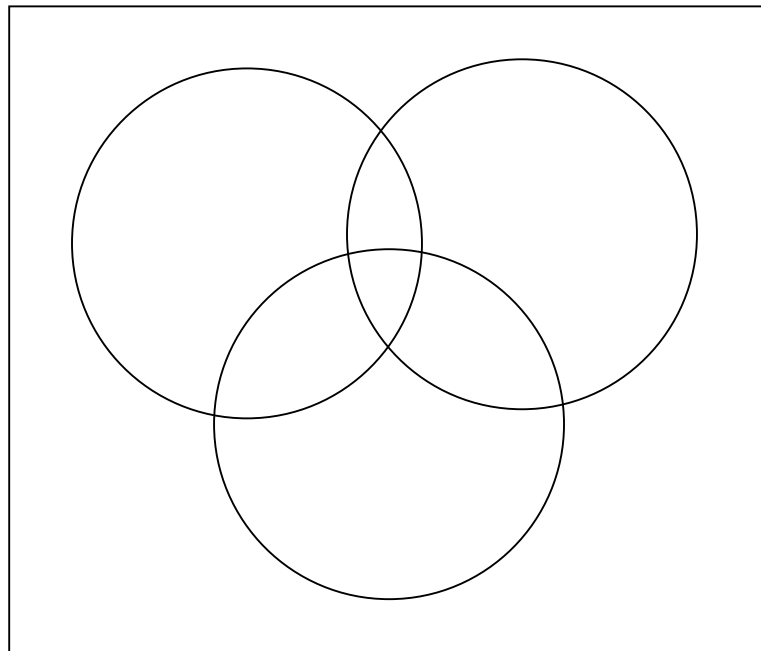
Justification:

II. Now create your own counting problem. Your problem should use three different sets and should give enough information for a student to fill in each region of the Venn diagram below. Work as a group and be creative!
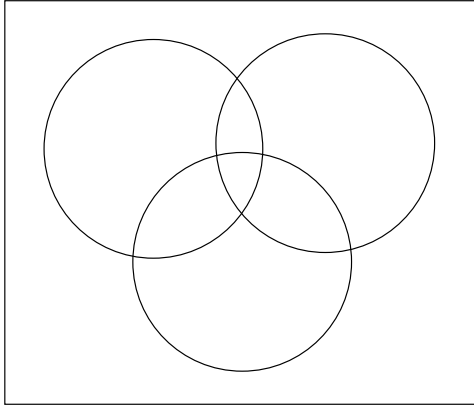
Problem Statement:

Chapter 2

Numeration and Operation

What are numbers? What is the nature of arithmetical truth?
– Friedrich Ludwig Gottlob Frege

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.
– Pierre-Simon Laplace

Contents

2.1 Whole Numbers and Counting

The whole numbers provide a basis for arithmetic and will be the starting point for our exploration of number sets. It is therefore important that we develop a good understanding of the whole numbers. For most, the first introduction to the whole numbers comes through counting objects. In this activity, you will explore the process of counting and how it relates to the whole numbers.

Objectives:

- To understand how the whole numbers are used in counting.
- To understand the characteristics of counting and set cardinality.

I. **Connecting Counting and Sets with Cardinality**
   If a child can correctly say the first five counting numbers, "one, two, three, four, five," will the child necessarily be able to determine how many books are in a collection of five books? To help you think about this question, complete the following tasks.

   (a) The following pictures describe errors that very young children commonly make when first learning to count objects. After understanding these errors, write down the characteristics of correctly counting a set of objects. How does this process connect to the cardinality of a set?
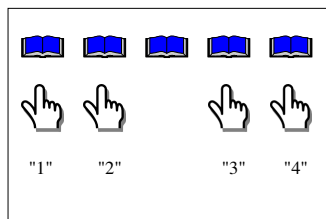
Child 1
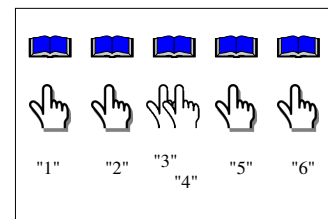
Child 2

   (b) Suppose a teacher asks two students to determine how many blocks there are in a set. Both students count as shown below.
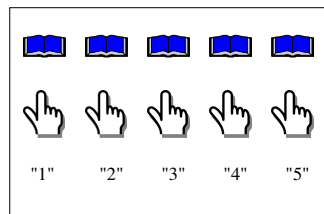
   Then the teacher asks the students "So how many blocks are there?" They respond differently this time, as shown on the next page.

18
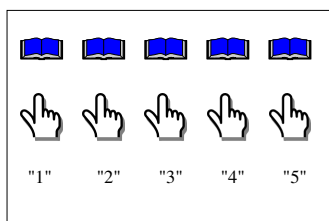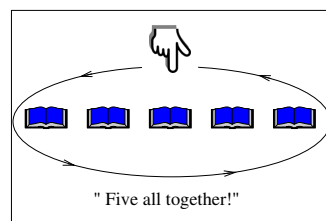
Child 1   Child 2

Which student understands counting better? What is the difference?

(c) Suppose that the teacher takes the original five books and asks the student to count that there are five of them. The teacher then covers these books up and puts one more beside them, as shown below. The teacher asks "Now how many books are there in all?"
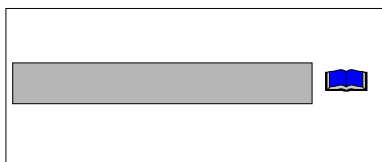
Here are some possible responses.

- Child 1 is unable to answer.
- Child 2 says "1,2,3,4,5" while pointing to the covered books, and then says "6" while pointing to the new book. Finally, the child says "there are 6 books."
- Child 3 says "5" while pointing to the covered books, then points to the new book and says "6." Finally, the child says "there are 6 books."

Compare these different responses. Which response shows a better understanding of counting and why?

2.2   Modeling Addition and Subtraction

Many students find even the simplest mathematics operations, like addition and subtraction, to be very abstract and hard to understand. Seeing these operations represented in a concrete way, called modeling, is often a useful step in understanding them. In this activity, you will explore several models for addition and subtraction of whole numbers.

Objectives:

- To become familiar with various models of addition and subtraction.
- To appreciate how concrete representations can aid in understanding abstract mathematical concepts.

I. **Modeling Addition**
   Addition can be modeled in many different ways. Here you see two ways to model the addition problem $2 + 3 = 5$. Follow these examples as you model each of the addition problems below.
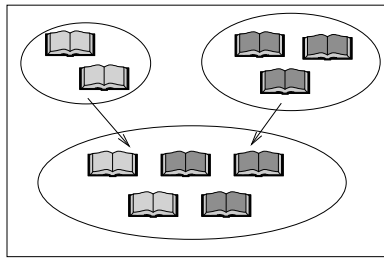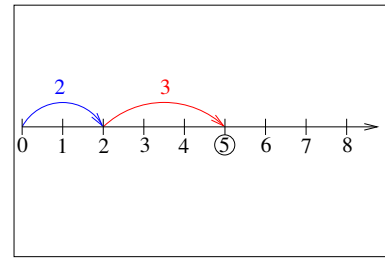
Set Model   Measurement Model

(a) Model $5 + 3 = 8$ using both models.

(b) Model $3 + 5 = 8$ using both models. How are your models different from the ones above?

II. **Modeling Subtraction**

The models we used for addition can also be adapted for subtraction. Here you see two of these models for the subtraction problem $6 - 4 = 2$. Follow these examples as you model each of the subtraction problems below.
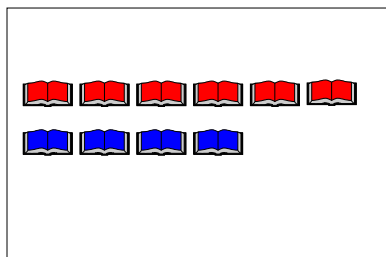
Comparison Model
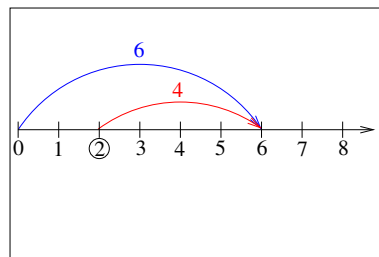
Measurement Model

(a) Model $7 - 2 = 5$ using both models.

(b) Devise your own model of subtraction, different from the two above, and use it to model $7 - 2 = 5$.

III. **Illustrating Properties with Models**

Models can also be very useful for illustrating and confirming properties of operations. Use one of the models seen in this activity to illustrate the following properties.

(a) The additive identity is 0.

(b) Addition is associative $(a + b) + c = a + (b + c)$.

2.3   Patterns and Circle Clocks

The analog clock has historically been a very familiar object that can be used to understand patterns in mathematics. In this activity, you will use circle clocks to help you appreciate visually the patterns in the multiplication operation.

Objectives:

- To observe a visual representation of multiplication tables.
- To see some of the structure underlying the multiplication operation.

I. **Connecting The Dots**
Connect the dots in the provided circle clocks starting with 0 and skip-counting using the given value. Keep connecting dots until you start repeating the connections. For example, in the "Counting by 3" clock, you would connect 0 to 3, 3 to 6, 6 to 9, etc.
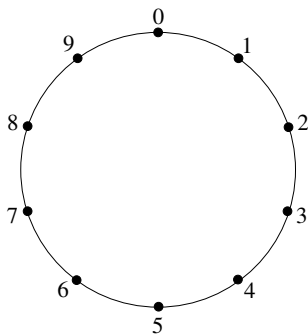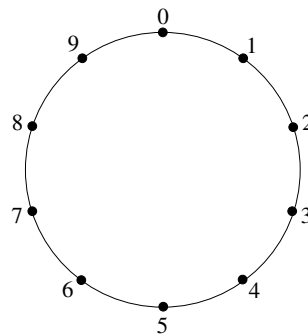
Counting by 1   Counting by 2   Counting by 3
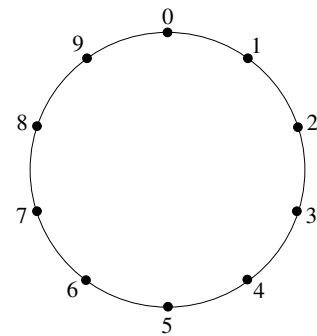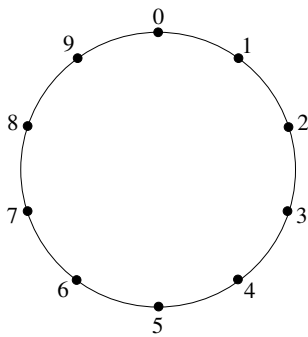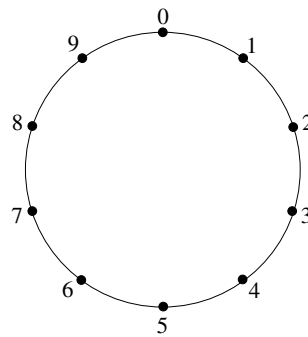
Counting by 4   Counting by 5   Counting by 6
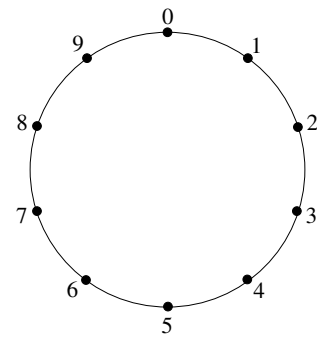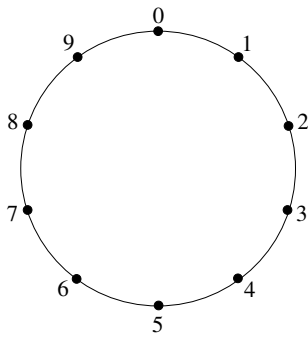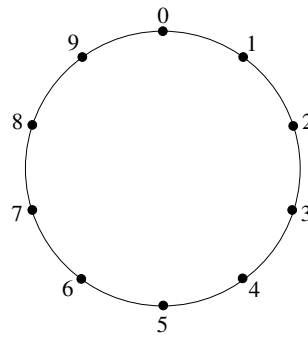
Counting by 7   Counting by 8   Counting by 9
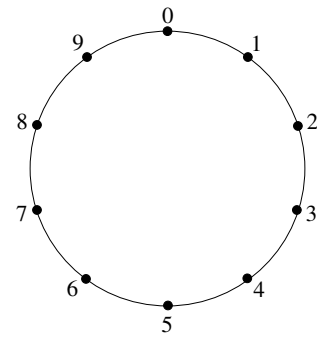
II. **Exploring Patterns**
    By looking for patterns in the circle clocks you've just filled out, we can better understand the structure of the multiplication operation. Answer the following questions about your circle clocks.

    (a) Describe the pattern in the "counting by 2" clock as if you were talking to someone on the phone.

    (b) Group the clocks into sets that have similar patterns and describe how these patterns are similar.

    (c) Now explain how the numbers that go with the similar clocks you found in the last problem are related to each other.

III. **Making Predictions**
     Once you have established the pattern for the first nine circle clocks, you can use it to make predictions. Predict the shape of the clock for multiples of 11. Explain your reasoning and then draw the pattern to see if your predictions are accurate.
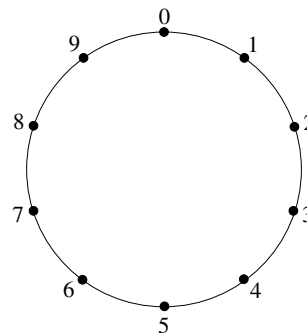
2.4   Interpreting Remainders

If mathematics is to be useful in the real world, we need to not only know how to solve a particular type of mathematical problem, but also how to correctly interpret the answer. When dividing whole numbers, we will often have a remainder left over. In this activity we explore various ways to interpret that remainder.

Objectives:

- To learn that there are many ways to deal with the remainder in a division problem.
- To appreciate the need to make sense of remainders.

I. **When Does a Remainder Make Sense?**
   Solve the following problem individually, and then discuss your answer with your group members. How were remainders dealt with in this problem?

   A client has ordered 415 doughnuts from your bakery. Each of your doughnut boxes holds 24 doughnuts. How many doughnut boxes will you need to package the order?

II. **Dealing with Remainders**
   Below you will find several answers to story problems. It is your job to individually write separate story problems involving 28 divided by 5 having each of these answers. After you have written the problems, share them with your group.

   (a) The Answer is 5

   (b) The Answer is 6

(c) The answer is $5\frac{3}{5}$

(d) The answer is 5 remainder 3

III. **Summarizing**

After discussing your problems as a group, come up with a single paragraph describing what you learned in this activity about the division process and remainders.

2.5 Playing with Operation Sense

As you practice with the four basic operations, you soon develop a sense for how they work. In this activity, you will play three different games involving dice and the operations addition, subtraction, multiplication, and division. Your goal is to not only have fun playing the game, but to also develop a general strategy for getting the highest score in each game.

Objectives:

- To develop a sense of the effect of the four arithmetic operations on numbers.
- To practice deductive, inductive, and/or intuitive reasoning.

I. **The Game of Maximize**

In the game of maximize, your goal is to use the numbers rolled on four dice to fill in the blanks in each expression to make the value of the expression as large as possible. Roll the four dice at the same time and then decide where to place the numbers.

(a) Play each of the games below several times as a group.

Game 1:

$$\square \times \square + \square - \square$$

$$\square \times \square + \square - \square$$

$$\square \times \square + \square - \square$$

$$\square \times \square + \square - \square$$

Game 3:

$$(\square \times \square) \div (\square - \square)$$

$$(\square \times \square) \div (\square - \square)$$

$$(\square \times \square) \div (\square - \square)$$

$$(\square \times \square) \div (\square - \square)$$

Game 2:

$$\square \times \square \div \square + \square$$

$$\square \times \square \div \square + \square$$

$$\square \times \square \div \square + \square$$

$$\square \times \square \div \square + \square$$

Game 4:

$$(\square - \square) \times (\square + \square)$$

$$(\square - \square) \times (\square + \square)$$

$$(\square - \square) \times (\square + \square)$$

$$(\square - \square) \times (\square + \square)$$

(b) Now discuss strategies for each of the games. What is the best way to make sure that you win a given game? How does this relate to the operations in the game?

II. **The Game of Target**

In the game of target practice, your goal is to get as close as possible to the target using three numbers and two operations rolled on the provided dice. You may use these in any order.

(a) Play the game several times with each of the targets given below.

Game 1: Target = 27

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

Game 3: Target = 81

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

Game 2: Target = 1

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

Game 4: Target = 0

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

    Numbers: ____ ____ ____ Ops: ____ ____

(b) Again, after playing the game several times for each expression, describe a general strategy that will help you get as close as possible to the target number. Try your strategy on at least one target you create yourself.

2.6 Properties of Operations

Operations such as addition and subtraction are an important part of elementary school mathematics. Many students are introduced to the various properties of these operations (i.e. commutative property, associative property, etc.), but do not really understand them. In this activity you will review these familiar properties and apply them in an abstract setting.

Objectives:

- To deepen student understanding of the properties of operations by applying them to abstract operations.
- To understand how tables are used to define abstract operations.

I. **Properties of Addition**

The addition operation on the set of whole numbers is one of the most basic operations. However, it is rich with examples of operation properties. Some of these are stated below using whole numbers a, b, and c. Give an example of each property using specific whole numbers.

(a) Closure: If a and b are whole numbers, then $a + b$ is a whole number.

(b) Commutative Property: $a + b = b + a$.

(c) Associative Property: $a + (b + c) = (a + b) + c$

(d) Identity: 0 is the additive identity.

II. **A Non-Standard Operation**

Use the new operation $\oplus$ defined for whole numbers a and b to answer the following questions.

$$a \oplus b = a - b + 2$$

(a) Find $1 \oplus 3$

(b) Is the set of whole numbers closed under $\oplus$? Explain.

(c) Find $5 \oplus 6$ and $6 \oplus 5$ then decide if you think that $\oplus$ is a commutative operation.

(d) Is there an identity element for $\oplus$? If so, what is it? If not, why not?

III. **Addition and Subtraction Tables**
    Complete the following addition and subtraction tables for whole numbers. If no answer exists for a
    given cell, enter an X. Place the result of $a - b$ in the cell with row label a and column label b. Then
    use these tables to answer the questions that follow.

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

−	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

(a) Are the whole numbers closed under subtraction? Describe how you can tell from the table.

(b) Is subtraction a commutative operation? Again, describe how the table shows the answer.

IV. **An Abstract Operation Table**
    The table below defines the operation Λ on $S = \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$. Use it to answer the following questions.

Λ	$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$
$\heartsuit$	$\clubsuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$
$\spadesuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$
$\clubsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$
$\diamondsuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$

(a) Is S closed under Λ? How can you tell?

(b) Find $\heartsuit \Lambda \clubsuit$ and $\clubsuit \Lambda \heartsuit$. Is Λ commutative? How can you tell by looking at the table?

(c) Is there an identity element for Λ on S? If so, what is it and how can you tell?

29

V. **Constructing Your Own Operation Table**

Given the set $S = \{A, B, C, D\}$ fill in the table below for the operation # in such a way that the following conditions are all met.

Conditions:

- S closed under #.

- # a commutative operation.

- # an associative operation.

- B the identity element for #.

- D and A inverses of each other.

- C its own inverse.

#	A	B	C	D
A				
B				
C				
D				

2.7 Numeration Systems

It is important to understand the difference between numbers and numerals. This is somewhat like the difference between a person and his or her name. A number is an abstract concept–an idea about quantity. On the other hand, a numeral is a symbol used to represent a number concept. We are used to thinking of numbers and numerals as being the same. However, in this activity we will begin to see that there are many numerals that can be used to represent numbers.

Objectives:

- To realize that numbers and numerals are not the same thing.
- To better understand why our numeration system is not simple for many young children to master.

I. **Out with the Old. . .**

Imagine that your group are members of a small tribe that lived thousands of years ago before our current numeration system was invented. Your tribe's numeration system has been alphabetically based. That is, you used letters to represent numbers as shown below. Your system is finite, meaning for values greater than Z you have no symbol. You just call those amounts "many."

| A | B | C | D | E | . . . |
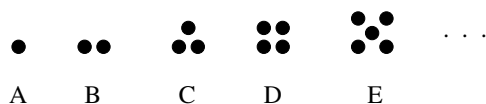
The tribe has now settled down and needs a more advanced system to keep track of the ever-growing herds as well as farm production. Luckily, one of your most brilliant mathematicians announced the discovery of a new system that could represent any number using only the symbols A, B, C, D, and a new symbol called zero written as 0. Before the mathematician could teach this system to anybody, she was kidnaped by a jealous neighboring tribe. However, the following artifacts where discovered in her hut. They seem to be related to the new system.

flats   longs   units
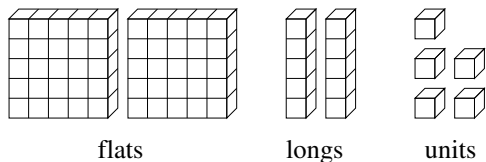
Work as a group to rediscover the new numeration system. It must have the following properties:

- It uses only the symbols A, B, C, D, and 0.
- It can represent any whole number.
- There is a relationship to the pictures above.

Describe your system below.

II. **In with the New!**

Now that you have a new numeration system, it is time to take it for a test drive. Complete the following activities in your new system.

(a) In the old system, one counted to ten like this:

A   B   C   D   E   F   G   H   I   J

Show how to count to ten in your new system.

(b) What numeral comes immediately after after BAD in your new system?

(c) What numeral comes immediately before AA in your new system?

(d) Make up a story problem involving addition that would be appropriate to your village. Show how you would use your system (not our modern system) to solve the problem.

(e) Make up a story problem involving division that would be appropriate to your village. Show how you would use your system (not our modern system) to solve the problem.

III. **Group Project: Present Your Numeration System**

Next week all of the local tribes are gathering for a mathematics conference. Your group must prepare a poster describing your numeration system. This poster should contain the following:

- A description of your numeration system.
- Examples comparing the old alphabetical system with the new system.
- Examples showing how the flat, long, and unit artifacts are related to your system.
- A list of advantages that the new system has over the old system.

Posters will be judged by the conference coordinator as well as the local tribes.

2.8 Modeling Numerals in Other Bases

The idea of place value can be quite abstract. Using models, however, can help students to understand why the position of a digit affects its value. In this activity you will model numerals in various bases using units, longs, flats and cubes that you build yourself out of centimeter cubes.

Objectives:

- To model whole numbers in various bases.
- To use these models to convert between bases.

I. **Modeling Numerals**

Use the provided centimeter cubes to construct physical models for the following numerals in the indicated bases. Record a sketch of your model in the space provided.

(a) 1101_{two}

(c) 240_{five}

(b) 37_{eight}

(d) $2D_{\text{sixteen}}$

II. **Using Models to Convert**

Now use models to convert each numeral from the base in which it is written to the given base. Record a sketch of how your models are used in the conversion process.

(a) Convert 322_{four} to base eight

(b) Convert 47_{eight} to base four

(c) Convert 138_{nine} to base six

(d) Convert 21_{three} to base six

III. **Finding The Base with Models**

Use models to help you determine the proper base "b" for each of the problems below. Record a sketch of the models that help you solve each problem.

(a) $12_{\text{ten}} = 1100_b$

(c) $230_{\text{four}} = 54_b$

(b) $234_{\text{ten}} = 176_b$

(d) $112_{\text{six}} = 1122_b$

2.9   Modeling Addition and Subtraction

One of the major advantages of our positional numeration system over many ancient systems is that calculations such as addition and subtraction are much easier to carry out in our system. In this activity, you will model these operations using units, longs, flats, and cubes to justify the standard algorithms for addition and subtraction.

Objectives:

- To interpret and model addition and subtraction operations in other bases.
- To gain a deeper understanding of the standard addition and subtraction algorithms.

I. **Addition**

You should be familiar with the standard addition algorithm and in particular the process of carrying tens from one column to the next. The addition problems below are not in base ten. But the same concepts still apply.

(a) Use units, flats, longs, etc. to model the addition **operation** for each problem.

i.
$$\begin{array}{r} 2 \quad 3_{\text{four}} \\ +\quad 3 \quad 1_{\text{four}} \\ \hline \end{array}$$

ii.
$$\begin{array}{r} 2 \quad 2 \quad 2_{\text{three}} \\ +\quad 1 \quad 2 \quad 2_{\text{three}} \\ \hline \end{array}$$

iii.
$$\begin{array}{r} 2 \quad 3 \quad 4_{\text{five}} \\ +\quad 3 \quad 0 \quad 2_{\text{five}} \\ \hline \end{array}$$

(b) Now without models, complete the problems below using the standard algorithm without converting to base ten.

i.
$$\begin{array}{r} 3 \quad 3_{\text{four}} \\ 3 \quad 2_{\text{four}} \\ +\quad 1 \quad 3_{\text{four}} \\ \hline \end{array}$$

ii.
$$\begin{array}{r} 6 \quad 2 \quad 5_{\text{seven}} \\ +\quad 3 \quad 4 \quad 2_{\text{seven}} \\ \hline \end{array}$$

iii.
$$\begin{array}{r} 4 \quad A \quad A_{\text{eleven}} \\ 3 \quad A \quad 6_{\text{eleven}} \\ +\quad 1 \quad 0 \quad 9_{\text{eleven}} \\ \hline \end{array}$$

II. **Subtraction**

You are doubtlessly familiar with the standard subtraction algorithm including the process of borrowing. The problems below are done in different bases, but this same concept still applies.

(a) Use units, flats, longs, etc. to model the subtraction **operation** for each problem. You may need several pictures to show the borrowing process.

i. $\begin{array}{ccc} 3 & 0_{\text{ four}} \\ - \quad 2 & 2_{\text{ four}} \\ \hline \end{array}$

ii. $\begin{array}{cccc} 2 & 0 & 1_{\text{ three}} \\ - \quad 1 & 2 & 2_{\text{ three}} \\ \hline \end{array}$

iii. $\begin{array}{cccc} 4 & 1 & 2_{\text{ five}} \\ - \quad 1 & 3 & 4_{\text{ five}} \\ \hline \end{array}$

(b) Perform the following subtraction problems without models in the given base.

i. $\begin{array}{ccc} 3 & 3_{\text{ four}} \\ - \quad 1 & 2_{\text{ four}} \\ \hline \end{array}$

ii. $\begin{array}{cccc} 4 & 2 & 2_{\text{ seven}} \\ - \quad 2 & 6 & 5_{\text{ seven}} \\ \hline \end{array}$

iii. $\begin{array}{cccc} 9 & 3 & 4_{\text{ eleven}} \\ - \quad 1 & A & A_{\text{ eleven}} \\ \hline \end{array}$

2.10   Skeletal Arithmetic

If the standard addition and subtraction algorithms are well understood, they can provide information about the structure of an addition or subtraction problem. In this activity, we will use our knowledge of these standard algorithms to find missing digits in addition and subtraction problems.

Objectives:

- To better understand the process of carrying in addition and borrowing in subtraction.
- To use the standard addition and subtraction algorithms to work backwards.

I. **Addition**
   Use your knowledge of the standard addition algorithm to find the missing information in each of the following problems. After completing each problem, write a sentence describing how you found the missing information.

   (a) Fill in the missing digits in the following base-ten addition problem.

			__	6	2
		3	9	4	__
+			__	8	__	7
	__	3	3	2	1

   (b) The following base-six addition problem is also missing several digits. Find those digits.

		1	__	__ $_{\text{six}}$
+		__	2	2 $_{\text{six}}$
	__	0	1	1 $_{\text{six}}$

   (c) In the base-ten addition problems below, find the unique single digit represented by each letter.

   i.
   ```
       A  H  A  H  A
   +      T  E  H  E
   -----------------
       T  E  H  A  W
   ```

   ii.
   ```
       W  R  O  N  G
   +   W  R  O  N  G
   -----------------
       R  I  G  H  T
   ```

II. **Subtraction**

In the problems below, use your knowledge of the standard subtraction algorithm to find the missing information. Again, describe how you found that missing information in a short sentence or paragraph.

(a) Fill in the missing digits in the following base-ten subtraction problem.

$$
\begin{array}{r}
\underline{}\;\; 8 \;\; 6 \;\; \underline{} \\
-\quad \underline{} \;\; 8 \;\; 3 \\
\hline
8 \;\; \underline{} \;\; 9
\end{array}
$$

(b) The following base-five subtraction problem is also missing several digits. Find those digits.

$$
\begin{array}{r}
2 \;\; \underline{} \;\; \underline{}_{\text{five}} \\
-\quad 2 \;\; 2_{\text{five}} \\
\hline
\underline{} \;\; 0 \;\; 3_{\text{five}}
\end{array}
$$

(c) In the base-ten subtraction problems below, find the unique single digit represented by each letter.

i.
$$
\begin{array}{r}
A \;\; B \;\; A \\
- \quad C \;\; A \\
\hline
A \;\; B
\end{array}
$$

ii.
$$
\begin{array}{r}
B \;\; A \;\; T \;\; S \\
- \quad W \;\; E \;\; R \;\; E \\
\hline
S \;\; P \;\; A \;\; S
\end{array}
$$

2.11   Mental Arithmetic

While the standard algorithm for addition and subtraction work well with pencil and paper, they are often not the best way to do mental computations. However, it is sometimes easy to write these strategies down incorrectly. In this activity, you will practice several short-cuts that aid in mental addition and subtraction and you will learn to write correct equations for these shortcuts.

Objectives:

- To develop a repertoire of mental arithmetic strategies.
- To understand and demonstrate the correct usage of the equals sign.

I. **Mental Addition**

Below you will find addition problems along with a proposed solution. While the proposed solution demonstrates a valid strategy to make mental addition easier, it may not be stated correctly. If required, correct the equation. Then write a sentence describing the strategy.

Problem	Solution	Correction and Strategy
(a) $37 + 42$	$\begin{aligned} 30 + 40 &= 70 \\ &= 7 + 2 \\ &= 70 + 9 \\ &= 79 \end{aligned}$	
(b) $59 + 35$	$\begin{aligned} 59 + 30 &= 89 + 5 \\ &= 94 \end{aligned}$	
(c) $186 + 125$	$\begin{aligned} 186 + 125 &= 200 + 111 \\ &= 311 \end{aligned}$	
(d) $656 + 225$	$\begin{aligned} 656 + 225 &= 656 + 230 \\ &= 886 - 5 \\ &= 881 \end{aligned}$	

II. **Mental Subtraction**

Below you will find subtraction problems along with a proposed solution. While the proposed solution demonstrates a valid strategy to make mental subtraction easier, it may not be stated correctly. Rewrite the equation correctly then write a sentence describing the strategy.

Problem	Solution	Correction and Strategy
(a) 65 - 28	$28 + 40 = 68 - 3$ $= 65$ So the answer is: $40 - 3 = 37$	
(b) 42 - 19	$42 - 19 = 43 - 20$ $= 23$	
(c) 184 - 125	$184 - 125 = 184 - 100 - 25$ $= 84 - 30 + 5$ $= 54 + 5$ $= 59$	
(d) 235 - 65	$235 - 65 = 235 - 5$ $= 230 - 30$ $= 200 - 30$ $= 170$	

2.12 Modeling Multiplication and Division

As was the case with addition and subtraction, the positional notation system makes multiplication and division algorithms much easier. To better understand why these systems work, we can model them using units, flats, longs, etc. In this activity you will use models to justify the standard multiplication and division algorithms in bases other than ten.

Objectives:

- To interpret and model multiplication and division operations in other bases.
- To gain a deeper understanding of the standard multiplication and division algorithms.

I. **Multiplication Models**
You should be familiar with the standard multiplication algorithm. The multiplication problems below have been worked out for you in various bases. Model these operations (not just the numerals involved) using units, flats, longs, etc. Sketch a picture of your model in the space provided.

$$
\begin{array}{rrr}
 & 3 & 2\,_{\text{four}} \\
\times & & 3\,_{\text{four}} \\
\hline
2 & 2 & 2\,_{\text{four}} \\
\end{array}
$$

$$
\begin{array}{rrr}
 & 1 & 3\,_{\text{four}} \\
\times & 2 & 1\,_{\text{four}} \\
\hline
 & 1 & 3 \\
3 & 2 & \\
\hline
3 & 3 & 3_{\text{four}} \\
\end{array}
$$

II. **Division Models**

The standard division algorithm is also one that should be familiar. Perhaps, however, you've never taken the time to figure out why it works. Consider a division problem such as $12_{six} \div 3_{six}$. We can model this using either repeated subtraction or partitioning.

$12_{six} \div 3_{six}$ is the number of blocks in each group when we divide 12_{six} into 3_{six} equal groups.

$12_{six} \div 3_{six}$ is the number of groups of 3_{six} blocks which can be made out of 12_{six} blocks with which we started.

Use units, flats, longs, etc. to model the long division problems that have been completed for you in base three below. Sketch pictures of your model in the space provided.

$$
\begin{array}{r}
2\quad 1_{\,three} \\
2_{\,three}\ \overline{\big)\ 1\quad 2\quad 0_{\,three}} \\
-\ \underline{1\quad 1} \\
1\quad 0 \\
-\ \underline{\quad 2} \\
1_{\,three}
\end{array}
$$

$$
\begin{array}{r}
2\quad 1\quad 2_{\,three} \\
10_{\,three}\ \overline{\big)\ 2\quad 1\quad 2\quad 0_{\,three}} \\
-\ \underline{2\quad 0} \\
1\quad 2 \\
-\ \underline{1\quad 0} \\
2\quad 0 \\
-\ \underline{2\quad 0} \\
0_{\,three}
\end{array}
$$

2.13   Multiplication and Division Base Six

You are no doubt experts at the multiplying and dividing multi-digit numbers in base 10. However, your skill in these areas depends on your familiarity with the base 10 numeration system and with base 10 single-digit multiplication tables. In this activity, we will perform multiplication and division in a different base to help you become more aware of the underlying processes.

Objectives:

- To better understand multi-digit multiplication and division algorithms.
- To appreciate the struggle students go through when learning these algorithms.

I. **Base Six Multiplication Table**
   Before you can efficiently multiply and divide multi-digit numerals in a given base, you must be familiar with the single-digit multiplication tables for that base. Use models and/or the idea of multiplication as repeated addition to fill in the single-digit base six multiplication table below.

$\times_{\text{six}}$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

II. **Base Six Multiplication**
   With the table above, you are now ready to practice several multi-digit base six multiplication problems. Use the indicated algorithm to perform each multiplication.

(a) Use the standard multiplication algorithm to complete each base six problem.

$53_{\text{six}} \times 34_{\text{six}}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $320_{\text{six}} \times 152_{\text{six}}$

(b) Use the lattice multiplication algorithm to complete each base six problem.

$35_{\text{six}} \times 24_{\text{six}}$ $503_{\text{six}} \times 143_{\text{six}}$

III. **Base Six Division**
Using the multiplication table in reverse can also help you solve division problems. For example, do find $32_{\text{six}} \div 4_{\text{six}}$, read across the 4's row in the table on the last page. You should find 32 in the 5's column, meaning that $4_{\text{six}} \times 5_{\text{six}} = 32_{\text{six}}$. But this also tells us that $32_{\text{six}} \div 4_{\text{six}} = 5_{\text{six}}$. Use strategies such as this to complete the following division problems with your favorite algorithm.

$140_{\text{six}} \div 4_{\text{six}}$ $11412_{\text{six}} \div 52_{\text{six}}$

2.14 Doubling Algorithms for Multiplication

Before the discovery of our modern positional numeration system, multiplication of whole numbers was extremely difficult. However, using the idea that multiplication is repeated addition, several similar algorithms were developed that made multiplication much easier even in non-positional numeration systems such as the Egyptian system. In this activity we will explore two similar multiplication algorithms that make use of the idea of repeated addition.

Objectives:

- To understand the connection between multiplication and addition in the context of a multiplication algorithm.
- To be able to justify alternative multiplication algorithms.

I. **Egyptian Duplation**

Even though the Egyptians did not have a positional numeration system, they developed an algorithm for multiplication. Their algorithm was based on the idea of multiplication as repeated addition, and in particular doubling a value. Consider the following example.

To multiply 11×17, or ∩| by ∩||||||| create the following table.

|	∩|||||||
||	∩∩∩||||
||||	∩∩∩∩∩∩||||||||
||||||||	⌐∩∩∩||||||

From the table, the product is: ⌐∩∩∩|||||| + ∩||||||| + ∩∩∩|||| which after regrouping equals ⌐∩∩∩∩∩∩∩∩||||||

(a) Use this method to find each product.

$$\|\|\|\|\| \times ∩∩\|$$

$$∩∩\|\| \times ⌐\|\|\|$$

(b) Describe how the duplation algorithm is related to repeated addition.

II. **Russian Peasant Algorithm**

This alternative algorithm for multiplication uses a method similar to the Egyptian duplation algorithm to multiply base ten numerals. An example of the algorithm, along with a description of the process, is provided below.

We wish to find 26×41.

halve	double
~~26~~	~~41~~
13	82
~~6~~	~~164~~
3	328
1	656

1. Create a table with two columns: halve and double.

2. Write the smaller factor in the halve column and the larger in the double column.

3. Repeatedly divide the number in the halve column by 2, ignoring remainders, and multiply the number in the double column. Stop when the halve column entry is 1.

$$26 \times 41 = 82 + 328 + 656$$
$$= 1066$$

4. Cross out any rows with an even number in the halve column.

5. Add together the remaining numbers in the double column.

(a) Use this method to find each product.

24×31 37×73

(b) **Group Project: Justify the Russian Peasant Algorithm**

As your group lab report for this week, write a one-to-three page paper describing why this algorithm works. Include examples to illustrate your justification.

As a hint, you might try using this method to multiply 25×31 and 36×73 and compare this process with the multiplications you did above. You might also look at the base-two representation of the numbers in the halve column.

2.15   Mental Multiplication and Division Tricks

Many of the algorithms we've seen for multiplication and division are best suited for pencil and paper work. When trying to multiply or divide mentally, we often need to use tricks that allow us to quickly multiply or divide without running through an entire multi-step algorithm. In this activity we will explore two of these simplified algorithms and then extend them to be useful in other circumstances.

Objectives:

- To learn shortcuts for mental multiplication and division.
- To see how to justify and extend those shortcuts.

I. **Multiplying By Eleven**

Our first trick is very specific to the number eleven. In order to multiply any two-digit numeral by eleven, follow these steps.

1. Separate the digits, placing the first in the hundred's and second in the one's place.
2. Add the two digits placing the sum in the ten's place, carrying if necessary.

For example, to find 11×32:

1. separate the 3 and the 2, giving 3__2.
2. Add the 3 and the 2 to get 5 and fill this in the blank to get 352.

(a) Practice this algorithm by multiplying each of the numerals below by 11. Verify your answer using one of the pencil-and-paper algorithms seen in class.

63   49

(b) Now extend this algorithm to multiplying three-digit numerals by 11. Show how you would use your extended algorithm to multiply 142 by eleven.

(c) Why does this algorithm work? Give a justification using either base-ten blocks (units, longs, flats, etc.) or expanded notation ($11 = (1 \times 10^1) + (1 \times 10^0)$).

II. **Dividing By Nine**

Another shortcut algorithm allows us to quickly divide any numeral by nine. To do this, follow the steps below.

1. Write down the leading digit of the dividend as the first digit of your answer.

2. To get the next digit in your answer, add the previous digit of your answer to the next digit of the dividend.

3. Repeat this process until you add to the last digit of the dividend. This is your remainder. If nine divides into the remainder, add the quotient to your answer.

For example, to find $2694 \div 9$:

1. Write down a 2 as the first digit of your answer.

2. Add 2 to the 6 to get the next digit of your answer, giving you 28...

3. Add 8 to 9 giving 17. Carry the 1, adding it to the 8 and write down 297.

4. The last sum is $17+4 = 21$. This is the remainder, but 9 still goes into it two times with three left over. Add the 2 to your answer of 297 to get 299 with remainder 3.

(a) Practice this algorithm by dividing each of the numerals below by 9. Verify your answers using one of the pencil-and-paper algorithms seen in class.

13204   201132

(b) Use a small example, such as $106 \div 9$ to show why this method works using base-ten blocks.

(c) Can you extend the algorithm to division by eight? How would you divide 2016 by 8?

2.16 String Problems

Those who are just learning multiplication and division often find longer problems intimidating. One useful tool for solving such problems is to string together a sequence of smaller problems that lead to the solution of the larger problem. In this activity you will practice recognizing valid strings and constructing your own.

Objectives:

- To understand how strings of smaller problems can be used to develop the solution to a larger problem.
- To be able to construct a string of smaller problems that lead to the solution to a larger problem.

I. **Multiplication Strings**

With a knowledge of basic multiplication facts and an understanding of place value, strings of smaller problems can be used to solve larger multiplication problems. Consider the following example.

$2 \times 7 = 14$	multiplication fact
$3 \times 7 = 21$	multiplication fact
$20 \times 7 = 140$	multiplication by 10 adds a 0
$23 \times 7 = 161$	23 sevens = 20 sevens + 3 sevens

(a) Explain how each of the strings of smaller problems below can be used to solve the final computation.

2×6	1×4
5×6	100×4
20×6	3×4
25×6	30×4
	7×4
	137×4

20×5	3×8
7×5	30×8
27×5	15×8
27×50	15×16
27×49	150×16
	151×16

(b) Now make up your own multiplication strings to solve the following problems. Use multiplication facts up through 10×10 as well as your knowledge of addition, subtraction and place value. Justify each part of your strings.

372×11 124×86

II. **Division Strings**

Just as we did with multiplication, we can string together smaller division problems to help us solve larger problems. Consider the example shown below.

$24 \div 12 = 2$	division fact
$2400 \div 12 = 200$	multiplication by 100 adds two 0's
$12 \div 12 = 1$	division fact
$120 \div 12 = 10$	multiplication by 10 adds a 0
$36 \div 12 = 3$	23 sevens = 20 sevens + 3 sevens
$2556 \div 12 = 213$	2400+120+36 = 2556 and 200+10+3 = 213

Create a division string to solve the following problems. Use division facts, multiplication facts, place value, addition and/or subtraction to make your strings. Justify each part of your strings.

$938 \div 7$ $32526 \div 13$

Chapter 3

Number Theory

Mathematics is the queen of the sciences and number theory is the queen of mathematics.

– Carl Friedrich Gauss

Perfect numbers like perfect men are very rare.

– Rene Descartes

Contents

3.1   The Locker Problem

Factoring appears unexpectedly in the solution to many different problems. In this activity, you will solve such a problem. It may be helpful to try solving a smaller version of the problem (perhaps with only ten lockers) to get you started.

Objectives:

- To better understand factoring of whole numbers.
- To identify and describe a pattern.

I. **The Problem**

Students at an elementary school decided to try an experiment. When recess is over, each student will walk into the school one at a time. The first student will open all of the first 100 locker doors. The second student will close all of the locker doors with even numbers. The third student will change all the locker doors with numbers that are multiples of three. (Change means closing locker doors that are open and opening lockers that are closed.) The fourth student will change the position of all locker doors numbered with multiples of four; the fifth student will change the position of lockers that are multiples of five, and so on.

(a) After 100 students have entered the school, which locker doors will be open?

(b) Why are these lockers the ones left open? Give a *mathematical* explanation.

3.2 Exploring E-Primes

By the time they reach college, most students are well familiar with the concept of a prime number and prime factorizations. In order to help you appreciate the difficulty some of your students might have in grasping these concepts for the first time, it can be helpful to start from scratch. In this activity you will work with a brand new "prime-like" property to see what you can discover.

Objectives:

- To better understand the properties of primes by exploring a similar definition.
- To appreciate the difficulty students will have understanding the properties of primes.

I. **A New Type of Prime**

Consider the set $E = \{1, 2, 4, 6, 8, \ldots\}$ consisting of one and all the even numbers. If we are only allowed to use factors from E, some of the numbers in this set can only be written as a product of 1 and themselves. For example, $6 = 1 \times 6$. While it is also true that $6 = 2 \times 3$, 3 is not in the set E, so it can not be used.

Suppose we decide to call such numbers E-**primes**. Any number in E that can be written as a product of two distinct numbers in E, other than 1 and itself, will be called E-**composite**. 1 is neither E-prime nor E-composite. Use this definition to answer the following questions.

(a) Find the first 10 E-primes.

(b) Can every E-composite number be factored into a product of E-primes? Justify your conclusion.

(c) List at least three even numbers that have a unique factorization into *E*-primes.

(d) Find an even number whose *E*-prime factorization is not unique.

(e) Come up with a test to decide whether an even number is an *E*-prime.

3.3   A Prime Numeration System

In previous weeks we explored several ancient numeration systems. In this activity, we will study a numeration system that was never actually used, but may still be of interest.

Objectives:

- To better understand the fundamental theorem of arithmetic by exploring representations of whole numbers.
- To appreciate the challenges of representation in a numeration system.

I. **A Mystery Numeration System**
   Th numeration system outlined below is based on factorizations and number theory. You have been given the representation of the first ten natural numbers in this system.

Base Ten	Mystery System		Base Ten	Mystery System
one	0		six	11
two	1		seven	1000
three	10		eight	3
four	2		nine	20
five	100		ten	101

Once you have studied and understood this numeration system, answer the following questions.

(a) How would the next ten numbers (eleven through twenty) be represented in this system?

(b) Find the base ten representation of each of these mystery system numerals.

   i. 230_m ii. 1032_m

(c) Do numbers have unique representations in this system? Justify your answer.

3.4 Factoring Games

Playing games is often a good way to better understand a concept. In this activity, you will play several games involving factoring. As you play each game, think about developing a general winning strategy.

Objectives:

- To better understand the concepts of prime and composite numbers and factoring.
- To develop the ability to generalize a plan for specific games.

I. **Factor Feat**

The game of Factor Feat is a two-person game. The goal of the game is to have the largest sum of claimed numbers. After you randomly choose a starting player, follow these steps to play.

(a) Randomly pick a starting player.

(b) The player claims one number from the list, and then claims all of its unclaimed factors.

(c) Switch players and repeat step 2, continuing until all numbers have been claimed.

(d) The winner is the player with the largest sum of claimed numbers.

(a) Play several of games using the provided game boards. As you play, try to develop general winning strategies for the game.

Game 1

	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

Game 2

	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

Game 3

	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

(b) Why is the number "1" left off of the game board?

(c) If you were to play game with a board from 2 to 100, what would be the best first move? What would be the worst first move? Explain.

II. **The Tax Man**

The second game is an interesting variation of the first. In this game, you select one of your group members to be the "Tax Man," and the rest of your group plays as a team to beat him or her. Here are the instructions for this game.

(a) The team chooses a number to cross off that has not been crossed off or circled and has at least one proper factor that has not been crossed off or circled.

(b) The tax man then circles all the possible factors of that number.

(c) Repeat this process until all numbers are either crossed off or circled.

(d) If at the end of the game the sum of the crossed off numbers is greater than the sum of the circled numbers, the team wins. Otherwise, the Tax Man wins.

(a) Play several games using the provided game boards. As you play, try to develop general winning strategies for your team.

Game 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

Game 2

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

Game 3

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

(b) Why is "1" included on this game board?

(c) If your team were playing this game on a board that went up to 100, what would your best first move be? What would the worst first move be? Explain.

Adapted from:

- Masingila, Joanna et al, *Mathematics for Elementary Teachers via Problem Solving*, Prentice Hall, 2002, 113.

- Bassarear *Mathematics for Elementary Teachers Explorations*, Brooks/Cole, 2008, page 80.

3.5   Modular Arithmetic

The Division Algorithm states that if you have two numbers, a and b, with $b \neq 0$, then you can always write $a = bq + r$ where $0 \leq r < b$. Basically, this means that if you divide a number a by a number b, you will always get a remainder r that is less than b. In this activity, we study those remainders using the idea or modular or "clock" arithmetic.

> **Objectives:**
>
> - To better understand the concept of a remainder.
> - To become familiar with modular arithmetic.

I. **Introduction**

Modular arithmetic is sometimes called clock arithmetic because it can best be represented using a circular "clock face" with the numbers 0 through $n - 1$ shown around the edges. To say that $a \equiv b$ (mod n) (a is congruent to b modulo n) means that a and b fall on the same tick mark of this clock if we start counting from zero. So, for example, $2 \equiv 14$ (mod 12) because 2 and 14 both fall on the same tick mark on a clock with 12 ticks.

(a) Draw a picture of a modulo 6 clock and list the numbers 0 through 11 evenly spaced around the clock, starting with 0 at the top.

(b) Now collect the numbers between 1 and 50 into sets of numbers which are congruent modulo 6. Remember, this means that the numbers fall on the same tick mark of your clock above, or that they have the same remainder when you divide by 6. For example, $1 \equiv 7$ (mod 6), so one of the sets will start with $\{1, 7, \ldots\}$. You should have six sets of numbers.

II. **Addition Modulo 6**

Each of the sets you found in part (b) above is called a "congruence class" modulo six. It is typical to represent these classes using their smallest members, 0, 1, 2, 3, 4, and 5. When this is done, we can do arithmetic modulo six. In the case of addition:

$$4 + 5 = 3 \pmod 6 \text{ because } 4 + 5 = 9 \equiv 3 \pmod 6.$$

(a) Fill in the addition table modulo 6 shown below.

$+_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

(b) Compare this table to a base six addition table. How are they similar, and how are they different?

(c) Because 0 is the identity for $+_6$ (why?), the inverse of a number a for $+_6$ is a number b such that $a +_6 b = 0$ and $b +_6 a = 0$. Find the inverses of 0, 1, 2, 3, 4, and 5, if they exist.

(d) A number and its $+_6$ inverse are sometimes called "sixes-compliments." Explain why this term is used, and how this can be helpful in subtraction.

II. **Multiplication Modulo 6**
As with addition, we can do multiplication modulo 6. Consider the example shown below.

$$3 \times 5 = 3 \pmod 6 \text{ because } 3 \times 5 = 15 \equiv 3 \pmod 6.$$

(a) Fill in the multiplication table modulo 6 shown below.

$\times_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

(b) Compare your $\times_6$ table with the base six multiplication table you created in activity 2.13.

(c) What patterns do you notice in the rows of your $\times_6$ table? Explain why those patterns appear.

(d) The identity for multiplication is 1. So a and b will be inverses for $\times_6$ if $a \times_6 b = 1$ and $b \times_6 a = 1$. Find the inverses of 0, 1, 2, 3, 4, and 5, if they exist.

3.6 Modeling Divisibility Tests

Because the process of long division can be, well, long, people have over time developed tests to quickly determine if a number is divisible by a given single digit number. You are probably familiar with several of these tests. In this activity, you will use models to explain why these tests work.

Objectives:

- To use models to justify common divisibility tests.
- To understand how place value is related to divisibility tests.

I. **Divisibility by Five**
   One of the most universally known divisibility tests is for 5. This test states that:

   A number is divisible by 5 if its last digit is a 0 or 5.

   (a) Give several examples of this divisibility test using numbers greater than 100.

   (b) Use models (units, longs, flats, etc) to show why this test works.

   (c) Do you think this test would work the same way in bases other than ten? Explain.

2. **Divisibility by 9**
   Another commonly known divisibility test is for 9. The tests states that:

   A number is divisible by 9 if, when you add the digits of the number together, ignoring place value, the sum is divisible by 9.

   (a) Give several examples of this divisibility test using numbers greater than 100.

   (b) Use models (units, longs, flats, etc) to show why this test works.

   (c) Using models similar to those above, come up with and justify a divisibility test for 3.

3.7 Divisibility Tests in Other Bases

Divisibility tests are usually specific to a certain base. If you change the base in which you are working, you need to develop a new set of tests in that base. In this activity you will work in base six to develop several divisibility tests.

Objectives:

- To utilize inductive reasoning and models in justifying divisibility tests.
- To understand the connections between divisibility and place value.

I. **Divisibility in Base Six**
   You have previously done multiplication and division in base six. Building on those activities, and your knowledge of the models for a base six numeration system, complete the following activities.

   (a) Develop a divisibility test for 2 in base six. Give examples and justify your test.

   (b) Develop a divisibility test for 3 in base six. Give examples and justify your test.

(c) Develop a divisibility test for 4 in base six. Give examples and justify your test.

(d) Develop a divisibility test for 5 in base six. Give examples and justify your test.

II. **Group Project: Find a General Divisibility Test**

Devise a divisibility test for $n - 1$ in base n. For example, your test should be able to determine if a number is divisible by 2 in base 3, by 3 in base 4, etc. Use place-value models (units, longs, flats, etc) to justify your test. Summarize your test and its justification in a 1-2 page paper.

3.8   Modeling the GCD and the LCM

The greatest common divisor (GCD) and least common multiple (LCM) are important concepts that appear frequently. It may seem that they are just the result of some esoteric computation. However, they have a very real and concrete meaning. In this activity, we will reinforce this meaning by using models to represent GCDs and LCMs.

Objectives:

- To understand how GCDs and LCMs can be modeled with concrete representations.
- To understand the relationship between the GCD and LCM models.

I. **Greatest Common Divisors**

In the linear model for the greatest common divisor, we represent numbers visually with Cuisenaire rods. In this model, the GCD is the number associated with the longest rod length which can form trains equal to the length of the two given rods. Consider the following example.

Find GCD(6,4):

- The dark green rod is equal to a train of red rods.

Dark Green = 6		
Red = 2	Red = 2	Red = 2

- The purple rod is equal to a train of red rods.

- No longer rod can be used to build a train equal to both of these.

Purple = 4	
Red = 2	Red = 2

(a) Use the provided Cuisenaire rods to model and find the greatest common divisor of each set of numbers. Note that you can represent numbers larger than 10 by a train of rods.

   i. GCD(6,9)   iii. GCD(24,18)

   ii. GCD(12,8)   iv. GCD(10,15,24)

(b) If one builds a train from shorter rods that is as long as possible without being longer than the longest rod, then the rod equal to the difference between the two is called the difference rod. Consider the following example.

Consider 10 and 4:

- The orange rod represents 10

- The purple rod represents 4, and a train of two is sill less than orange.

- The difference rod is then red.
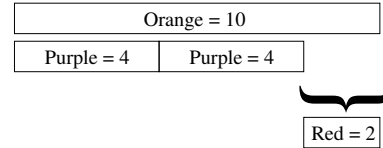
Find the difference rod for each of the following pairs of numbers. Again, you may need to represent larger numbers with a train of rods.

   i. 15 and 6   ii. 16 and 6

(c) Now find the GCD of the pairs of numbers above, and of the smaller number and the difference. In our example above, we would find GCD(10,4) and GCD(4,2).

(d) You should have noticed that the GCDs were the same. Use the model to explain why this will always be true.

(e) What method of finding GCDs is this process modeling?

II. **Least Common Multiples**

In the linear model for the least common multiple, we again use Cuisenaire rods to represent the various numbers. This time, however, we want to find the shortest rod possible that can be represented as a train of either starting rod. For example:

Find LCM(4,6):

- The dark green rod represents six.

- The purple rod represents four.

- The shortest length that can be build as a train of sixes or fours is 12.
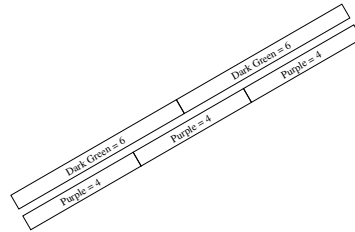
(a) Use the provided Cuisenaire rods to model and find the least common multiple of each set of numbers.

    i. LCM(3,5)   iii. LCM(8,18)

    ii. LCM(8,12)   iv. LCM(4,6,10)

(b) How do these models show the relationship between the GCD and LCM of two numbers? Consider the example below and then write a sentence explaining.

- GCD(4,6) = 2

- 2 divides dark green into 3 parts and purple rod 2 parts.

- LCM(4,6) = 12

- 12 is a train of 2 dark green rods, or a train of 3 purple rods.
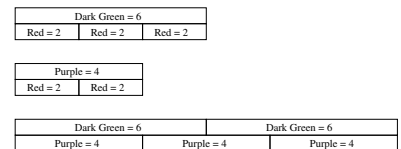
3.9 GCD and LCM Problems and Properties

The greatest common divisor and least common multiple both have many real-world applications. In this activity, you will practice solving problems that use either the GCD, the LCM, or perhaps both. In the second part of the activity, you will answer questions leading to general properties of GCDs and LCMs.

> **Objectives:**
>
> - To understand how GCDs and LCMs are used in solving real-world problems.
> - To generalize from specific examples to general properties of GCDs and LCMs.

I. **Solving Story Problems**

   Solve each of the following problems, explaining your solution. Does this problem involve the GCD or the LCM? Explain.

   (a) Pencils come in packages of 18; erasers that fit on top of these pencils come in packages of 24. What is the smallest number of pencils and erasers that you can buy so that each pencil can be matched with an eraser? How many packages of pencils will you need, and how many packages of erasers? Assume that you can only buy whole packages.

   (b) Sam has a bag with 45 red candies and another bag with 30 green candies. He wants to make identical snack bags containing both red and green candies, and using up his entire supply of both types of candy. What is the largest number of snack bags that Sam can make this way? How many red candies and how many green candies will go in each snack bag?

   (c) Mary has lots of 8-inch sticks that she is placing in a line end-to-end. Tom has lots of 12-inch sticks that he is also placing end-to-end to make a line. If Mary and Tom want their lines of sticks to be of equal length, how long could they be? What is the shortest such length?

II. **Properties of LCMs and GCDs**

Answer each of the following general questions about LCMs and GCDs. In each case, give a specific example if possible, and then state a general rule.

(a) What is the largest value that LCM(a, b) could be?

(b) Under what circumstances will LCM$(a,b) < ab$?

(c) What is the largest value that GCD(a, b) could be?

(d) Under what circumstances will GCD$(a,b) = 1$?

3.10 Circle Clocks and Greatest Common Divisors

In a previous activity we looked at circle clocks to better understand patterns in base 10 multiplication. Now that we have introduced the notion of a greatest common divisor, we will revisit the topic. In this activity, we explore patterns in modulo 18 circle clocks and their relationship to the greatest common divisor.

Objectives:

- To understand how patterns can depend on the GCD.
- To connect previous work with the GCD concept.

I. **18-Point Circle Clocks**
Below you will find several 18-point circle clocks. Starting with the "0" dot on top, draw lines connecting every Nth dot (where N will be given to you) until you come back to the starting point. What is the relationship between the number of dots on the clock (18), the value of N, and the number of dots that are connected by lines? The first clock is done for you.
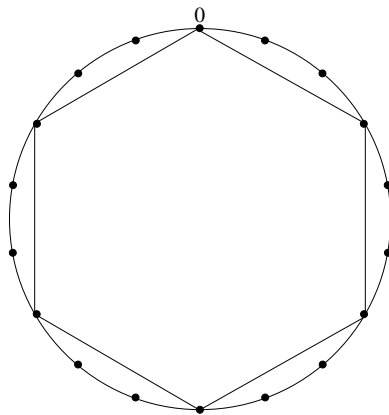
$N = 3$, 6 Dots Connected

$N = 5$

$N = 4$

$N = 6$

$$N = 8$$

$$N = 12$$
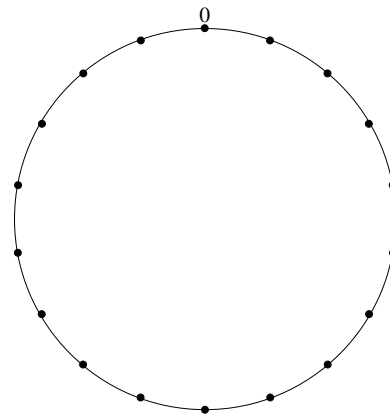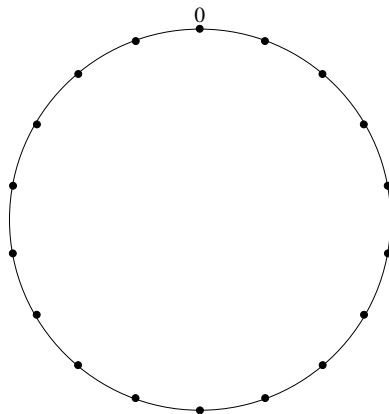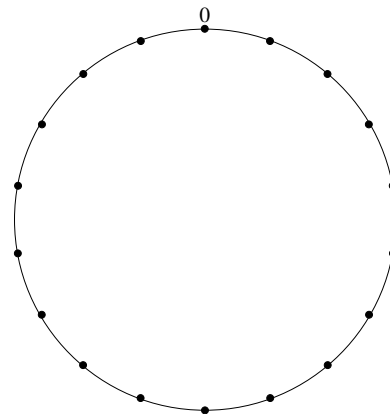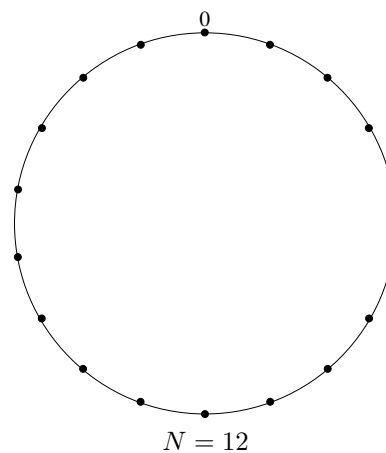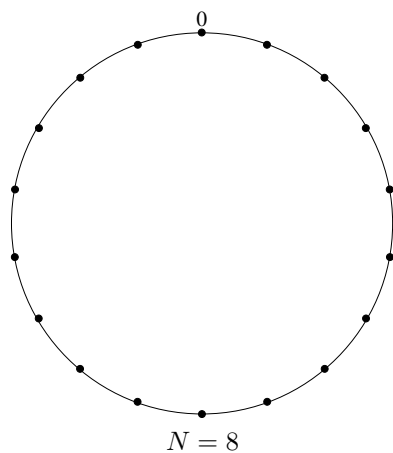
Fill in the first six rows of the table below showing the number of dots connected in each of your clocks. Then use this table to discover the relationship between the number of dots on the clock, the number N, and the number of dots connected.

Dots on Clock	N	Connected Dots
18	3	
18	4	
18	5	
18	6	
18	8	
18	12	

Based on the patterns you observed above, predict the number of dots that would be connected in each of the following situations.

Dots on Clock	N	Connected Dots
18	9	
18	15	
18	16	
24	2	
24	12	
24	15	

71

Chapter 4

Extending the Number System

God created the natural number, and all the rest is the work of man.
– Leopold Kronecker

Of what use is your beautiful investigation regarding pi? Why study such problems when irrational numbers do not exist?
– Leopold Kronecker

Contents

4.1 Modeling Integer Operations

When we introduce the concept of a negative number, several of our operation models need to be adjusted. In this activity, you will experiment with various models to see how negative numbers are represented.

Objectives:

- To realize that modeling integer operations require careful attention.
- To be able to justify operations on integers using various models.

I. **Integer Addition**
   Earlier in this class, you worked with the set and measurement models for addition. We now wish to develop rules to allow us to use these models with negative numbers as well. Work as a group to develop models for the following problems.

   For each of the problems below, give a set model, a measurement model, and then describe how you had to adjust the models to accommodate negative numbers.

   (a) Model the problem $5 + (-2)$.

   (b) Model the problem $(-7) + 5$.

   (c) Model the problem $(-3) + (-6)$.

II. **Integer Subtraction**
Previously we worked with the Comparison, Take-Away, and Measurement Models for subtraction. Again, adjust these models so that they will work with negative numbers.

For each of the problems below, give two of the three models above and then describe how you had to adjust those models to accommodate negative numbers. You should use each of the three models mentioned at least once.

(a) Model the problem $2 - 5$.

(b) Model the problem $(-3) - 6$.

(c) Model the problem $(-2) - (-4)$.

III. **Integer Multiplication**
Previously we worked with the repeated addition model for multiplication. Adjust this model so that it works with negative numbers. For each of the problems below, give the model and then describe how negative numbers were accommodated.

(a) Model the problem $4 \times (-3)$.

(a) Model the problem $(-4) \times 3$.

(b) Model the problem $(-4) \times (-3)$.

IV. **Integer Division**
   Previously we worked with the partitioning and repeated-subtraction models for division. Adjust these model so that it works with negative numbers. For each problem below, give both models and then describe how negative numbers were accommodated in each.

(a) Model the problem $(-8) \div 2$.

(b) Model the problem $8 \div (-2)$.

(c) Model the problem $(-8) \div (-2)$.

4.2 Absolute Value and Opposites

Now that we are using positive and negative numbers, our numeration system has become a little more complex. Numbers now have both a magnitude (think of this as the number of items in a set model of the number) and direction (think of this as the shading of those items). In this activity we will introduce the concept of absolute value and the opposite of a number.

Objectives:

- Students will understand that integers have both a magnitude and a direction.
- Students will be able to compute absolute values and find opposites.

I. **Absolute Value**
The absolute value of a number is its magnitude. Below are several interpretations of absolute value with respect to two of the models we have used for numeration and operation.

- **Set Model**
The absolute value is the smallest number of objects that can be used to represent the number.
- **Measurement Model**
The absolute value is the distance from the origin of the number line to the number.

In this first part of the activity, you will construct models to help you find and understand absolute values.

(a) Represent each of the values below using a set model.

    i. -2 iii. $2 - 6$

    ii. 3 iv. $7 - 7$

(b) The absolute value of a number n is written as $|n|$. Find the absolute value of the numbers above.

    i. $|-2| =$ ii. $|3| =$ iii. $|2 - 6| =$ iv. $|7 - 7| =$

(c) How can the set model help you find absolute values?

(d) Now use a measurement model to describe these values.

    i. -2 iii. $2 - 6$

    ii. 3 iv. $7 - 7$

(e) How can the measurement model help you find absolute values?

(f) Which model do you find most helpful for understanding absolute value. Explain why.

II. **Magnitude and Direction**
In the introduction to this activity, we said that integers have both a magnitude and direction. Based on your exploration of absolute value above, answer the following questions about this idea.

(a) Is the absolute value of a number its magnitude or its direction? Justify your answer.

(b) The term "opposite" refers to the direction of an integer. For example, the opposite of -3 is the number with the same magnitude, but in the *opposite* direction. Explain how the opposite of an integer is related to its absolute value.

4.3 Modeling Fractions
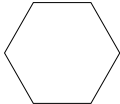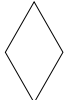
Whole numbers have a very concrete interpretation. But fractions are not as easy to understand. There are, however, several ways to model fractions. In this activity, you will use pattern blocks, Cuisenaire rods, and geoboards to better understand the meaning of the numerator and denominator of a fraction.

Objectives:
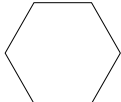
- Students will become familiar with several different manipulatives.
- Students will learn to apply fraction concepts with these different manipulatives.
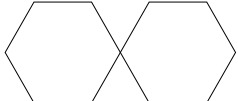
I. **Pattern Blocks**

Pattern blocks are mathematical manipulatives designed to allow students to see how shapes can be decomposed into other shapes. They are divided into two sets. The first set contains the green, blue, red, and yellow blocks. Each of these shapes can be built out of the green equilateral triangle. The second set contains the orange and beige blocks, which can not be built from the green triangle. These blocks can be used to represent fractions as shown in the examples below.
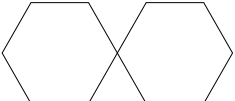
If This Represents 1	This Represents $\frac{1}{2}$	Explanation
		It takes two of the trapezoids to make one of the hexagons. Therefore, if the hexagon represents one, the trapezoid is $\frac{1}{2}$.
		It takes two triangles to make the diamond. So if the diamond represents one, the triangle represents $\frac{1}{2}$.

In each of the problems below, sketch the model for the given fraction using the representation of one given to you. Then write an explanation of your model.
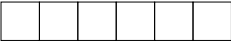
(a) If $\quad$ = 1, sketch $\dfrac{5}{6}$.

(b) If $\quad$ = 1, sketch $\dfrac{2}{3}$.

(c) If $\overline{}$ = $\dfrac{3}{4}$, sketch 1.

(d) If $\bigcirc\bigcirc$ = $1\dfrac{1}{3}$, sketch 1.

II. Cuisenaire Rods

Cuisenaire rods are a versatile mathematics manipulative used for modeling many different mathematical phenomena. Each colored rod is the same length as a certain number of white "unit" rods. These rods can be used to represent fractions in a linear model, as shown below.
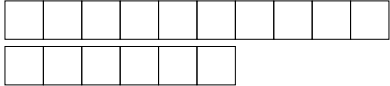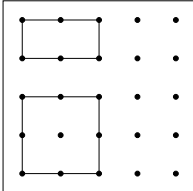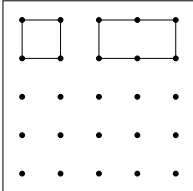
If This Represents 1	This Represents $\frac{1}{2}$	Explanation
▭▭	�	The unit rod is twice as long as the representation, so it must represent $\frac{1}{2}$.
▭▭▭▭▭▭	▭▭▭	It takes two of the three-white-long rods to make a six-white-long rod, which represents 1. So the three-long represents $\frac{1}{2}$.

In each of the problems below, sketch the model for the given fraction using the representation of one given to you. Then write an explanation of your model.

(a) If ▭▭▭▭▭▭ = 1, sketch $\dfrac{5}{6}$.

(b) If ▭▭▭▭▭▭▭▭▭ = 1, sketch $\dfrac{2}{3}$.

(c) If $\boxed{} = \dfrac{3}{4}$, sketch 1.
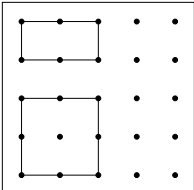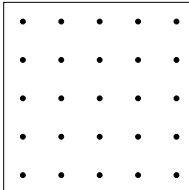
(d) If $\boxed{} = 1\dfrac{1}{3}$, sketch 1.
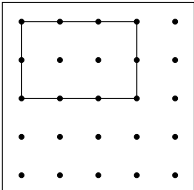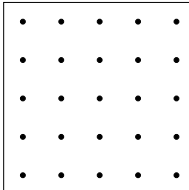
III. **Geoboards**

A geoboard is a manipulative often used to explore basic concepts in geometry. In this activity, however, we will use it to model fractions using an area model as shown below.

If This Represents 1	This Represents $\frac{1}{2}$	Explanation

If we take the original area enclosed in the unit picture and cut each region in half, we get the picture representing $\frac{1}{2}$.

In each of the problems below, sketch the model for the given fraction using the representation of one given to you. Then write an explanation of your model.

(a) If $= 1$, sketch $\dfrac{5}{6}$.

(b) If $= 1$, sketch $\dfrac{2}{3}$.

(c) If $\boxed{} = \dfrac{3}{4}$, sketch 1.

(d) If $\boxed{} = 1\dfrac{1}{3}$, sketch 1.
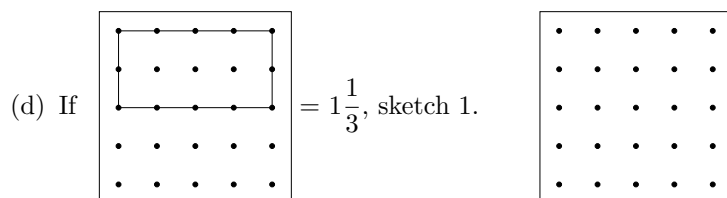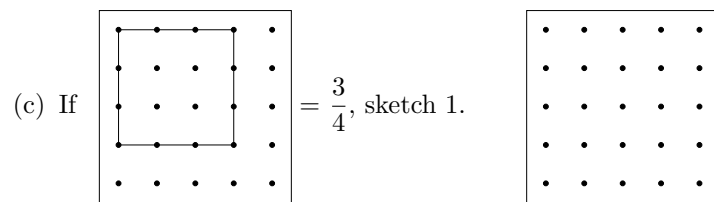
4.4   Developing Fraction Sense

Most of us have a good sense for whole numbers, and maybe even integers. However when it comes to fractions, we often find it hard to tell when one fraction is bigger than another, or what the approximate value of a given fraction will be. In this activity, you will work to develop your fraction sense by constructing fractions that have given properties.

Objectives:

- Students will develop a sense for the value and relation of fractions.
- Students will practice constructing fractions with given properties.

I. **Smallest and Largest**
   The problems below ask you to use a set number of digits (selected from 0-9) to construct a fraction that is as small or as large as possible. Assume all fractions represent positive rational numbers.

   (a) Using two different digits, construct the smallest positive fraction possible.

   (b) Using two different digits, construct the largest possible fraction not equal to a whole number.

   (c) Construct the smallest positive fraction possible using:

      i. three different digits.

      ii. four different digits.

      iii. five different digits.

      iv. eight different digits.

   (d) Construct the largest possible fraction not equal to a whole number using:

      i. three different digits.

      ii. four different digits.

      iii. five different digits.

      iv. eight different digits.

II. **Closest to a Target**
   The problems below ask you to construct a fraction using a given number of digits that is as close to a target as possible, or in a given range. Assume that all fractions need to be positive rational numbers.

   (a) Construct the fraction closest (but not equal) to 2 using two different digits.

   (b) Construct a fraction between $\dfrac{1}{4}$ and $\dfrac{3}{8}$.

   (c) Construct the fraction half-way between $\dfrac{7}{8}$ and 1.

   (d) Construct the fraction half-way between $\dfrac{1}{3}$ and $\dfrac{1}{2}$.

   (e) Construct the fraction closest to, but not equal to $\dfrac{9}{10}$ using four not-necessarily different digits.

   (f) Construct the fraction closest to $\dfrac{1}{2}$ using six different digits.

III. **Comparing Fractions**
   In the final part of this activity, suppose that you are given two fractions, $\dfrac{a}{b}$ and $\dfrac{c}{d}$ and told that $\dfrac{a}{b} < \dfrac{c}{d}$. Answer the following questions.

   (a) If $b = d$, what do you know about a and c? Explain.

   (b) If $a = c$, what do you know about b and d? Explain.

Adapted from Frank Lester, Jr. *Mathematics for Elementary Teachers via Problem Solving*, pages 159-161.

4.5   Interpreting Fractions

Fractions can be interpreted in several different ways. In this activity, we will use the linear model in the form of Cuisenaire rods to examine each of those interpretations.

Objectives:

- Students will be able to interpret and model fractions in two different ways.
- Students will understand the difference between units and wholes.

I. **Cuisenaire Rod Models**

There are at least two ways to interpret a fraction whose numerator is not one. The fraction $\frac{3}{5}$, for example, can be thought of as either (a) three one-fifths, or (b) three divided into five parts of equal size. These two interpretations are modeled quite differently.

- Begin with one unit, divide it into five pieces of equal size. Three of those pieces represent $\frac{3}{5}$.
- Begin with three units, divide that into five pieces of equal size. One of those pieces represents $\frac{3}{5}$.

Use Cuisenaire rods to model **both interpretations** of the following fractions. Be sure that you use the same color rod to represent one unit in both interpretations. Draw sketches of your models.

(a) $\frac{2}{3}$

(c) $\frac{5}{4}$

(b) $\frac{5}{6}$

(d) $\frac{10}{3}$

2. **Other Models**

As a group, pick your favorite model (area model or set model) other than the linear model, and then use pictures or a manipulative to show **both interpretations** of a fraction are equivalent. Make up your own examples and sketch your models below.

4.6   Units vs. Wholes

While fractions are most commonly thought of as relating parts and wholes, this thinking is somewhat simplistic. When we try to solve more complicated problems involving fractions, it can lead to confusion. In this activity you will solve several story problems involving fractions. In each problem, you will run into fractions that may not be given in terms of the same wholes.

Objectives:

- Students will practice solving story problems involving fractions.
- Students will learn that the part-whole relationships in fractions are relative.

I. **Absent Students**

A teacher friend of yours tells you that yesterday a particularly large number of students were absent from her class. She says that "$\frac{2}{5}$ of the boys and $\frac{1}{5}$ of the girls were absent."

(a) What fraction of her class was absent? Explain how you solved this problem.

(b) What if you knew there were the same number of boys as girls in her class. Would this change your solution and answer?

II. **How Much is Your Share?**
   You are renting a house with four other students. The utility bills are due every two months. The four other residents moved in on January 1st, and you moved in on February 1st. In March you receive a utility bill for January and February.

   (a) What fraction of the bill should you pay? What fraction will the other four residents pay?

   (b) What assumptions did you make in order to solve this problem?

4.7   Fraction Sense on a Number Line

It can be difficult to accurately compare fractions when they have different denominators. In this activity, you will practice making these comparisons and estimating the location of fractions on a number line.

> **Objectives:**
>
> - Students will understand how fractions are represented on a number line.
> - Students will practice estimating and discover multiple ways to compare fractions.

I. **Estimation on a Number Line**
   Number line models are especially useful for comparing fractions or estimating relative positions. The goal of this exercise is to get you to estimate the position of a fraction on a standard number line **without converting the fraction to decimals or performing the given operation**. For each of the lettered values below,

   - Mark the location of the value with its letter (A-H).
   - Write a sentence explaining how you estimated the location of the value.
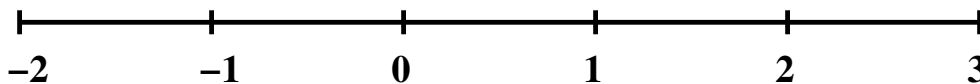
A. the number $\frac{1}{4}$

B. $\frac{1}{4}$ of the number 2

C. the number $\frac{3}{4}$

D. $\frac{3}{4}$ of $\frac{1}{2}$

E. the number $\frac{7}{3}$

F. the number $-\frac{3}{2}$

G. $\frac{1}{2} + \frac{2}{3}$

H. $\frac{2}{3} - 1$

II. **A Number Line Without Numbers**

The numbers a and b are indicated on the number line below. Mark each of the following values with the appropriate letter A through H. Do not try to determine the numerical values of a and b, rather estimate using their relative positions. Again, give a short description of how you estimated the location for each value.

$$\overset{\textstyle a \qquad \textstyle 0 \qquad\qquad \textstyle b}{\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!+\!\!\!-\!\!\!+\!\!\!-\!\!\!-\!\!\!-\!\!\!+\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv}$$

A. $\dfrac{3}{4}a$

B. $\dfrac{1}{2}b$

C. $-\dfrac{1}{3}a$

D. $\dfrac{7}{6}b$

E. $\dfrac{6a}{5}$

F. $-\dfrac{11a}{12}$

G. $\dfrac{b-a}{2}$

H. $\dfrac{a+b}{3}$

4.8   Discovering Base $\frac{1}{2}$

Positional notation is important not only for whole numbers, but also for numbers with digits to the right of the "decimal" point. In this activity we explore how place value affects such numbers by first looking at "decimals" in base two, and then flipping things around to examine base one-half.

Objectives:

- To develop a deeper understanding of base and place value.
- To understand place value to the right of the "decimal."

I. **"Decimals" in Base Two**
Recall that in the base ten system, the digits to the right of the decimal point have the following place values. Observe the relationship between these place values and the base 10.

Decimal Place	Name	Value - Fraction	Value - Exponent
first	tenths	1/10	10^{-1}
second	hundredths	1/100	10^{-2}
third	thousandths	1/1000	10^{-3}
fourth	ten-thousandths	1/10000	10^{-4}

Use this information to help you answer the following questions about the base two numeration system.

(a) What are the first four place values to the right of the "decimal point" in the base two system?

(b) Convert the following base two numerals to base ten.

   i. 0.1_{two} ii. 10.11_{two} iii. 111.011_{two}

(c) Convert the following base ten numerals to base two.

   i. 10.5   ii. 0.75   iii. 2.25

II. **Base One-Half**

An interesting thing happens if we use a unit fraction as a base. For example, in base one-half, the numeral $101_{\text{one-half}}$ would have the following expanded representation.

$$101_{\text{one-half}} = 1 \times \left(\frac{1}{2}\right)^2 + 0 \times \left(\frac{1}{2}\right)^1 + 1 \times \left(\frac{1}{2}\right)^0 = \frac{1}{4} + 0 + 1 = 1.25$$

Use this example to help you complete the following activities.

(a) Write out the first ten counting numerals in base one-half.

(b) Convert the following base one-half numerals to base ten.

    i. $11_{\text{one-half}}$ ii. $10.1_{\text{one-half}}$ iii. $0.101_{\text{one-half}}$

(c) How is the base one-half representation of a number related to the base two representation? Provide an example to support your claim.

III. **Group Project: Explore Base $\frac{1}{10}$**

Now that you have some experience with fractional bases, your project is to apply that to the more familiar base 10 setting. Your task is to create a poster describing and demonstrating the base $\frac{1}{10}$ numeration system. You poster must contain the following:

- a description of the base $\frac{1}{10}$ numeration system.
- a list of the first ten counting numerals in base $\frac{1}{10}$.
- an example of converting a number with and without a "decimal" from base 10 to base $\frac{1}{10}$.
- an example of converting a number with and without a "decimal" from base $\frac{1}{10}$ to base 10.
- a description of the relationship between numerals in base 10 and in base $\frac{1}{10}$.

Adapted from Frank Lester, Jr. *Mathematics for Elementary Teachers via Problem Solving*, 40.

4.9   Modeling Operations on Fractions

Operations on fractions can be confusing. This is partially due to the fact that fractions are more complicated, carrying information about both the parts and the whole. In this activity, you will extend the models of fraction values that we used earlier to show how to perform addition, subtraction, multiplication, and division with fractions.

Objectives:

- Students will review and better understand the procedures for addition, subtraction, multiplication, and division of fractions.
- Students will explore a variety of models for explaining and justifying these procedures.

I. **Modeling Addition and Subtraction**
   Use the manipulatives of your choice (Cuisenaire rods, pattern blocks, geoboards, sets, etc.) to illustrate how and why each of the following addition and subtraction operations work. You should not only illustrate the specific example, but also describe a general procedure for performing the operation.

   (a) Adding fractions with the same denominators. For example: $\frac{3}{7} + \frac{2}{7}$.

   (b) Adding fractions with different denominators. For example: $\frac{2}{3} + \frac{2}{5}$.

   (c) Subtracting fractions with different denominators. For example: $\frac{3}{4} - \frac{1}{3}$.

II. **Modeling Multiplication**
   The actual procedure for multiplying fractions is often easier to remember than the procedure for adding or subtracting fractions. Use the manipulatives of your choice to illustrate the specific example and describe in general how to model and perform fraction multiplication.

   (a) $\dfrac{1}{2} \times \dfrac{1}{3}$

   (b) $\dfrac{4}{5} \times \dfrac{3}{8}$

III. **Modeling Division**
   Fraction division is often the hardest operation to model and understand. Recall that there are several ways to interpret the division of integers For example, $6 \div 3$ can be thought of as:

   i. The number in each group if six items are divided into three groups of the same size.
   ii. The number of groups if six items are divided into groups of three.
   iii. The number of times that three can be subtracted from six.

   Use any of the interpretations above and Cuisenaire Rods to model the following division problems. Give the quotient and remainder (if any) for each problem and sketch your models.

   (a) $4 \div \dfrac{1}{2}$

93

(b) $\dfrac{2}{3} \div \dfrac{1}{3}$

(c) $\dfrac{5}{6} \div \dfrac{2}{3}$

(d) $\dfrac{3}{4} \div \dfrac{2}{3}$

4.10   Fraction Games

In a previous activity, you had the opportunity to develop operation sense by playing games with whole numbers and operations. In this activity, we will repeat this using fractions.

Objectives:

- Students will develop a sense of the effect of the four arithmetic operations on fractions.
- Students will practice deductive, inductive, and/or intuitive reasoning.

I. **The Game of Maximize**
   Recall that in the game of maximize, your goal is to use the numbers and operations rolled to make as large a value as possible. Roll your fraction die three times to generate three random fractions, and roll your operations die twice. Then construct an expression with as large a value as possible.

(a) Play this game several times as a group, recording your rolls and your expression value below.

   Game 1: Fractions: _____, Operations: _____   Maximum Value: _____

   Game 2: Fractions: _____, Operations: _____   Maximum Value: _____

   Game 3: Fractions: _____, Operations: _____   Maximum Value: _____

   Game 4: Fractions: _____, Operations: _____   Maximum Value: _____

   Game 5: Fractions: _____, Operations: _____   Maximum Value: _____

   Game 6: Fractions: _____, Operations: _____   Maximum Value: _____

(b) Now come up with a general strategy for playing this game. Does your strategy depend on the specific fractions rolled? Does it depend on the operations rolled?

2. **The Game of Target**
   Another game that we played in a previous activity was the game of target. In this game, your goal is to get as close as possible to the given target using three numbers, rolled on your fraction dice, and two operations.

   (a) Play the game several times with each of the targets given below.

   Target $= \frac{1}{2}$

   Game 1: Fractions: _____, Operations: _____,   Closest Value: _____

   Game 2: Fractions: _____, Operations: _____,   Closest Value: _____

   Game 3: Fractions: _____, Operations: _____,   Closest Value: _____

   Target $= 1$

   Game 1: Fractions: _____, Operations: _____,   Closest Value: _____

   Game 2: Fractions: _____, Operations: _____,   Closest Value: _____

   Game 3: Fractions: _____, Operations: _____,   Closest Value: _____

   Target $= 10$

   Game 1: Fractions: _____, Operations: _____,   Closest Value: _____

   Game 2: Fractions: _____, Operations: _____,   Closest Value: _____

   Game 3: Fractions: _____, Operations: _____,   Closest Value: _____

   (b) Again, after playing the game several times for each target, describe a general strategy that will help you get as close as possible to the target value.

4.11 Decimals and Base 10 Blocks

Many students have only a procedural understanding of decimals. This means that while they can perform operations, they don't really understand why these operations work the way they do. One of the best ways to understand why something works, and thus be able to remember how it works, is to have a picture or model to associate with the process. In this activity you will practice using models to perform and justify various operations on decimals.

Objectives:

- Students will apply their understanding of base 10 block models for whole numbers to decimal representations.
- Students will be able to justify each step of decimal computation algorithms using manipulatives.

I. **Connecting Decimals to Base 10 Blocks**
In this first activity you will use base 10 blocks to model various operations on decimals. These base 10 blocks are pictured below.
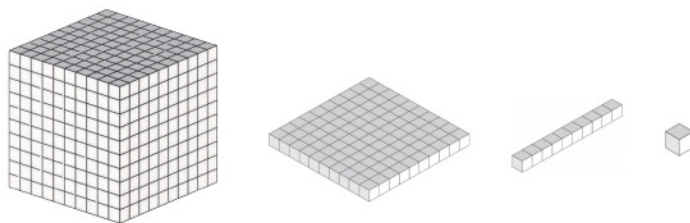
Use these models to answer each of the questions below. Include a sketch of the model you used to help you answer the question, as well as a sentence justifying your answer.

(a) If the flat has a value of 1, represent 0.13 in two different ways.

(b) Represent 2.13 in two different ways by changing the manipulative you designated as the unit.

(c) Represent 0.0412 in three different ways. Justify your answers.

(d) Demonstrate $0.132 + 0.308$ with base 10 blocks.

(e) Demonstrate $0.402 - 0.135$ with base 10 blocks.

(f) Show why $0.1 \times 0.1 = 0.01$ with base 10 blocks.

(g) Demonstrate the equality of 0.2, 0.20, and 0.200 with base 10 blocks.

II. **Comparing Decimals**
   Now use base ten blocks to help you decide which of each pair of numbers below is larger. Include a sketch of your representation as well as an indication of what the unit is in your model.

   (a) 0.3 and 0.15

   (d) 0.05 and 0.042

   (b) 2.4 and 2.11

   (e) 4.31 and 4.3101

   (c) 3.24 and 3.2151

4.12 The Right Bucket

Most people have a reasonably good ability to estimate how large a product of two whole numbers will be. Decimals, however, are usually more difficult because of the change in decimal place. For example, 1×0.1 is larger than 2×0.02 even though 1×1 is smaller than 2×2. In this activity, you will play a game to help you improve your decimal estimation skills.

Objectives:

- Students will develop their ability to estimate with decimals.
- Students will better understand decimal multiplication.

I. **The Right Bucket Game**

In the "right bucket" game, you pick numbers from a list, trying to get their product to be between 1 and 10 and scoring points for how close you actually make the product. As the purpose of this game is to help you learn to estimate decimal products, it is important that you **not use a calculator** during the estimation phase of the game. You may use a calculator when computing the actual product.

Game Rules

1. Select two numbers from the list whose product you estimate will be in the highest scoring bucket shown below.

2. Record your numbers, estimate their product, and explain your reasoning. Then cross the numbers off of the list.

3. Find the actual product and then determine your score for that round of the game by finding the bucket into which your product fell.

4. Repeat this process until you run out of numbers in your list.

5. Total your points to find your score for the game.

Consider the following example in which the numbers 52.3 and 0.09 were selected.

Decimals	Estimate	Reasoning	Product	Points
52.3 and 0.09	5.23	0.09 is about 1/10, and I know that 1/10 of 52.3 is 5.23, which puts us between 1 and 10.	4.707	3

(a) Game 1:

Decimals for Game 1

| 3.6 | 0.03 | 13.1 | 29.6 | 11.9 | 0.7 | 33.7 | 0.04 | 21.9 | 0.125 |
| 10.1 | 0.42 | 0.07 | 2.9 | 0.29 | 19.5 | 5.52 | 23.1 | 9.6 | 1.8 |

Decimals	Estimate	Reasoning	Product	Points
			Total:	

(b) Game 2:

Decimals for Game 2

| 0.024 | 0.9 | 5.74 | 0.245 | 7.3 | 1.2 | 23.8 | 16.5 |
| 0.068 | 8.7 | 0.12 | 1.8 | 0.78 | 42.7 | 0.2 | 3.6 |

Decimals	Estimate	Reasoning	Product	Points
			Total:	

4.13   Understanding Ratios

Fractions and ratios are often confused by beginning mathematics students because rations can be written as fractions. A true fraction, however, represents a part-whole relationship. On the other hand, ratios represent part-part relationships. In this activity you will gain experience working with and modeling ratios.

Objectives:

- Students will know what is meant by the *ratio* of two quantities.
- Students will be able to construct physical models of ratios.

I. **Understanding Ratios**

Suppose that two-thirds of the students in a particular class are female. Another way to describe this is to say that there are two females for every one male. In mathematical language, we say that the *ratio* of females to males is 2 to 1. This could be written as "2:1" or as the fraction "$\frac{2}{1}$."

In each of the situations described below, decide which of the statements that follow are true. Justify your answers with examples.

(a) The Humane Society has kittens and puppies available for adoption. On a given day, the ratio of the number of kittens to puppies available for adoption is 3:5. Which of the following statements are true? Justify your answers with examples.

  i. Three-fifths of the pets available for adoption are puppies.

  ii. There are three-fifths as many kittens available as puppies.

  iii. The ratio of the number of puppies to the number of kittens is five to three.

  iv. Five-eights of the pets available for adoption are puppies.

  v. There are three puppies for every five kittens available.

  vi. If there are a total of 30 pets in the shelter, 18 of them are kittens.

(b) A punch recipe calls for lemonade, limeade, and club soda in a ratio of 3 to 2 to 4. Which of the following statements are true? Again, justify your answers.

    i. Eight cups of limeade are used in making 36 cups of punch.

    ii. Two-sevenths of each batch of punch is limeade.

    iii. There is twice as much club soda in each batch of punch as limeade.

    iv. One-third of every batch of punch is lemonade.

II. **Modeling Ratios**
Many of the same models that can be used for general fractions work with ratios. In this activity, you will use Cuisenaire rods to construct models for several different ratios. Sketch a picture of your model in each situation.

(a) You have a problem. Your swim team is composed of boys and girls in a ratio of 5 to 7. You need to know what fraction of the team is male. Use Cuisenaire rods to show why the answer is $\frac{5}{12}$.

(b) A trail mix calls for granola, nuts, and apple chips in a ratio of 4 to 1 to 2. Use Cuisenaire rods to show that $\frac{4}{7}$ of every batch of trail mix is granola, $\frac{1}{7}$ nuts, and $\frac{2}{7}$ apple chips.

III. **Generalizing Ratios**

Generalize your solutions and models from the previous problems by answering the following questions. Use models or algebra as appropriate, but do not assume specific values for your variables.

(a) A mixture consists of two ingredients A and B in a ratio of a to b. What fraction of the mixture is A? What fraction of the mixture is B?

(b) Suppose that a mixture consists of three ingredients, A, B, and C in a ratio of a to b to c. What fraction of the mixture is A? What fraction is B? What fraction is C?

(c) Is it possible to find two whole numbers in a ratio of 7 to 9 whose sum is 562? If so, find them. If not, explain why.

4.14 Solving Proportions

A proportion is a statement that two ratios are equal. While a proportion may look like an equation involving two numbers, it actually gives a relationship between four numbers. In this activity you will learn to model proportions, and then learn to solve proportions in which one or more of the four numbers is unknown.

Objectives:

- Students will know what is meant by a proportion.
- Students will learn several ways to solve proportions.

I. **Modeling Proportions**

Consider the following example of a proportion. As you read through the example, think carefully about what it means.

A class has 10 boys and 20 girls. The ratio of boys to girls is therefore 10:20. However, it would also be correct to express this ratio as 1:2. Equating these two ratios gives the proportion: 10:20 = 1:2.

To explore this idea, select the color of Cuisenaire rod which makes the following proportions true.

(a) purple : brown = yellow : _____

(b) red : brown = white : _____

(c) blue : light green = _____ : _____, or

_____ : _____

(d) red : light green = _____ : dark green

(e) dark green : brown = _____ : purple

(f) red : yellow = purple : _____

II. **Solving Proportions without Algebra**

Proportions can be used to solve a wide variety of problems. Even if you do not have the tools of algebra (as would be the case with elementary school children), you can use proportions "by inspection." Consider the following example.

A proposed state regulation for pre-schools would require two licensed teachers be hired for every five children under the age of three. How many teachers would be needed for a class of 20 two-year-olds? Letting t represent the number of teachers, we get the proportion:

$$\frac{2}{5} = \frac{t}{20}.$$

Since the denominator of right-hand fraction is four times the denominator of the left-hand fraction, we need to multiply the numerator of the left-hand fraction by four to get the numerator of the right-hand fraction, meaning $t = 2 \times 4 = 8$.

Use this method to solve each of the problems on the next page.

(a) David read 30 pages of a book in 35 minutes. How many pages can he read in 70 minutes?

(b) If 7 grapefruit sell for $1.40, how many grapefruit can you buy for $4.00?

(c) Donna scored 75 goals in her soccer practice. If her success-to-failure ratio is 5:4, how many attempts were unsuccessful?

(d) Miss Shake weighs 142 pounds on Earth and 426 pounds on Jupiter. What is Amy's weight on Earth if her weight on Jupiter is 387 pounds?

(e) What percent of 78 is 13?

(f) Thirty-five percent of the students at Jefferson High drive to school. If 105 parking stickers were issued to the students who drive, how many students are in the school?

III. **Solving Proportions by Cross Multiplying**
The proportions in the preceding problems always had one ratio which was an even multiple of the other. This is rarely the case when solving real-life problems. The algebraic method for solving proportions, as you probably remember, is to "cross multiply." Students in middle school can be taught this method, but their teacher must be prepared to explain *why* it works.

(a) Prove that the proportion $\frac{16}{24} = \frac{x}{8}$ can be rewritten as $128 = 24x$.

(b) Prove that the proportion $\frac{a}{b} = \frac{c}{d}$ can be rewritten as $ad = bc$.

4.15   Percent Change

The idea of a percent change is one that is not very well understood. This can lead to a lot of confusion in important aspects of every day life, such as pricing, taxes, population growth, etc. In this activity you will work through several problems involving percent change.

Objectives:

- Students will be exposed to different meanings of percent change.
- Students will understand that a percent change is relative to some starting point that must be defined.

I. **Additive or Multiplicative Percents**
When we talk about a change in a percent, we must understand the difference between an additive change and a multiplicative change.

In an additive change, we add or subtract one percent to or from another. For example, if 10% of students in a particular class get an A and 20% get a B, then a total of $(10+20)\% = 30\%$ have a grade of B or better.

In a multiplicative change, we actually multiply the percents together to compute the new percent. For example, if a store marks down its original price by 10% and then you have a coupon for an extra 20% of the sale price, then you end up paying $(80\% \times 90\%) = 72\%$ of the original price, so your discount is 28%, not 30%.

Answer the following questions, keeping these differences in mind. In each case, identify if the percents should be combined additively or multiplicatively.

(a) Becky works at Target and gets a 15% employee discount on all purchases–even items on sale. During their recent Black Friday sale, Target had some items priced 85% off. Does this mean that Becky could get them for free?

(b) The current sales tax rate in your state is 10% (just to make things simple). A ballot measure proposes to raise this rate by 5%. A group of anti-tax citizens claims that this means you will pay 15% in sales tax. Another group of pro-government citizens claims that your sales tax would only go up to 10.5%. Who is right?

II. **Combining Percents**

Many older copy machines allowed you to enlarge or reduce a copy by a small number of set percents. Suppose that a copy machine had the following buttons:

$$\boxed{50\%} \quad \boxed{75\%} \quad \boxed{80\%} \quad \boxed{100\%} \quad \boxed{150\%} \quad \boxed{200\%}$$
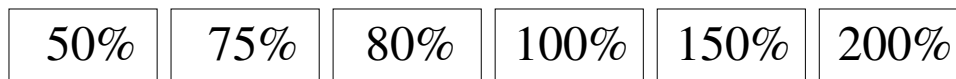
Answer the following questions about this copy machine.

(a) If you made a copy that was 150% the size of the original, and then made a copy of your enlargement that was 50% its size, would your final product be the same size as the original? Explain.

(b) If the 100% button were broken, how could you make a copy that is the same size as your original?

(c) How could you make a copy that is three times as large as the original?

(d) How could you make a copy that is 37.5% the size of the original if the 75% button is broken?

(e) If the 80% button were broken, would it be possible to make a copy that is 80% the size of the original? If so, explain how. If not, explain why not.

Chapter 5

Algebra, Functions, and Graphs

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

– Joseph Louis Lagrange

In mathematics the art of proposing a question must be held of higher value than solving it.

– Geoge Cantor

Contents

5.1 Developing a Calculator Policy

Technology has become an important part of classroom teaching. This can be both a good thing, and it can cause problems. In this activity, you will get the opportunity to read a short story illustrating the potential drawbacks of over-reliance on technology and then respond by developing a school-wide calculator policy.

Objectives:

- Students will consider the implications of calculator usage in elementary classrooms.
- Students will develop a comprehensive calculator policy for an elementary (K-8) school.

I. **The Feeling of Power**

Your instructor will provide you with a copy of *The Feeling of Power*, a short story by Isaac Asimov. After reading this story, you may keep your copy if you like. If you do not want it, please return it to your instructor for later use.

II. **Formulating a Calculator Policy**

Imagine that you are a group of teachers from a K-8 school. Your school's principle has heard that all 'modern" mathematics teachers should have their students use calculators. On the other hand, teachers complain that their students don't know how to do basic arithmetic without a calculator. Students regularly use calculators to compute the value of expressions as simple as '1+1."

The principle has asked your group to come up with a policy on the use of calculators. Your policy should address calculator usage at every grade level from Kindergarten through eighth grade. To help you formulate this policy, answer the following questions as a groop.

(a) What are some potential benefits of having students use calculators?

(b) What are some potential problems with allowing students to use calculators?

(c) Should the calculator policy differentiate between grade levels? If so, how?

(d) Should the calculator policy depend on student performance? If so, how?

III. **Group Project: Write a Calculator Policy**

As a group, write a one-page letter to parents describing the calculator policy. Your letter needs to reflect your answer to the questions above, and must cover all grade levels, K-8. The letter should be professionally formatted with correct grammar, spelling, and punctuation.

5.2 Modeling Algebra with Tiles

Many people find algebra difficult to remember because it can be very abstract. It is not always easy to remember various rules for adding, multiplying, and factoring polynomials which are just lists of symbols. For that reason, it is useful to connect the algebraic concepts with a real-world physical representation. In this activity you will gain experience working with tiles as a model for algebra.

Objectives:

- To practice adding, multiplying and factoring polynomials of degree two or less.
- To model these operations using manipulatives.
- To see that algebra and geometry are connected.

Types of Tiles:
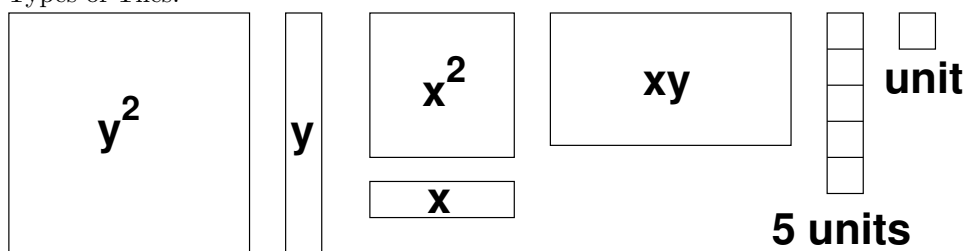
I. **Addition of Polynomials**

We have seen that addition can be modeled by "putting things together" and subtraction by "taking things away." Use these ideas together with the provided algebra tiles to model the following problems. Sketch your model in the space provided. Note that you can designate a "negative" by turning a tile over so that it is red (or shading in your picture).

(a) $6 + (-3)$

(c) $(x^2 - x + 4) + (x^2 + 2x - 6)$

(b) $(x + 3) + (2x - 1)$

(d) $(3x^2 + 2xy + y^2) - (x^2 - xy + 2y^2)$

II. **Solving Equations**

An important concept to keep in mind when solving equations is that each side of the equals sign must remain "balanced" throughout the process. Use algebra tiles to solve each of the following equations.

(a) $2x = 6$

(d) $x^2 = 9$

(b) $3x - 2 = 7$

(e) $(x + 1)^2 = 9$

(c) $2x + 5 = x - 3$

III. **Multiplying Polynomials**

Multiplying polynomials is a little more difficult to model. See if you can develop a method to model these multiplications using algebra tiles (Hint: think two-dimensionally). You instructor can provide further guidance if you get stuck.

(a) $3 \times (x + 1)$

(b) $x \times (y - 2)$

(c) $(x + 1) \times (x + 4)$ (d) $(x + y) \times (x - y + 1)$

IV. **Factoring Polynomials**

Factoring polynomials is undoing multiplication problems such as the ones you worked on above. You should use a similar model in reverse to factor each polynomial below.

(a) $4y - 2$ (b) $x^2 + 3x + 2$

(c) $x^2 + xy + 3x + y + 2$

(d) Not all polynomials can be factored. Use your model to show why $x^2 + x + 1$ can not be factored.

5.3 From Tables to Functions

When mathematics students think of a function, they usually think of the expression that gives the rule for the function. However, functions can be represented in many different ways. In this activity you will explore how tables, graphs, and formulas can be used to represent the same function.

Objectives:

- Students will be exposed to functions represented as tables, graphs, and formulas.
- Students will be able to convert between these function representations.

I. **Functions as Tables**
A certain movie theater has discovered that by changing the price of a small bag of popcorn, they will change the number of bags purchased on a given evening. The table below shows the relationship between price and number sold. Use it to answer the following questions.

Price	Number Sold	Profit
$3.00	100	
$3.50	85	
$4.00	70	
$4.50	55	
$5.00	40	

(a) Describe the relationships shown in this table without using a formula.

(b) Explain why the relationships you observed above seem reasonable (or unreasonable).

(c) Is the information in the table enough to determine the price that the theater should choose for a bag of popcorn? Explain.

(d) Suppose that is costs the theater $1.50 in supplies and labor to produce each bag of popcorn. We can then expand our table to include a third column for the **total profit** the theater would make at each selling price. Complete the column provided for this information in the table above.

II. **Functions as Graphs**

While tables can provided basic input-output values of a function, it is often hard to see an overall picture of what the function does from a table alone. For this reason, graphs are often used to represent functions. Complete the graph below based on the table from the previous page and then use it to answer the following questions.

(a) Plot the profit data from the table on the previous page on the provided graph. Place units on your graph to match your table values.

Total Profit

Price Per Bag
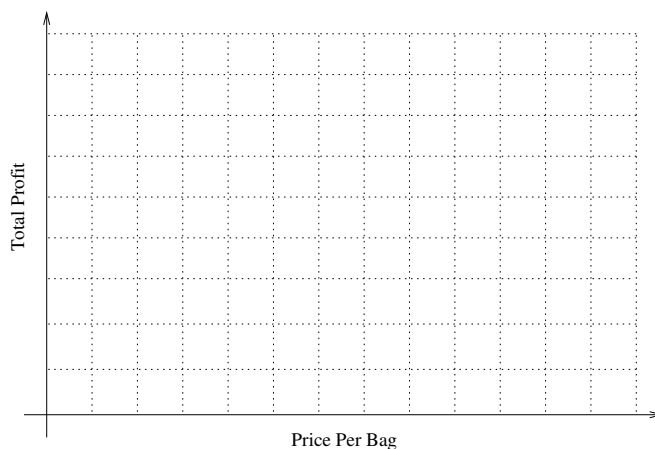
(b) Because the profit earned seems to depend on the cost of a bag, we could use *function notation* to describe the data in the table and graph. For example, because we make a total of $150 when we charge $3.00 per bag, we could write $P(3.00) = 150$ where P is the profit function. Use this notation, along with the table and/or graph, to answer each of the following questions

   i. What is the profit when a bag of popcorn costs $4.00?

   ii. What is the profit when a bag of popcorn costs $4.25?

   iii. What is the profit when a bag of popcorn costs $5.50?

III. **Functions as Formulas**

Functions are most commonly represented as formulas using variables. For example, $f(x) = x + 1$ is the function that adds one to whatever input is given. Use this notation to perform the following tasks.

(a) Write a function $N(c)$ for the number of bags of popcorn sold when the cost of a bag is c dollars.

(b) Write a function $P(c)$ for the total profit earned when the cost of a bag of popcorn is c dollars.

5.4   Interpreting Graphs

It is one thing to construct a graph of a function, but it is another thing entirely to be able to correctly interpret the information provided in a graph constructed by somebody else. In this activity you will be given statements involving a function relationship and asked to identify the graph which accurately represents this relationship.

Objectives:

- Students will think critically to eliminate possibilities.
- Students will able to describe how a graph represents a given relationship.

I. **Speed vs. Time**
   In each of the following situations, you are given a description that relates time to the speed at which something is moving. Determine which of the graphs shown best represents this relationship, and then justify your choice.

(a) A school bus drives up to a school and drops off the children.
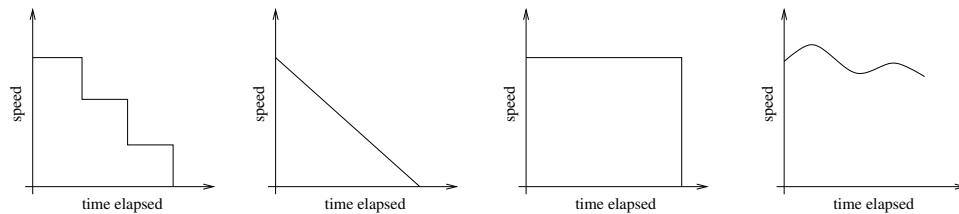
(b) A child climbs a ladder to the top of a playground slides and slides down.
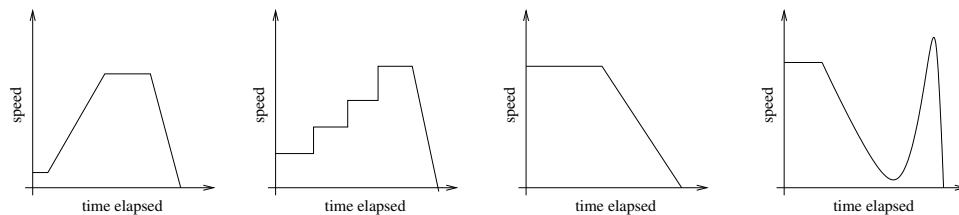
(c) A bicyclist pedals up a hill at a constant rate and then races down the other side.
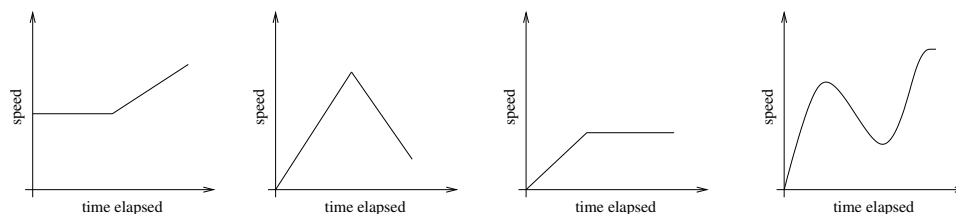
II. **Distance vs. Time**

In the final two examples, we examine a relationship between distance traveled and time. Be sure that you take this change into account as you select the most appropriate graph and justify your answers.

(a) A girl takes a ride on a Ferris wheel.
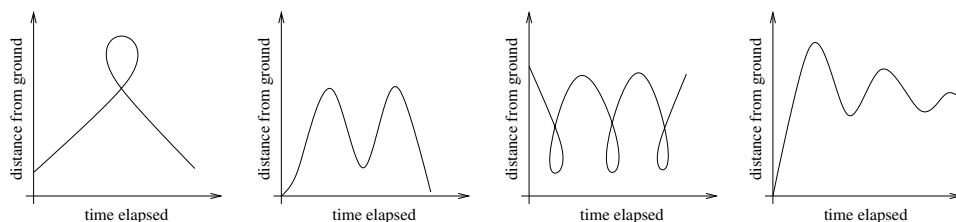
(b) a boy swings on a playground swing.
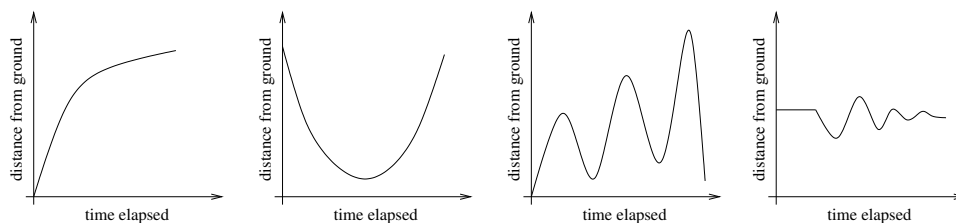
5.5 Growth Patterns in Figurate Numbers

Have you ever thought of numbers as having shapes? The ancient Greeks did. They were one of the first peoples to explore the idea of arranging numbers into regular shapes like triangles, squares, pentagons, etc. Numbers that can be arranged into these shapes are called figurate numbers. In this activity you will work to develop functions for several sets of figurate numbers.

Objectives:

- Students will see the connection between shape patterns and number properties.
- Students will express numbers with given shape patterns using function notation.

I. **Square Numbers**
One of the most basic sets of figurate numbers are the square numbers. These are the numbers that can be represented as perfect squares as shown below.
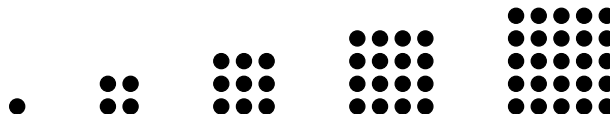
In the example above, you can see that the first five square numbers are 1, 4, 9, 16, and 25–corresponding to the number of dots in each square. Continue this pattern to solve the following problems.

(a) Sketch each of the following square numbers and determine its value (i.e. number of dots).

    i. The 6th square number   ii. The 8th square number

(b) Find a formula for the function $S(n)$ which gives the value of the nth square number. Your formula should make $S(1) = 1$, $S(2) = 4$, etc.

II. **Trinumbers**

A more interesting set of figurate numbers is called the trinumbers. These are the numbers that result from forming an ever larger outline of a triangle, as shown below.
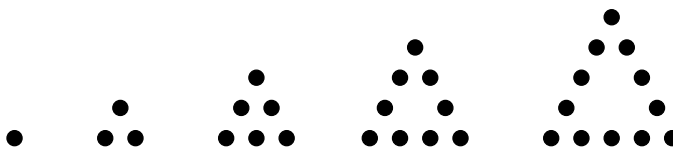
Use this pattern to help answer the following questions.

(a) Sketch each of the following trinumbers and determine its value (i.e. number of dots).

    i. The 6th trinumber   ii. The 8th trinumber

(b) Find a formula for the function $t(n)$ which gives the value of the nth trinumber.

III. **Triangular Numbers**

The final set of figurate numbers we will examine is similar to the trinumbers. Instead of leaving the triangles empty, however, we fill each triangle with dots as shown below.
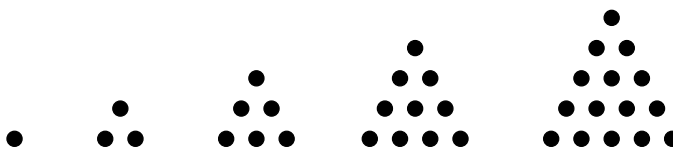
This set of numbers is more difficult to work with than the previous two since seeing the pattern does not lead right away to an obvious formula. You may wish to refer back to some previous activities from chapter one (stacking cereal boxes).

(a) Sketch each of the following triangular numbers and determine its value (i.e. number of dots).

    i. The 6th triangular number   ii. The 8th triangular number

(b) Find a formula for the function $T(n)$ which gives the value of the nth triangular number.

IV. **Recursive Functions**

Each of the functions you found so far in this activity is in "closed form." This simply means that to find the value of the nth figurate number, you can plug n into the function.

Another way to express these functions is recursively. A recursive function starts by giving the first number in the sequence, called the base case, and then gives general instructions for how to go from a given number in the sequence to the next number. For example, consider the square numbers.
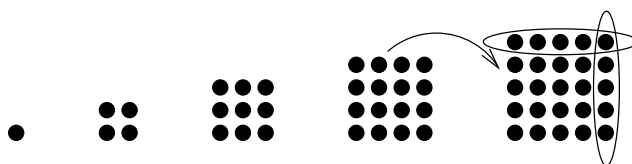
The base case is $S_1 = 1$, because the first square number has one dot in it. The recursive step is $S_n = S_{n-1} + 2n - 1$. This is because when we move to the next square, we add two rows of dots that are n dots long and share one dot in common. So, for example, $S_5 = S_4 + 2(5) - 1 = 16 + 10 - 1 = 25$.

(a) Give a recursive function t_n for the sequence of trinumbers.

(b) Give a recursive function T_n for the sequence of triangular numbers.

5.6   Iterated Functions

When we evaluate a function on a number n to get a value $f(n)$, and then plug that into the same function to get $f(f(n))$ and continue this process to get $f(f(\cdots f(n)\cdots))$ we are iterating the function. Iterating a function can lead to some interesting results with useful applications. In this example we will explore iterating linear functions.

Objectives:

- Students will explore iterating linear functions.
- Students will develop an iterated function for a specific application.

I. **Iterating Linear Functions**

A linear function is one of the form $f(x) = ax + b$. What happens when you iterate a function like this? Does it depend on the values of a and b? Does it depend on the x that we start with? In this first part of the activity, you will get the chance to explore these questions.

(a) Consider the function $f(x) = 2x - 3$. Suppose we want to iterate this function starting with either $x = 2$, $x = 3$, or $x = 4$ as our initial value. Will the starting value make a difference in what the iteration does? Fill in the rest of the following tables to help you make this decision.

Starting with $x = 2$		Starting with $x = 3$		Starting with $x = 4$	
Iterations	Result	Iterations	Result	Iterations	Result
0	2	0	3	0	4
1	1	1	3	1	5
2	-1	2	3	2	7
3		3		3	
4		4		4	
5		5		5	
6		6		6	

You should find that starting with $x = 3$ will always give 3 back, no matter how many times you iterate the function. But starting with $x = 2$ (or any number less than 3), gives smaller and smaller values. Starting with $x = 4$ (or any number bigger than 3) will give larger and larger values. The value $x = 3$ is called a **fixed point** for $f(x)$. More specifically, it is a **repelling fixed point** because for numbers other than $x = 3$, iterating the function gives values that move away from 3.

(b) Will every liner function have a fixed point? Will they all be repelling? Consider the function $g(x) = \frac{1}{2}x + 3$. Starting with $x = 10$, $x = 6$, and $x = -10$ examine the results of iterating the function by again filling in the tables below.

Starting with $x = 10$		Starting with $x = 6$		Starting with $x = -10$	
Iterations	Result	Iterations	Result	Iterations	Result
0	10	0	6	0	-10
1	8	1	6	1	-2
2	7	2	6	2	2
3		3		3	
4		4		4	
5		5		5	
6		6		6	

123

In this case $x = 6$ is the fixed point for $g(x)$, but it is an **attracting fixed point** because if you start with values other than 6 you find a sequence of numbers that get closer and closer to 6.

(c) Now we want to generalize a bit. Suppose we are given a function $h(x) = ax + b$ How can we find the fixed point of this function? Hint: if x is a fixed point of $h(x)$ then $h(x) = x$ which means $ax + b = x$. Find the fixed points for each of the following functions.

   i. $f_1(x) = 2x - 3$ iii. $f_3(x) = 3x + 4$

   ii. $f_2(x) = \frac{1}{2}x + 2$ iv. $f_4(x) = \frac{1}{3}x - 4$

(d) Can you tell if each fixed point is attracting or repelling without plugging in values? Determine if the fixed points you found above are attracting or repelling and describe your strategy.

II. **Applications of Iterated Functions**
   Suppose that a doctor determines that a patient would benefit from having 120 mg of a certain drug in his system over an extended period of time. Research has shown that the patient's body will eliminate 40% of this drug each day. What would be the ideal dosage for the patient to take in a daily pill? In this section of the activity, you will use iterated functions to determine this.

   (a) Our first task is to develop a linear function $f(x) = mx + d$ that, when iterated, will model this situation. To help you do this, answer the following questions.

      i. Suppose the patient took a 120mg pill on Monday. How much is left in his system on Tuesday?

ii. If the patient again took 120mg on Tuesday, how much would be in his system on Wednesday?

iii. From your work above, will taking 120mg pills result in the 120mg being in the patient's system? Explain.

(b) The change in the amount of drug in the patient's system can be described as follows: Each day, the patient has 60% of the previous amount left in his system, plus whatever the pill he takes adds to that. This suggests the the amount of drug in the patient's system can be found by iterating $f(x) = 0.6x + d$ where d is the dosage amount he takes each day.

What then is the appropriate value of d to make sure that the long-term drug level is 120mg? Put another way, what value of d makes 120 the fixed point of the function $f(x) = 0.6x + d$?

(c) Check your results by filling in the table below showing the amount of drug in the patient's system each day after he starts taking his pills.

Number of Days	Amount of Drug in System
0	0
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

(d) Suppose that the patient has taken this drug for a long time, so the amount in his system is 120mg. However, he will be excluded from an upcoming athletic event if the amount of drug in his system exceeds 10mg. If he stops taking his pill (so $d = 0$), how many days will it take for him to be eligible for competition? Fill in the table below to help with this.

Number of Days	Amount of Drug in System
0	120
1	
2	
3	
4	
5	
6	
7	
8	

III. **Group Project: Develop Your Own Iterated Function Application**

The above application is just one of many possible ways in which iterated functions and fixed points could be useful. As a group, come up with a real-world example, other than drug dosing, in which iterated functions would be useful. Create a poster showing your application and bring it with you to the next lab meeting. Your poster should contain the following:

- a brief description of your real-world application.
- a specific example (i.e. 120mg of the drug in the patient's system).
- the iterated function you found to meet the requirements of your example.

Chapter 6

Geometry

Geometry is one and eternal shining in the mind of God. That share in it accorded to men is one of the reasons that Man is the image of God.
– Johannes Kepler

This (axiomatic math) is no longer mathematics, it is theology.
– Paul Albert Gordan

Contents

6.1 Axiomatic Systems

An axiomatic system is a set of terms and rules from which other results may be derived. Some of the most basic terms are left undefined, and these are used in the definition of other terms. The rules, which are called *axioms*, are taken on faith. The results, which are logically deduced from the axioms and definitions, are called *theorems*. In this activity, you will practice proving theorems in a simple axiomatic system.

Objectives:

- Students will learn what is meant by an *axiomatic system*.
- Students will become acquainted with a simple axiomatic system and prove simple theorems based on the given axioms and definitions.

I. **An Axiomatic System to Order Integers**

Suppose that we are given the set of integers, and that all of the familiar arithmetic operations are defined on them in the usual way. We will take the relation "$<$" to be an undefined term.

Let a, b, and c represent integers. The following axioms are given:

i. If a and b are distinct integers, either $a < b$ or $b < a$, but not both.

ii. If $a < b$ and $b < c$ then $a < c$.

iii. If $a < b$ then $a + c < b + c$

iv. $1 < 2$

Use these axioms to prove each of the following theorems. You may use theorems that you have proven in the proofs of subsequent theorems. Justify each step by citing the axiom or theorem you are using.

(a) Theorem 1: $2 < 3$

(b) Theorem 2: $1 < 3$

(c) Theorem 3: $1 < 5$

(d) Theorem 4: $0 < 1$

(e) Theorem 5: For any integer a, if $-a < 0$ then $0 < a$.

II. **Inconsistent Axiomatic Systems**
   If by the use of valid reasoning you are able to produce contradictory results from your axioms, the system is called *inconsistent*. The axiomatic system above is consistent. However, if the following axiom is added, it becomes inconsistent.

   v. If $a < b$ then $ac < bc$.

   Reasoning with the original 4 axioms plus this new one, produce a contradiction.

6.2 Geometric Proof

In formal geometry, relationships between points, lines, and angles are not just taken for granted, they are proved in an axiomatic system. In this activity, you will get the chance to prove several basic geometric properties relating angles using the axioms of Euclidean geometry.

Objectives:

- Students will practice producing formal geometric proofs.
- Students will learn or refresh the common relationships between angle measures.

I. **Axioms of Euclidean Geometry**
   The following axioms are a subset of typical axioms for Euclidean geometry.

   i. Exactly one line may be drawn through any two distinct points.
   ii. Any line may be extended indefinitely.
   iii. All right angles are congruent.
   iv. If two parallel lines are cut by a transversal, then consecutive interior angles are supplementary.

   Using these axioms, the definitions seen in class, previously proved theorems (from this activity), and the basic rules of arithmetic, prove each of the following theorems. You may find it helpful to draw and carefully label a sketch for each theorem. You may then refer to your sketch, as long as you do not assume anything from your drawing.

   (a) **Theorem 1**: Angles that are supplementary to the same angle are congruent.

   (b) **Theorem 2**: Vertical angles are congruent.

(c) **Theorem 3**: If two parallel lines are cut by a transversal then corresponding angles are congruent.

(d) **Theorem 4**: If two parallel lines are cut by a transversal, alternate interior angles are congruent.

(e) **Theorem 5**: If two parallel lines are cut by a transversal, alternate exterior angles are congruent.

(f) **Theorem 6**: The sum of the interior angles in any triangle is 180°.

6.3   Proofs of the Pythagorean Theorem

The Pythagorean theorem is probably the most famous theorem in all of mathematics. The simple equation $a^2 + b^2 = c^2$ would likely be recognized by anyone whose taken a high school level mathematics class. The theorem is named for Pythagoras, a Greek philosopher who lived around 500 B.C. Although he is credited with the theorem, many different cultures discovered this relationship independently of each other.

Although the Pythagorean theorem is quite famous, what is not so well known is that there are over 370 different proofs of this theorem. In this activity, you will be asked to explain how various diagrams prove that $a^2 + b^2 = c^2$.

Objectives:

- Students will gain experience working with geometric proofs
- Students will be able to explain a proof of the Pythagorean Theorem.

I. **A Classical Proof**
   The following proof represents the same square in two different ways in order to show that the area of two incongruent figures is the same. Use the figure below to prove the Pythagorean theorem.
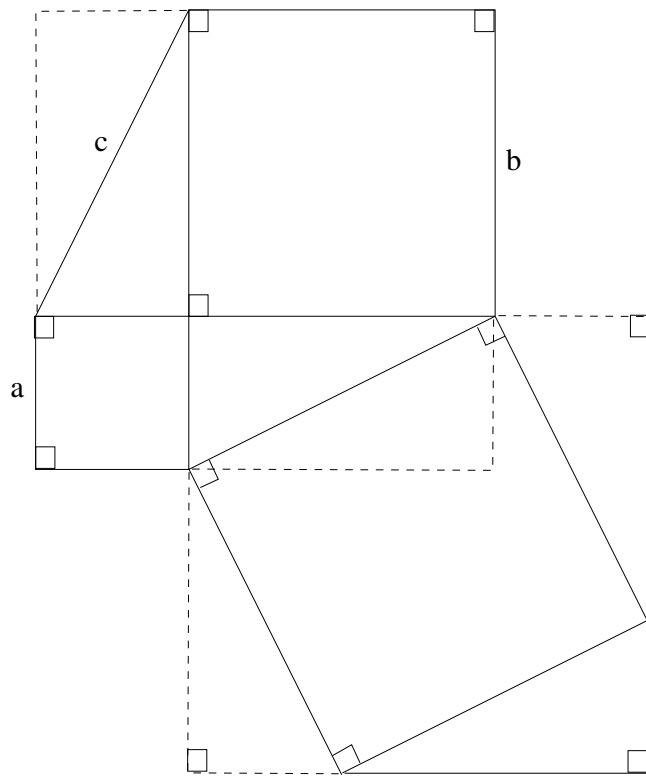
2. **Perigal's Proof**
Many proofs of the Pythagorean theorem have ancient origins but were rediscovered later by people unfamiliar with the older sources. This proof, called Perigal's Proof, was "discovered" by mathematician Henry Perigal in 1873, but was probably known to the Arabian mathematician Iabit ibn Qorra a thousand years before.
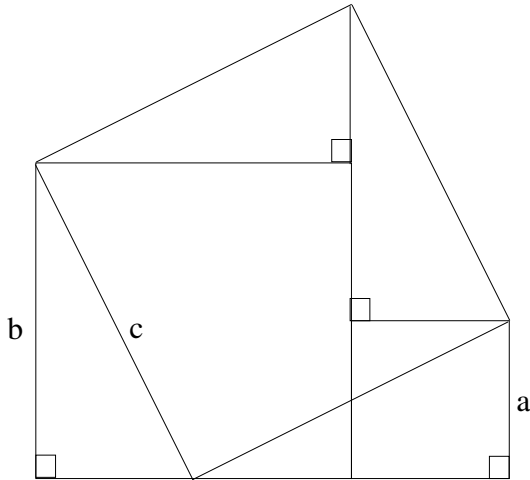
3. **Saunderson/Bhaskara Proof**
This figure is attributed to Saunderson (1682-1739) but probably came from the 12th-century Hindu mathematician Bhaskara. Can you use this figure to find a proof of the Pythagorean theorem?
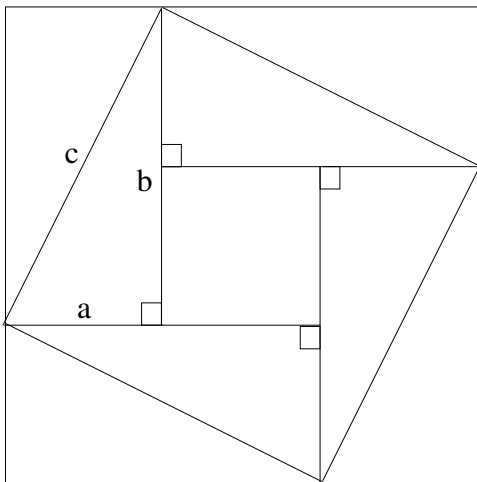
4. **The Behold! Proof**

   The 12th-century Hindu scholar, Bhaskara, simply wrote "Behold!" along side this figure demonstrating the Pythagorean theorem. He must have felt that the figure spoke for itself! Incidentally, this figure is also found in an ancient Chinese text, making it a candidate for a proof known to Pythagoras.
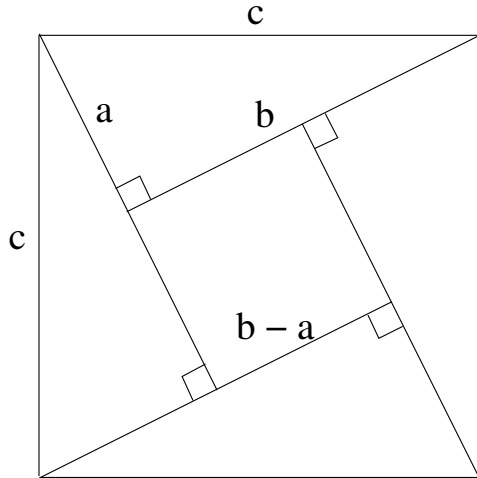
5. **Similar Triangle Proof**

   The similar triangle proof found using the figure below has the dual distinction of being the shortest when written out, as well as being the proof most commonly found in geometry books. Use the similar triangles found below to prove the Pythagorean theorem.
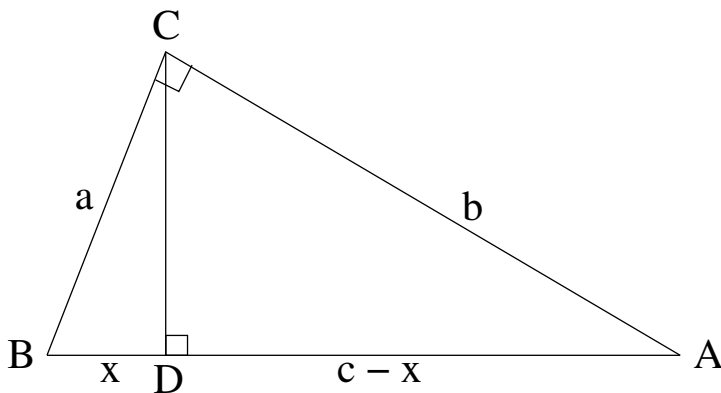
6. **A Presidential Proof**
   James A. Garfield, the country's 20th president, discovered a proof of the Pythagorean theorem in 1876 while serving in the US House of Representatives. Garfield's proof of the theorem is illustrated with the trapezoid shown below. Remember that the formula for the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$ where b_1 and b_2 are the bases and h is the height.
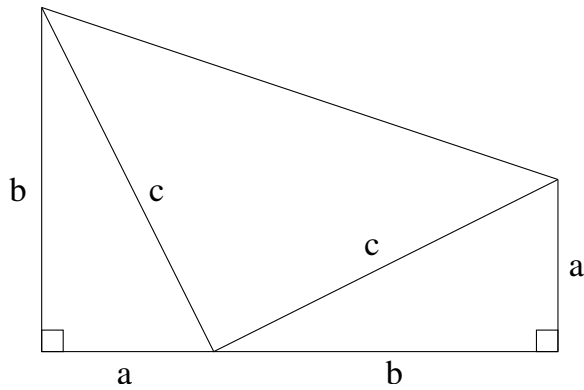
7. **A Quaint Proof**
   The rectangle below is divided in half by the thick black line. Congruent shapes are shaded similarly. Can you use this information to prove the Pythagorean theorem?
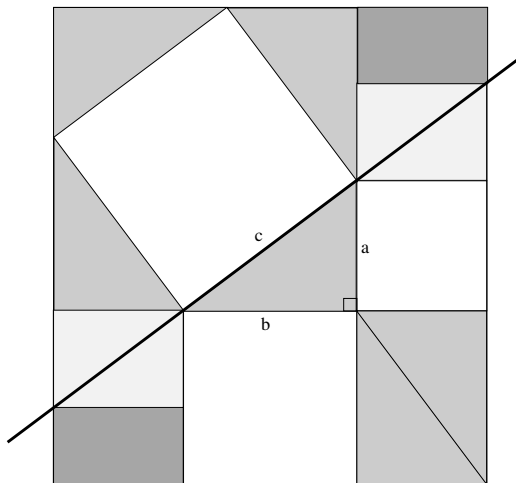
6.4   Interior Angles of a Polygon

Many people remember that "the sum of the interior angles of a triangle is 180°" from their their high school geometry class. However you may or may not recall that this is just one specific instance of a more general formula for the sum of the interior angles of any polygon. In this activity, you will get a chance to explore this situation and (re)discover the formula.

Objectives:

- Students will develop a formula for finding the sum of the measures of the interior angles of a polygon.
- Students will prove the formula for quadrilaterals and pentagons.

I. **Developing a Formula**

In activity 6.2, you proved that the sum of the measures of the interior angles of any triangle is 180°. In this first section, you will extend this to other polygon shapes.

(a) Use pattern blocks to help you complete the table below listing the number of sides and the sum of the measures of interior angles. Hint: You may find it useful to combine several pattern blocks of the same kind.

Polygon	# Slides	Sum of $\angle$
Triangle		
Square		
Rectangle		
Parallelogram		
Trapezoid		
Rhombus		
Pentagon		
Hexagon		

(b) Now suppose that a polygon has n sides. What will the sum of the measures of its interior angles be? How can you justify your claim?

2. **Proving the Formula**
   You will now be asked to construct a geometric proof that the formula you came up with at the end of the last section works in two specific cases. You may use the axioms and theorems from activity 6.2 in your proofs. Of particular use may be Theorem 6, which states that the sum of the measures of the interior angles of a triangle is 180°.

   (a) Prove the formula for any quadrilateral. Include a labeled sketch to help explain your proof.

   (b) Prove the formula for any pentagon. Include a labeled sketch to help explain your proof.

3. **Applying the Formula**
   Use the formula you developed and proved in the last section to assist you in answering each of the following questions.

   (a) A student claims that the sum of the measures of the interior angles of a decagon is 1800°, and even presents you with a picture to prove the claim. Describe what the student did, and why the method is incorrect.
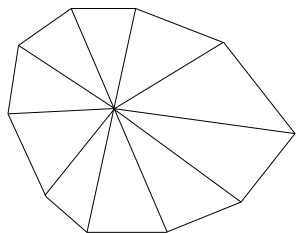
(b) Find the sum $\measuredangle a + \measuredangle b + \measuredangle c + \measuredangle d + \measuredangle e + \measuredangle f$ from the figure below.
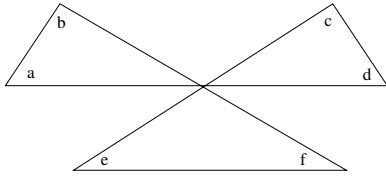
6.5   Possible Polygons

We have previously seen that "two points determine a line" and "three points determine a plane." We can also ask what particular conditions *determine* a polygon. That is, how many polygons can we make that have a particular attribute. In this activity you will explore this question in more detail.

Objectives:

- Students will develop an understanding of the properties that determine a polygon.
- Students will develop intuition by making predictions and then testing that prediction.

I. **Triangles**

How many triangles are possible using a given set of side lengths and angles? In this section you get to explore that question. For each situation, first make one of the following predictions:

- No triangles exist with these properties.
- Exactly one triangle exists with these properties.
- More than one triangle exists with these properties.

Record your prediction in the space provided, and then use the provided protractor to test your prediction. If you are wrong, explain what the correct answer should have been.

(a) Side lengths: 2 cm, 3 cm, and 4 cm

(c) Sides 4 cm and 4 cm, and angle: 90°

(b) Side lengths: 3 cm, 5 cm, and 9 cm

(d) Sides 3 cm and 4 cm with included angle 45°

(e) Angles: 40°, 60° and 80°   (f) Sides 3 cm and 4 cm with non-included angle 45°

I. **Quadrilaterals**
Repeat the same process for the following descriptions of quadrilaterals.

(a) Sides 2 cm, 3 cm, 4 cm, and 5 cm   (c) Angles all equal, but sides not all equal

(b) Sides all equal, but angles not all equal   (d) Exactly one right angle

(e) Exactly two right angles

(h) Two pairs of congruent sides of different lengths

(f) Exactly three right angles

(i) Exactly three congruent sides

(g) Exactly two congruent sides

6.6 Tangram Puzzles

Tangrams are a versatile manipulative that can be used to develop spatial sense. They were invented in China at least two hundred years ago and quickly became a popular puzzle because of the many different combinations. The word *tangram* probably came from American sailors who referred to all things Chinese as "Tang." In this activity, you will practice creating puzzle shapes with a tangram set.

Objectives:

- Students will develop their spatial abilities.
- Students will be able to name shapes and see relationships among shapes.

I. **Solving Puzzles**

Below are several famous shapes that can be made using the entire set of tangrams. For each shape:

(a) use all seven tangram pieces to make the figure shown.

(b) sketch your solution on the figure below.

(c) describe any new insight you gained from the puzzle (i.e. patterns, discoveries, questions, etc).
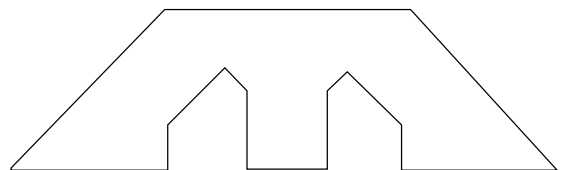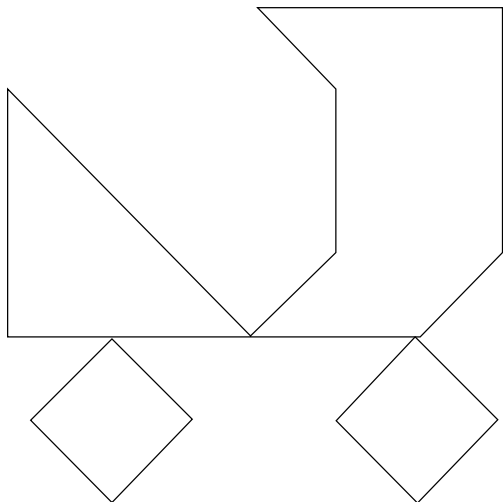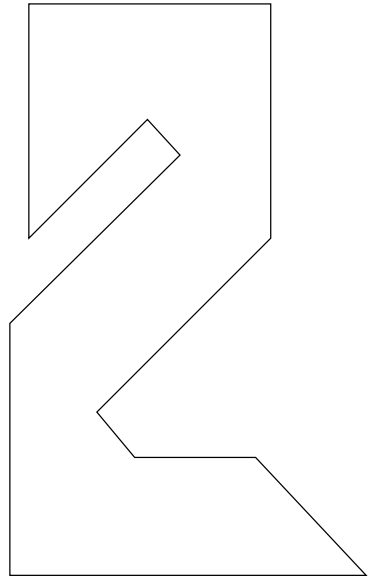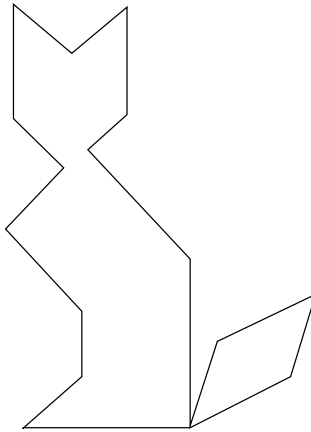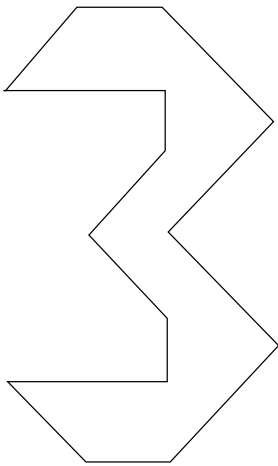
II. **A Tangram Paradox**

The two shapes below form a famous tangram paradox. Each of the figures shown has been made with one set of seven tangram pieces. Yet it appears that the figure on the right has one more piece. Solve these two puzzles and sketch your solution. In one or two sentences, describe why this is not actually a paradox.
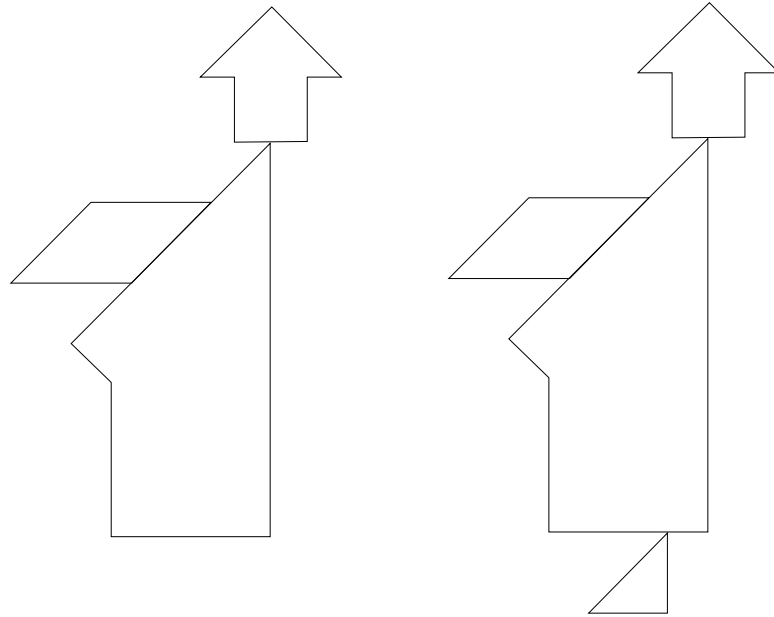
III. **Communicating Puzzles**

Each member of your group should now make their own tangram puzzle and sketch it in the space provided below. Do not show your puzzle to any of your group members. When everybody in your group is done sketching their puzzle, complete the following steps, making sure that each person gets to play the role of persons A, B, and C.

(a) Person A turns around so that he or she can not see the table or person B.

(b) Person B sits facing away from person A so that he or she can not see person A's sketch.

(c) Person C sits so that he or she can observe both person A's figure and person B's work.

(d) Person A now describes their figure to person B who attempts to reproduce it using a set of tangram pieces. Person C can not make any comments.

(e) Once the description has been finished and person B has their pieces arranged, see how closely the two shapes match. At this point person C can give feedback on the communication process.

6.7 Congruence

One of the major ideas in a conceptual geometry course is that of congruence of shapes. This idea also has important applications in industry, science, and the arts. After all, mass production simply means making lots of congruent copies of an item. In this activity, you will explore the idea of congruence using manipulatives.

Objectives:

- Students will develop their spatial abilities.
- Students will gain a deeper understanding of congruence.

I. **Congruence on a Geoboard**
   In our first set of problems, we will work with congruence questions on a Geoboard. Your group will be provided with Geoboards and rubber bands with which to work. Sketch your results on the geoboard figures provided below.

(a) Divide each of the following regions into the given number of congruent triangles.
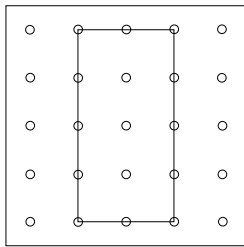
|Four congruent triangles | Three congruent triangles | As few congruent triangles as possible|
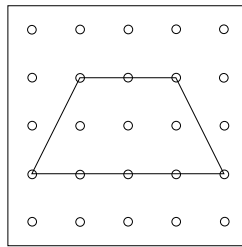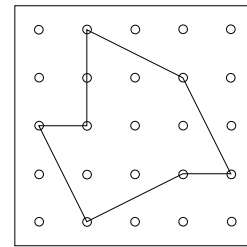
(b) Determine if each pair of figures is congruent. Describe how you can tell without physically cutting one figure out and placing it on top of the other.

144
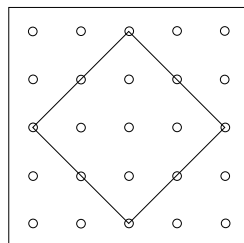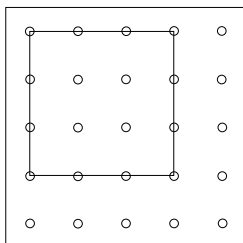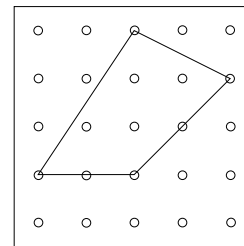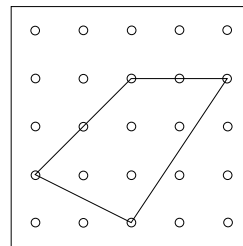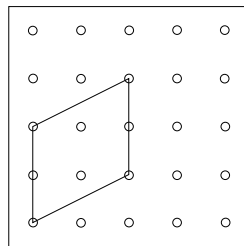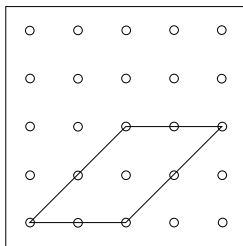
II. **Congruence with Tangram Sets**
In the following questions, you are asked about congruence between shapes made from a single set of
tangram pieces. In each case, use the pieces described to construct the given shapes and determine if
they are congruent.

(a) A trapezoid is constructed using the two little triangles and the parallelogram. Another is con-
structed using the medium triangle and the two little triangles. Are these two trapezoids congruent?
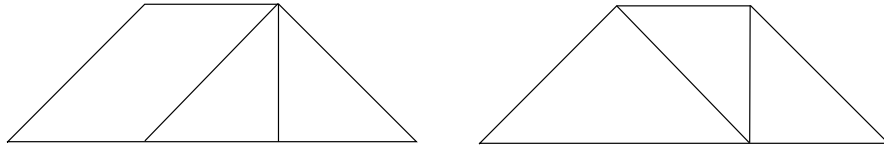
(b) Another pair of trapezoids is constructed as shown below. Are these congruent? Explain.
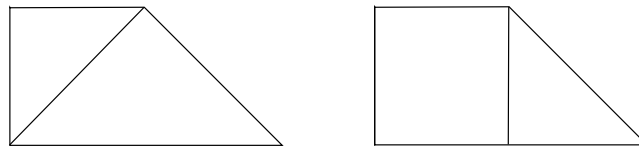
(c) One final pair of trapezoids is constructed as shown below. Are these congruent? Explain.
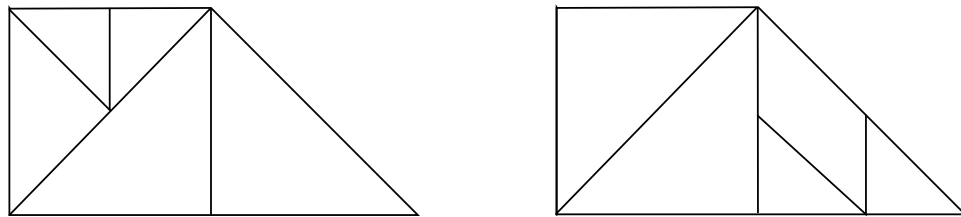
6.8 Relationships Between Polygons

There are many names for polygons in geometry. This is especially true of polygons with four sides, called quadrilaterals. In this activity you will practice differentiating between the various types of quadrilaterals both in describing them to others and in classifying them using Venn diagrams.

Objectives:

- Students will understand that all shapes have multiple attributes.
- Students will be able to classify quadrilaterals.

I. **Name That Quadrilateral**

You've probably played the game "20 questions" at some point. To goal of this game is for one person to guess what object another is thinking of by asking questions that have yes/no answers. In the first part of this activity, your group will play a modified version of this as follows:

- One person picks a quadrilateral from the list below without revealing their selection.
- The other members of your group take turns asking yes/no questions until somebody guesses the correct quadrilateral.

A good strategy is to focus your questions on properties which distinguish sets of quadrilaterials.

(a) Play the game several times so that each group member has a turn picking the quadrilaterial.
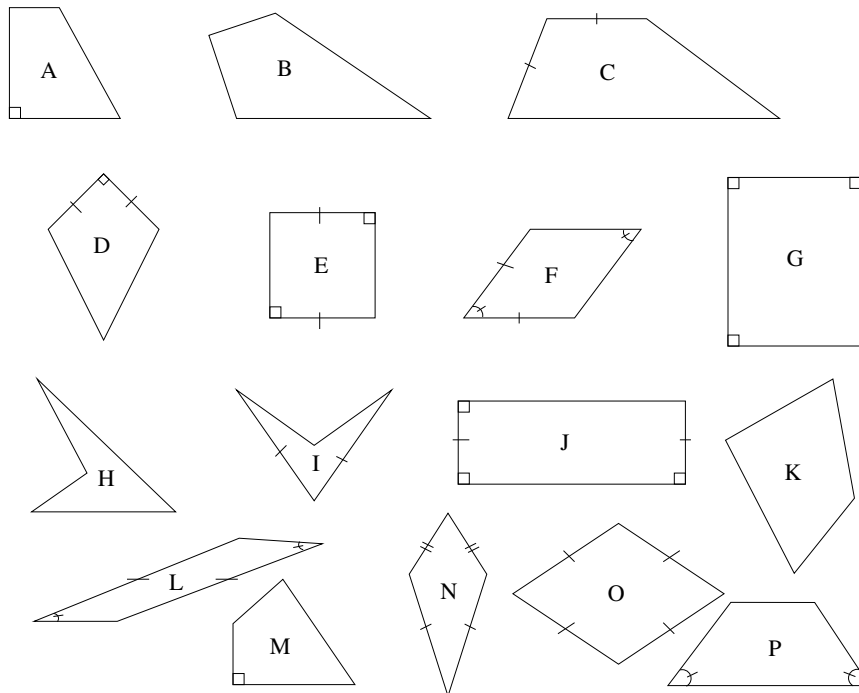
(b) As a group, pick the three questions which you think are most usefuly for quickly identifying the correct quadrilaterial. List them below.

146

II. **Classifying Quadrilaterals**

Recall that Venn diagrams can be used to visually represent related sets of objects. The following activities ask you to create Venn diagrams for the quadrilaterials shown on the previous page, and/or to identify properties of sets shown in Venn diagrams.

(a) In the Venn diagrams shown below, identify the characteristics shared by quadrilaterials in the given sets and label the sets accordingly.

   i. Left Set:

      Right Set:
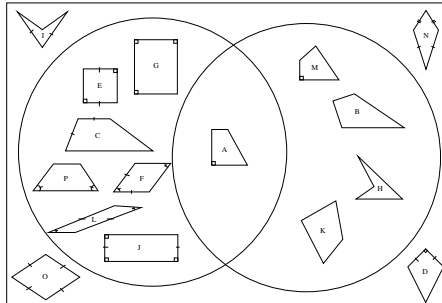
   iii. Left Set:

      Right Set:
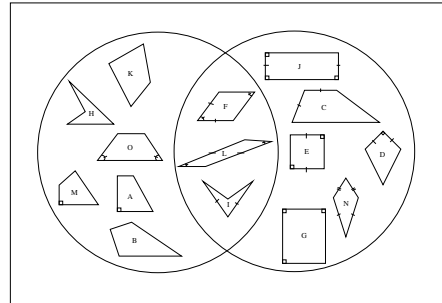
   ii. Inner Set:

      Outer Set:
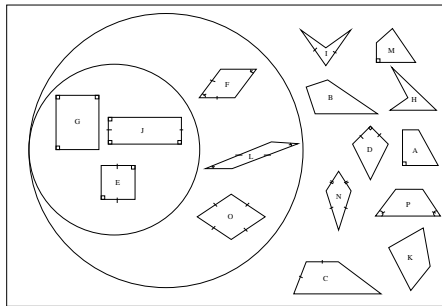
   iv. Left Set:

      Right Set:
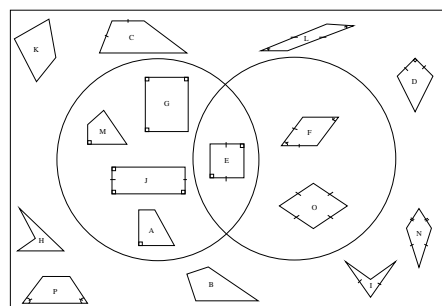
(b) For each group of properties given below, construct a Venn diagram with one set per property. Show the location of each of the quadrilaterials from the previous page in your diagram.

   i. Left Set: Has a Right Angle
      Right Set: Has a Pair of Congruent Sides

   ii. Left Set: Convex
      Right Set: Has an Obtuse Angle
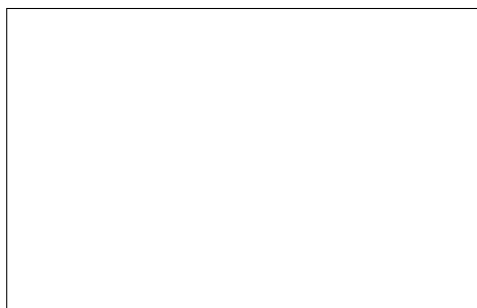
6.9 Sketching Rectangular Prisms

Representing three-dimensional objects with a two-dimensional sketch requires both knowledge and practice. Sketching figures with given dimensions takes even more finesse. In this activity you will practice sketching rectangular prisms of given dimensions.

Objectives:

- Students will understand how perspective relates to measurements and drawings.
- Students will be able to sketch rectangular prisms.

I. **Sketching Cubes**

The simplest rectangular prism is a cube. In this first part of the activity, you will learn how to sketch a two-dimensional representation of a cube and how to determine perspective. Consider the following step-by-step instructions for sketching a cube.

1) Draw a square to represent the front face of the cube.

2) Draw a slightly smaller square behind it to represent the back face.

3) Join the corresponding vertices using straight line segments.

4) Indicate the edges that cannot be seen by a viewer by making them dotted lines.

Steps 1 and 2   Steps 3 and 4
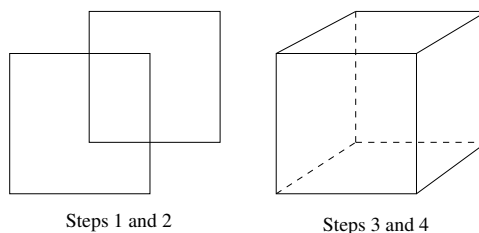
Use these instructions to help you complete the following tasks involving cubes.

(a) For each of the following cube sketches, indicate the position of the viewer relative to the cube. Use phrases such as "above," "below," "left of," "right of," or "in front of" in your description.

i.   ii.   iii.   iv.
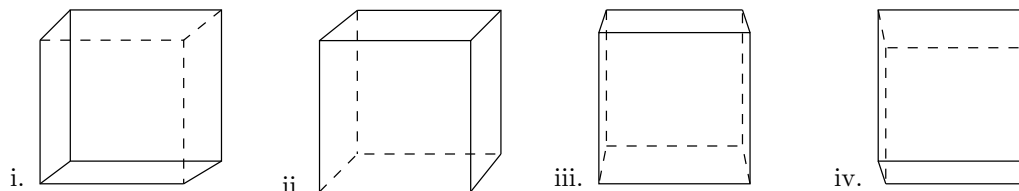
(b) Now sketch a cube of edge length 4 cm viewed from slightly above and to the left.

(c) The drawings on the previous page trick your eye into a seeing a three-dimensional cube by making objects that should be farther away (like the back of the cube) appear smaller. Drawing objects like this is called *perspective drawing*.

The examples you've seen so far are all *one-point perspective* drawings in which the face of the cube was taken to be parallel to the plane of the paper so that all lines that are not contained in that plane meet at a single point when extended. Show these perspective lines for each of the drawings from the previous page. The first one is done for you.
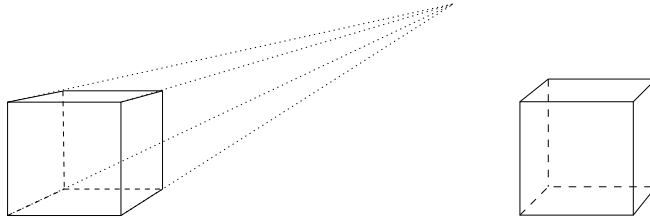
(d) Another form of perspective drawing is *two-point perspective* drawings. Consider the drawing of the cube below. Are any of the faces of the cube parallel to the plane of the paper?
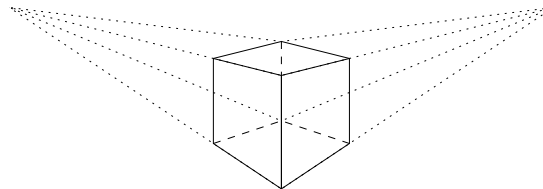
(e) What part of the cube is closest to the viewer?

II. **Sketching Other Rectangular Prisms**
A technique similar to the that used for cubes can be used to draw any rectangular prism. Use it to draw each of the following.

(a) A rectangular prism of dimensions 2 cm by 4 cm by 4 cm in which the square faces are parallel to the plane of the paper and the rectangular prism is viewed from slightly below and to the right.

(b) A rectangular prisms of dimensions 2 cm by 4 cm by 4 cm in which a pair of non-square faces is parallel to the plane of the paper and the rectangular prism is viewed from slightly above and directly in front.

(c) A rectangular prism of dimension 2 cm by 4 cm by 4 cm using two-point perspective in which an edge connecting two non-square faces is closest to the viewer.

6.10 Building Blocks

Even simple compound three-dimensional objects, such as those made by stacking congruent cubes, which we will call blocks, can be challenging to accurately represent in three dimensions. In this activity, you will explore various two-dimensional representations of such three-dimensional objects.

Objectives:

- Students will develop spatial sense.
- Students will be able to translate physical objects into a description or picture.

I. Isometric Projections
   The pictures below show a two-dimensional representation of a three-dimensional object made by stacking blocks on top of each other–without any holes underneath a visible block–which we will call a building. This type of representation is called an isometric projection. In this activity, you will practice interpreting and creating isometric projections of buildings.

   (a) Determine the number of cubes needed to make each structure.
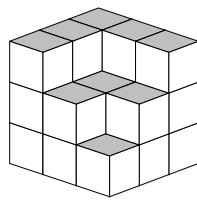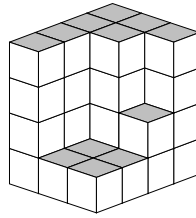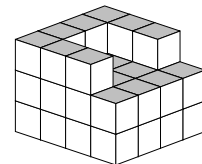
i.   ii.   iii.

   (b) Can you determine with certainty the number of blocks in the figure below? If so, explain why. If not, give a minimum and maximum number of possible blocks.
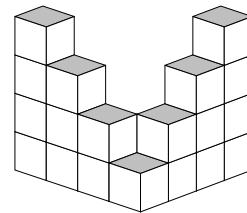
   (c) Each member of your group should now construct a building using 12-24 centimeter cubes. Choose two of these buildings and sketch an isometric projection of each on the grid provided below.
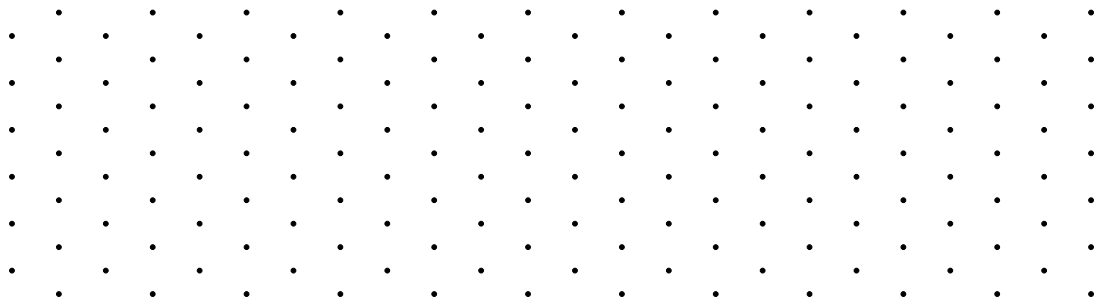
II. **Elevations and Cross Sections**

Another common way to represent block buildings is to use what are called elevations and cross sections. An elevation is two-dimensional image of one side of the building, and a cross section is a two-dimensional image of the top (or any "floor") of the building, as shown below.
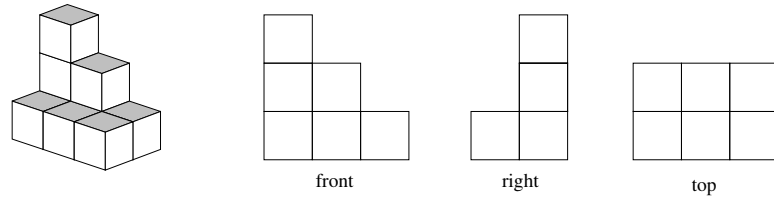

This second part of the activity asks you to work with these types of sketches.

(a) Use centimeter cubes to construct the building shown in each set of elevation and cross sections below. Give an isometric projection of the building on the provided grid.

	Front	Right	Top	Projection
i.				
ii.				
iii.				
iv.				

(b) Now give the front and right elevations, and a top cross section for each of the buildings whose isometric projections are shown below. Indicate any of sketches about which the projection leaves ambiguity.

	Projection	Front	Right	Top
i.				
ii.				
iii.				
iv.				
v.				
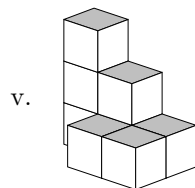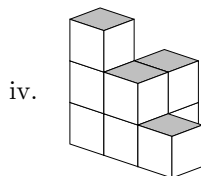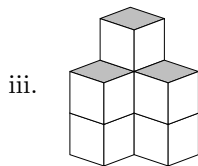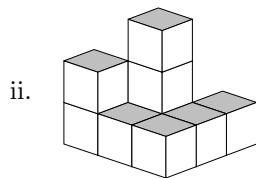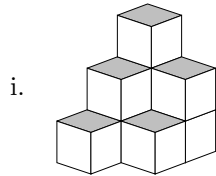
6.11 Nets

It is always a good idea to look for connections between something you know and something new. For example, post people are fairly comfortable with two-dimensional geometric shapes such as squares, triangles, etc. But many struggle to visualize three-dimensional figures. In this activity, you will practice constructing two-dimensional figures that will fold up into a given three-dimensional figure. These are called nets.

Objectives:

- Students will relate two- and three-dimensional figures together.
- Students will develop spatial sense.

I. **Constructing Nets**

There are many polyhedra in every day life that start off as nets. That is, they are constructed by taking a very flat material, such as cardboard, and folding it up to form a three dimensional surface.
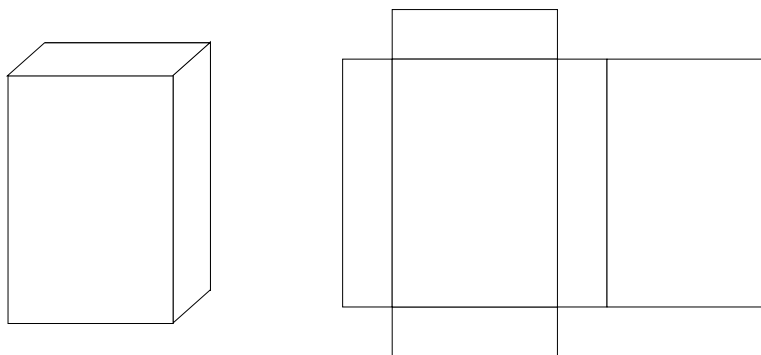
Take, for example, the standard cereal box shown below. Those of you who recycle regularly are probably already familiar with how to flatten this box back down into a net, as shown. Answer the following questions involving this, and other, three dimensional shapes and their nets.

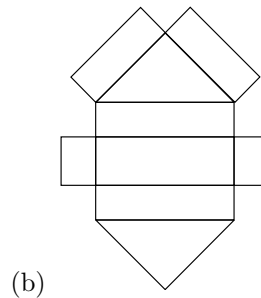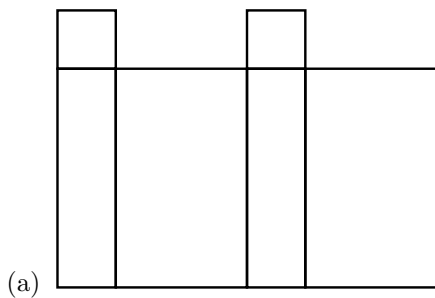(a) Sketch at least two more distinct nets for the cereal box shown above.

(b) If the cereal box were 10 inches wide, 16 inches tall, and 3 inches deep, what would the dimensions of your nets be? Label the pictures in part (a) with these dimensions.
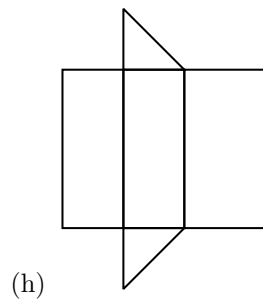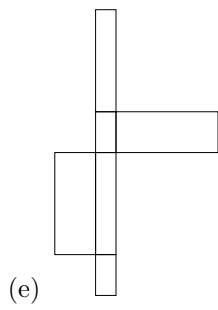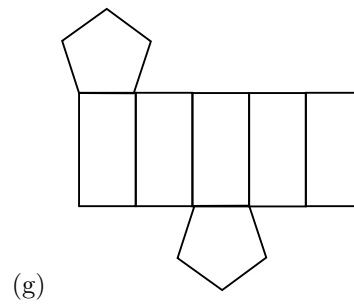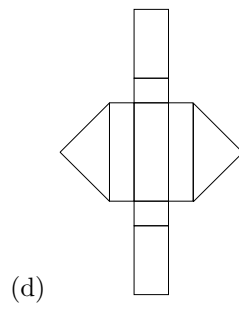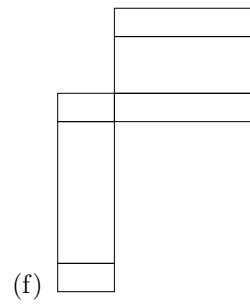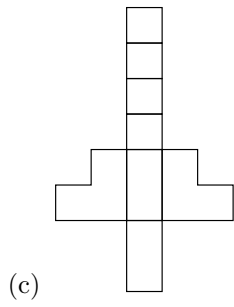
(c) Construct at least two nets for each of the following figures.

i.

ii.

iii.

iv.
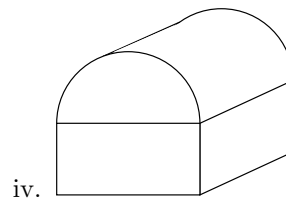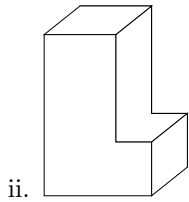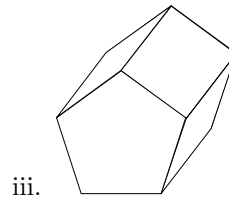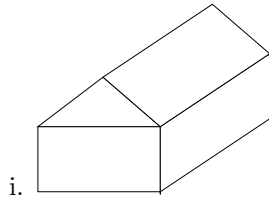
II. Deconstructing Surfaces

Not all two-dimensional figures are nets! If we try to work the other direction, we must determine if a given two-dimensional figure will fold up into a polyhedron, and if so what that polyhedron will look like. In each of the following examples, determine if the given figure is a net. If it is, sketch the three-dimensional figure into which it will fold.

(a)

(b)

(c)

(f)

(d)

(g)

(e)

(h)

Adapted from Bassarear *Mathematics for Elementary Teachers Explorations*, 229-230.

6.12 Constructing Polyhedra

We will now put the concept of a net into practice. In this activity, you will construct several polyhedra from the provided nets.

Objectives:

- Students will relate two-dimensional and three-dimensional figures.
- Students will develop spatial sense.

I. **Creating Polyhedra**

Use the provided nets, straight-edges, scissors, and glue to construct polyhedra. Each member of your group should construct at least one polyhedron. For best results, follow the steps below.

(a) Using a straight-edge and pencil, score all the internal edges of the net to make it easier to fold.

(b) Cut along all of the external edges.

(c) Fold along the internal edges and match up numbered tabs.

(d) Glue these tabs in place. Note that you may have to hold each tab for a minute or two to give the glue a chance to dry.
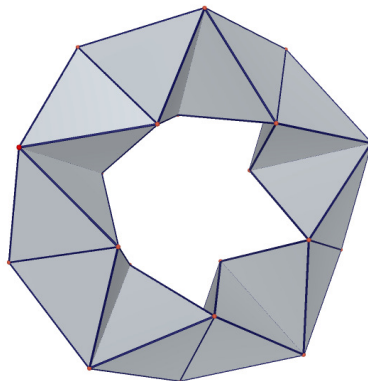
The polyhedra you have available to you are:
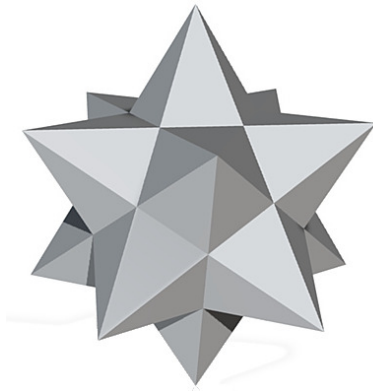
Small Stellated Dodecahedron
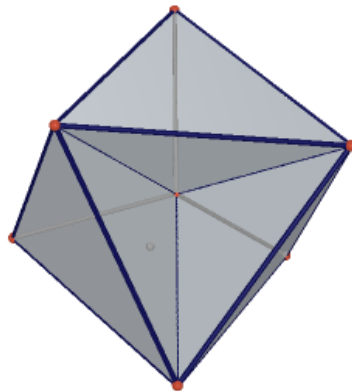
Tetrahemihexihedron

Rotating Ring of Tetrahedra

6.13   Spherical Geometry

To begin this activity, think about the following question.

> A wandering bear leaves home and walks 100 kilometers south. After a rest, she turns west and walks straight ahead for 100 kilometers. Then she turns again and walks north. To her surprise she finds that she arrives back home again. What color is the bear?

When geometry is done on a sphere (such as the surface of the earth) instead of on the idealized plane of Euclidean geometry, things change. In this activity you will explore some of the differences between standard Euclidean geometry, and spherical geometry. By the end of the activity, you should be able to confidently answer the question above.

Objectives:

- Students will appreciate that Euclidean geometry is an idealized system.
- Students will understand how changing the axioms of geometry will change the relationships with which they are familiar.

I. **Lines on a Sphere**
   In spherical geometry, a line is a great circle around the outside of a sphere. A great circle is the intersection of a sphere and a plane that passes through the center of the sphere. Answer the following questions related to lines in Euclidean and spherical geometry.

   (a) In standard Euclidean geometry, do lines have a length? Explain.
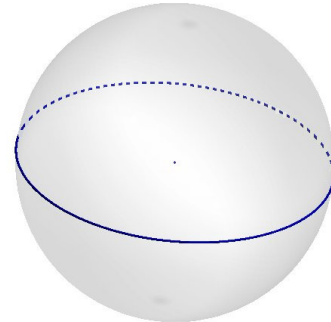
   (b) In spherical geometry, do lines have a length? Explain.

   (c) If two distinct lines in a plane intersect, at how many distinct points can they intersect?

   (d) If two distinct lines on a sphere intersect, at how many distinct points can they intersect?

II. **Relationships Between lines**

Now consider the two lines shown on the sphere below. Use these as examples to help you answer the following questions about the relationships between points on a line and lines on a sphere.

(a) Points A, B, and C on the figure to the right are colinear. Which of the three points is between the other two?
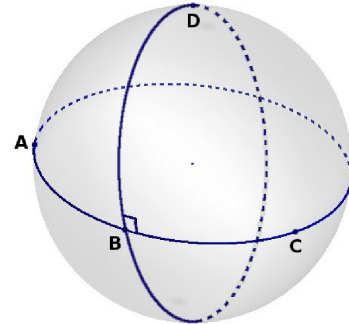
(b) Suppose the spherical line through D is perpendicular to the line through AB. Are there other lines through D perpendicular to AB? If so, how many?

(c) In Euclidean geometry, how many lines are there perpendicular to a given line through a given point not on that line?

(d) Referring again to the figure above, how many lines are parallel to the line through A and B?

III. **Spherical Triangles**

In the final figure below, a triangle $\triangle ABC$ is shown. Use this as a starting point to answer the following questions.

(a) Is the sum of the angle measures in $\triangle ABC$ 180°?

(b) How many right angles can a spherical triangle have?

(c) How many obtuse angles can a spherical triangle have?
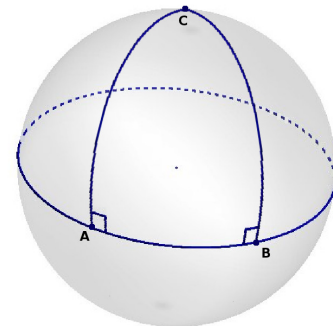
6.14 Basic Geometric Constructions

The rules and procedures for geometric constructions have their origins in the basic axioms of Euclidean geometry. The value of studying these constructions can be found in the analysis of the construction and understanding of why they work–not in any practical application. In this activity you will be asked to perform several basic geometric constructions.

Objectives:

- Students will be able to perform basic compass-and-straight-edge constructions.
- Students will understand the limitations of compass-and-straight-edge constructions.

I. **Line Segments and Angles**
   The basic tools for geometric constructions are:

- A compass – used to construct circles, arcs, and copy distances
- A straight edge – used to construct line segments between given points

While the straight edge you have been provided is actually a ruler, you may not use its measuring marks in your constructions. Complete each of the following constructions.

(a) Construct a line segment congruent to $\overline{AB}$.

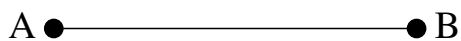A •————————————• B

(b) Construct an angle congruent to $\angle QRS$.
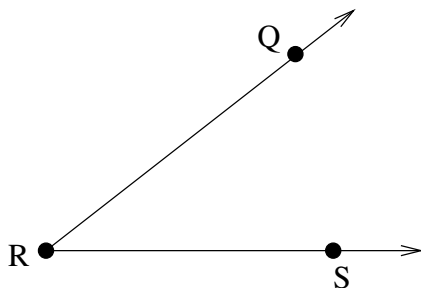
II. **Related Lines and Angles**
    Now that you know how to construct basic line segments and angles, can you figure out how to construct perpendicular and parallel lines? What about bisecting an angle? The following constructions ask you to do just that.

(a) Construct the perpendicular bisector of $\overline{AB}$.

A ●————————————————● B

(b) construct a line segment parallel to $\overline{AB}$.

A ●————————————————● B
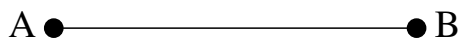
(c) Construct an angle bisector for $\angle XYZ$.

X ●

Y ●————————————● →
                Z

III. **Other Geometric Figures**
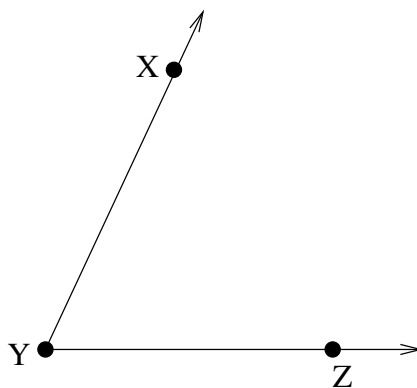   The constructions you just completed for the building blocks for most other constructions. Use them to help you construct the geometric figures described below.

   (a) Construct a triangle congruent to $\triangle EFG$.

F

E   G
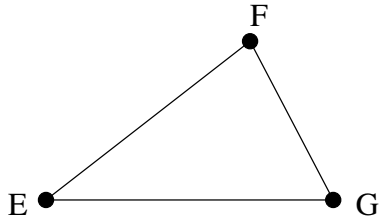
   (b) Construct a square with sides congruent to $\overline{AB}$.

A ●━━━━━━━● B

   (c) Construct a square with sides congruent to $\overline{AB}$ and inscribe a circle in that square.

A ●━━━━━━━● B

Chapter 7

Measurement and Transformation

Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only.

– Henri Poincare

He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.

– Plato

Contents

7.1   Unit Conversion

Units are an often ignored, but very important part of measurment. Without knowing what the units of a numerical measurement are, it is impossible to make sense of the number. In this activity, you will practice using units and performing unit conversions.

Objectives:

- Students will understand that units provide a necessary context for any measurement.
- Students will be able to convert between units.

I. **Conversion Methods**
   It is sometimes hard to understand when one should multiple and when one should divide when performing unit conversions. In each situation below, help the student to understand the correct method for conversion by:

   - Drawing a picture to show how the two units are related, and
   - Writing an explanation of how the unit conversion is correctly completed.

   (a) Lucy is confused about why one should *multiply* by 3 to convert 12 yards to feet. She thinks it makes more sense to *divide* by 3 because feet are smaller than yards. Explain to Lucy the relationship between the *size* of a unit and the *number* of units it takes to cover the length of an object.

   (b) Sam is confused about why one should *divide* by 100 to convert 300 centimeters into meters. He thinks that *multiplying* by 100 makes more sense because meters are bigger than centimeters. Help Sam understand the correct way to make this conversion.

II. **Creating Your Own Units**
In this section your group will construct your own system of measurement based on various lengths. For the purposes of this one activity, define the following units of length.

- A **Thumb**
  The length from the tip to the first knuckle of the thumb of the oldest person in your group.

- A **Phone**
  The length of the longest edge of the newest cellular phone currently carried by one of your group members when it is fully opened and extended.

- A **Humerous**
  The distance from the elbow to the shoulder of the left arm of your group's youngest member.

Answer each of the following questions. Be sure to include units in all computations and each of your answers.

(a) Determine the number of thumbs in a phone.

(b) Determine the number of phones in a humerous.

(c) Now find the height of your group's shortest member measured in humerous.

(d) Convert the height you found above to thumbs. Show your work!

(e) A new unit of length is to be introduced into your system. It's length must be a multiple of the thumb, phone, and humerous units. Name this unit, and give the conversions from thumbs, phones, and humerous to your new unit.

7.2 How _____ Is It?

How do we know the empire state building is 1250 feet tall? Can we be sure the distance between Portland and College Place is 236 miles? Neither of these measurements were made with a ruler or tape measure! In this activity you will get to consider various methods for measuring length, height, and thickness in challenging situations.

Objectives:

- Students will realize that measurements are, by definition, approximations.
- Students will see a connection between measurement and data collection and analysis.

I. **How Thick Is It?**
Most of our linear measurements are in units that are quite large, such as centimeters, inches, feet, etc. In many areas of science and manufacturing, units are needed for very small measurements such as the thickness of your skin, or the width of a molecule. Answer the following questions to explore how you might determine the thickness of a sheet of standard paper.

(a) Brainstorm several ideas for how to measure the thickness of a sheet of paper.

(b) Select one method and pursue it as a group. Write a detailed description of your plan.

(c) Conduct your measurement and determine the thickness of one standard sheet of paper. How precise is your answer?

II. **How Far Is It?**

Before modern measuring methods were invented, people determined distances by pacing. Armies kept track of how far they had gone using professional pacers who practiced taking equal-sized steps. Suppose you had to measure the distance from the southeast corner of Bowers Hall and the northwest corner of Kretschmar Hall without any modern methods. How would you do it? Discuss the following questions as a group and record your ideas and final consensus in the space provided.

(a) What does one pace mean, and how can you measure it?

(b) How can you ensure that a pace means the same thing for each group member.

(c) How will you ensure that the length of a pace does not change during or between measurements?

III. **Group Project: Measure the Distance from Bowers to Kretschmar**

Your group project for this week is to perform the measurement you discussed above. You will then write a one-to-two-page report with the following sections.

- **Summary of Procedures**
  Describe your measurement process. At a minimum, you must have two different group members measure the distance in paces at least two times each. Be sure to describe how you ensured that a pace for one group member was the same as a pace for another.

- **Summary of Results**
  Include your raw data as well as averages for individuals and an overall average in number of paces. Convert the overall average to an approximate distance in feet. Give your conversion formula.

- **Discussion of Possible Error**
  Describe the possible sources of error in the distance you found. Include ideas on how more accurate measurements might be taken.

Adapted from Bassarear *Mathematics for Elementary Teachers Explorations*, 293-294.

7.3 Relating Area and Perimeter in Rectangles

Both the area and the perimeter of a rectangle depend on its length and width. It makes sense then that they should be related, but in what way? In this activity you will explore the relationship between area and perimeter in rectangles using colored tiles.

Objectives:

- Students will more fully understand what is meant by a figure's *perimeter* and *area*.
- Students will investigate the relationship between the area and perimeter of rectangles.

I. **Building Rectangles**

Use the provided square units (colored tiles) to build rectangles to the following specifications. In some cases, there may be several possible ways to build a rectangle. In other cases it may not be possible at all. Include sketches of your rectangles in the space provided.

(a) Area = 10 (build all possible)

(b) Perimeter = 10 (build all possible)

(c) Area an even number

(d) Perimeter an even number

(e) Area an odd number

(f) Perimeter an odd number

(g) Area a prime number

(h) Perimeter a prime number

(i) Non-square with Area a perfect square

(j) Perimeter a multiple of 4, but sides $\neq$ 4

II. **Relating Area and Perimeter**
Each of the following descriptions asks you to build a rectangle where the area and perimeter have a given relationship. Again, some rectangles may not be possible, or there may be multiple rectangles.

(a) Area less than Perimeter

(e) Max Area using < 20 tiles

(b) Area equal to Perimeter

(f) Minimum Perimeter with Area = 16

(c) Area greater than Perimeter

(g) Largest Perimeter with Area = 16

(d) Area = Perimeter + 1

(h) Perimeter not a min/max with Area = 12

III. **General Relationships**
Now that you've had some experience creating rectangles with given properties, think about the following general questions. Write a sentence or two answering each question and justifying your answer.

(a) If two rectangles have the same perimeter, will they have the same area as well?

(b) What is the shape of the rectangle that will have the largest possible area for a given perimeter?

(c) What is the shape of the rectangle that will have the largest possible perimeter for a given area?

7.4 Discovering Pick's Formula

In 1899, Georg Pick discovered a beautiful formula for calculating the area of a polygon that can be formed on a geoboard using only the number of geoboard points on the interior of the polygon and the number of geoboard points on its boundary. Your task in this activity is to discover this relationship for yourself.

Objectives:

- Students will systematically explore a problem and practice recognizing patterns.
- Students will develop Pick's formula for finding the area of a polygon on a lattice.

I. **Systematically Exploring Boundary Points**

Consider the polygon drawn on the geoboard grid below. To the right you've been given the number of interior and boundary points of the polygon.
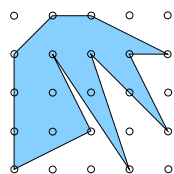
- The number of interior points is $I = 3$
- The number of boundary points is $B = 14$
- The area of the polygon is $A = 9$ units^2

It is not intuitively obvious how to combine $I = 3$ and $B = 14$ to get the area of 9. In order to discover this relationship, you will have to work systematically to see what effect changing these values has on the area. In this first part of the activity, we will explore changing B, the number of boundary points.

(a) Construct two different polygons on the provided geoboards that have the given number of boundary points and no interior points.
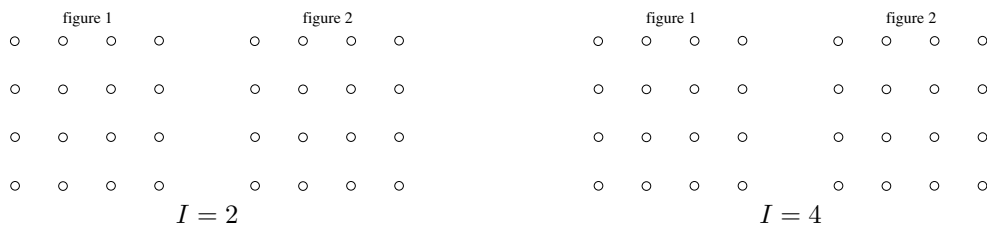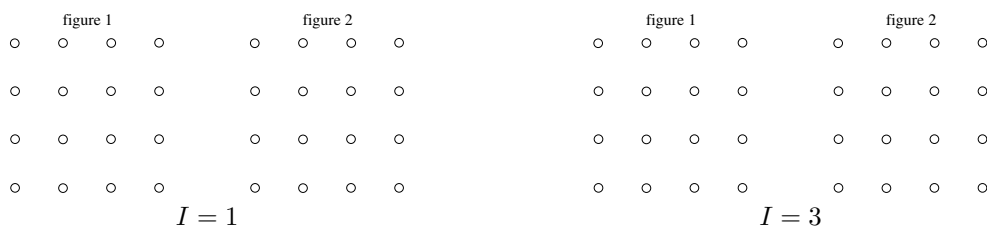

$B = 3$

$B = 6$

$B = 4$

$B = 7$

$B = 5$

$B = 8$

170
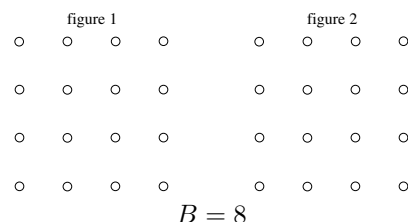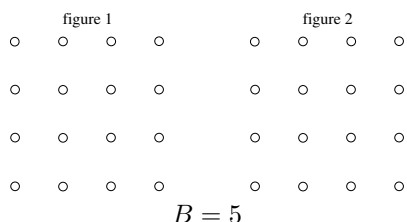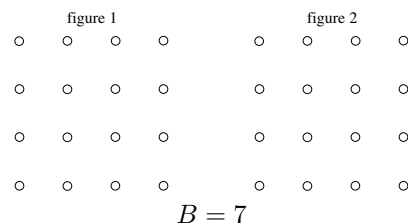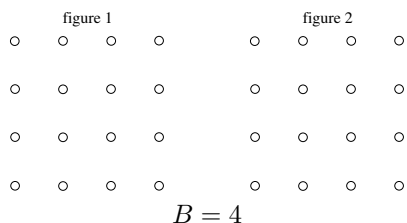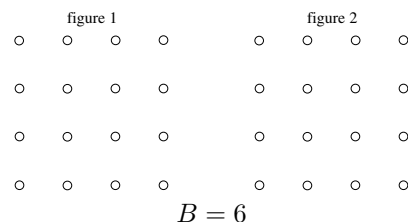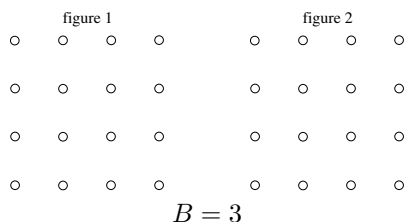
(b) Now compute the area of each of your figures and fill in the appropriate entries in the table on the next page.

(c) Write a sentence explaining any patterns you see. How does the number of boundary points affect the area of the polygon?

II. **Systematically Exploring Interior Points**
Now that you have a sense for what boundary points do to the area, it is time to explore I, the number of interior points.

(a) Keeping the number of boundary points fixed at $B = 8$, draw two different polygons with the given number of interior points.

figure 1   figure 2   figure 1   figure 2

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

$I = 1$ $I = 3$

figure 1   figure 2   figure 1   figure 2

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○   ○

$I = 2$ $I = 4$

(b) Again, compute the area of each of your figures and fill in the appropriate entries in the table on the next page.

(c) What patterns did you notice when changing the number of interior points? Write a sentence describing how the number of interior points affects the area of the polygon.

III. **Putting It All Together**
    You should now have the table below completely filled out, and you should have a good idea how the number of interior points I and the number of boundary points B affect the area of a polygon A. Use this information to complete the following.

Interior Points (I)	Boundary Points (B)	Area (A)
0	3	
0	4	
0	5	
0	6	
0	7	
0	8	
1	8	
2	8	
3	8	
4	8	

(a) Now use this information to develop a formula for the area of a polygonal region based (A) based on the number of interior points (I) and boundary points (B) of the region.

(b) Test your formula on the following polygonal regions.
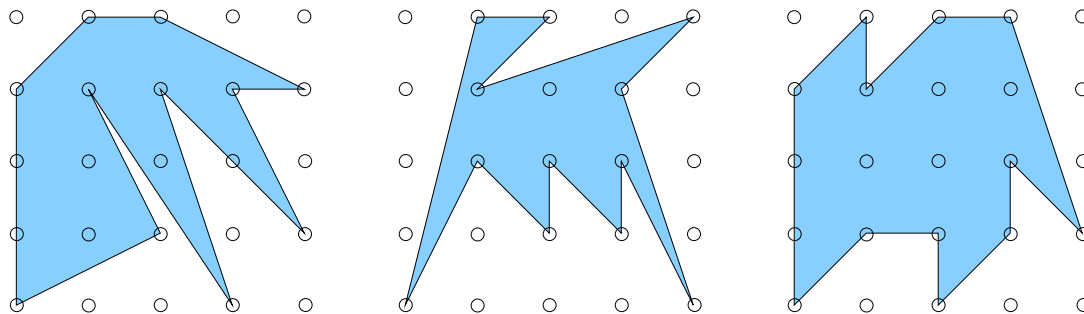
## 7.5   Area and Perimeter on a Geoboard

Working with polygons on a geoboard is often a good way to understand the concepts of area and perimeter, and how to best compute these measurements. In this activity, you will practice finding perimeters and finding areas in several different ways for polygons on a geoboard.

Objectives:

- Students will apply systematic thinking to recognize patterns and solve problems.
- Students will apply the notion of decomposition in computing area.
- Students will understand the notions of perimeter as length and of area as covering.

I. **Finding Area and Perimeter**
Finding the area and perimeter of a figure on a geoboard is often best accomplished by dividing the figure up into rectangles and triangles. Take, for example, the figure below.
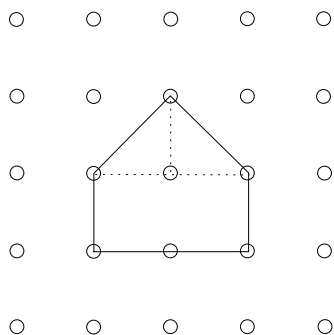
We divide the polygon into two triangles and a rectangle as shown. Since the distance between two adjacent pegs, measured vertically or horizontally, is one unit:

- the area of the rectangle is 2 units^2 (count the squares).
- The area of each triangle is $\frac{1}{2}$ units^2 (half of one square unit).
- The total area is $2 + \frac{1}{2} + \frac{1}{2} = 3$ units^2.

To find the perimeter, we must add the lengths of all sides together. Remember that diagonal distances from one peg to another are **not** one unit! Use the Pythagorean theorem to calculate these distances. Thus, we compute the perimeter of the figure above as follows.

- The perimeter along the bottom is 2 units.
- The perimeter of each vertical side is 1 unit.
- The perimeter c of each of the two diagonals is found using the Pythagorean theorem.
$$1^2 + 1^2 = c^2 \quad \Rightarrow \quad c^2 = 2 \quad \Rightarrow \quad c = \sqrt{2}$$
- The total perimeter is $2 + 1 + 1 + \sqrt{2} + \sqrt{2} = 3 + 2\sqrt{2}$ units.
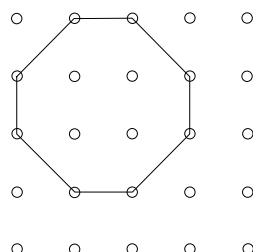
Using these methods, find the area and perimeter of each of the geoboard figures below and on the back of this page. If you are familiar with Pick's Formula, use it to verify your area computations.
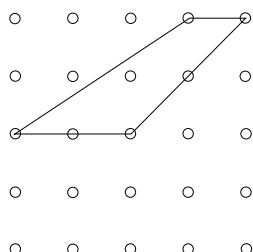
Area: _____

Perimeter: _____
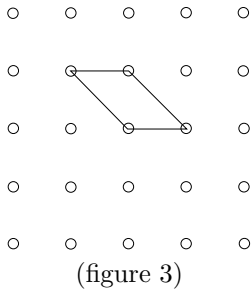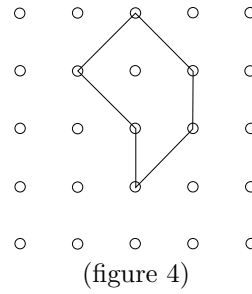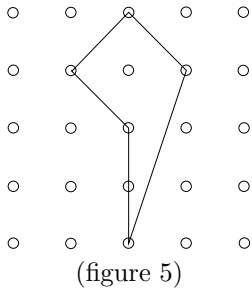
(figure 1)

Area: _____

Perimeter: _____

(figure 2)
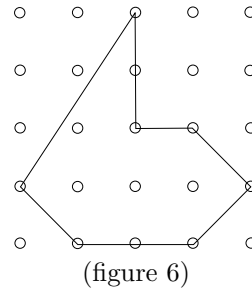
173

Area: _____

Perimeter: _____

(figure 3)

Area: _____

Perimeter: _____

(figure 4)

Area: _____

Perimeter: _____

(figure 5)

Area: _____

Perimeter: _____

(figure 6)

Area: _____

Perimeter: _____

(figure 7)

Area: _____

Perimeter: _____

(figure 8)

Area: _____

Perimeter: _____

(figure 9)

Area: _____

Perimeter: _____

(figure 10)

Area: _____

Perimeter: _____

(figure 11)

Area: _____

Perimeter: _____

(figure 12)
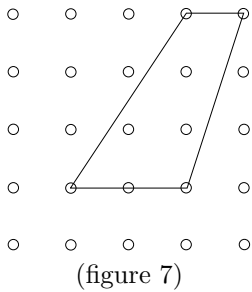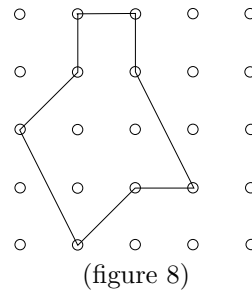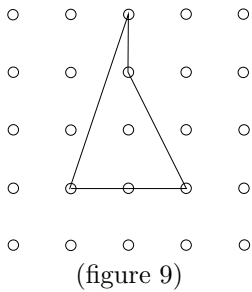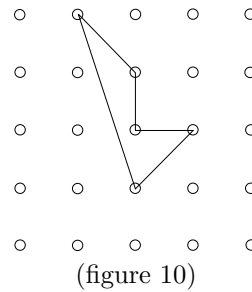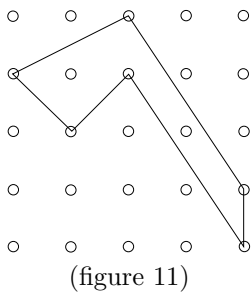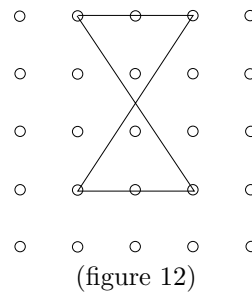
Adapted from Frank Lester, Jr. *Mathematics for Elementary Teachers via Problem Solving*, 79-80.

7.6 Exploring the Perimeter and Area of a Circle

The formulas for the area and perimeter of a circle are well known, but not very well understood. You likely remember that the perimeter of a circle of radius r (usually called its circumference) is $C = 2\pi r$, but do you know why? You probably learned that the area of a circle of radius r is $A = \pi r^2$, but where does this formula come from? In this activity we will explore these questions with actual circles.

Objectives:

- Students will develop an understanding of where the area and perimeter formulas for a circle come from.
- Students will become familiar with the concept of π.

I. **The Circumference of a Circle**
   The formula for the circumference of a circle is $C = 2\pi r$ if one knows the radius r, or $C = \pi d$ if the diameter of the circle, d, is known. In fact, this formula was originally written as $\pi = \frac{C}{d}$, defining the constant π as the ratio of a circle's circumference to its diameter. In this first part of the activity, you will estimate pi by measuring the circumference and diameter of several circles.

(a) Use the provided instruments to measure the circumference and diameter of each circle below.
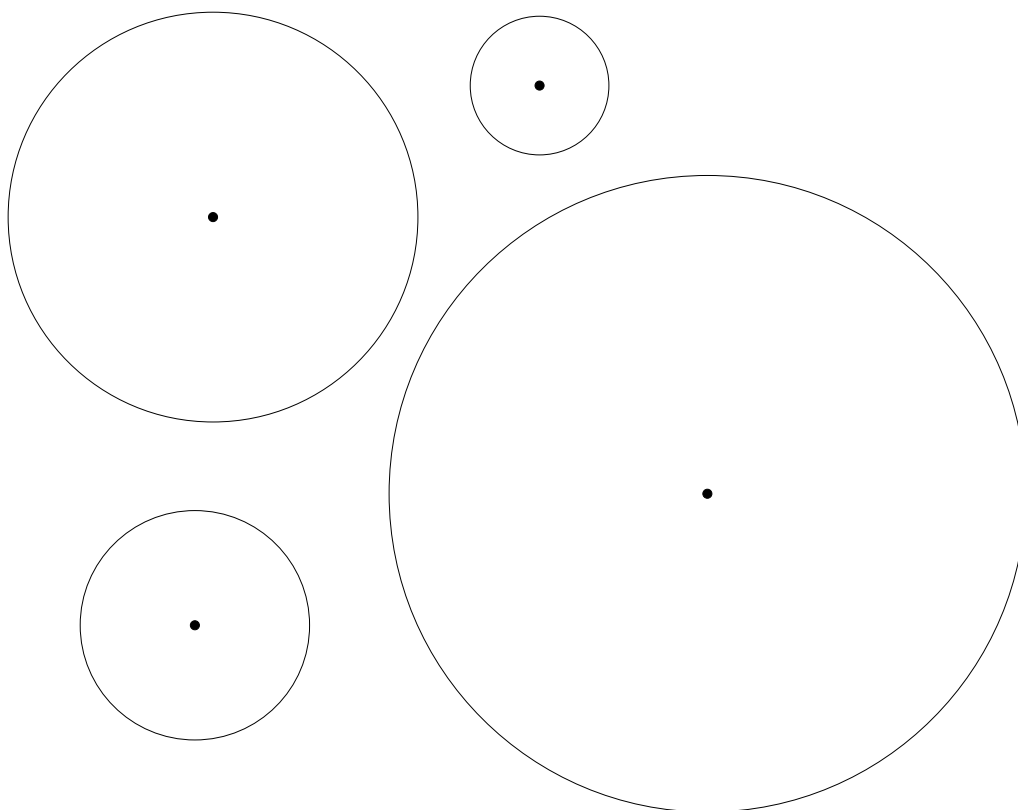
(b) Use each pair of measurements to estimate π to four decimal places. Explain any differences.

II. **The Area of a Circle**

The formula for the area of a circle is $A = \pi r^2$ where r is the radius of the circle. It is difficult to justify this formula by looking an a whole circle. In this section, you will cut a circle into sectors as shown below, and arrange them to justify the area formula.

(a) Cut the circle provided by your instructor into sectors, and then arrange them as shown below. Describe the shape into which you arrange the sectors.
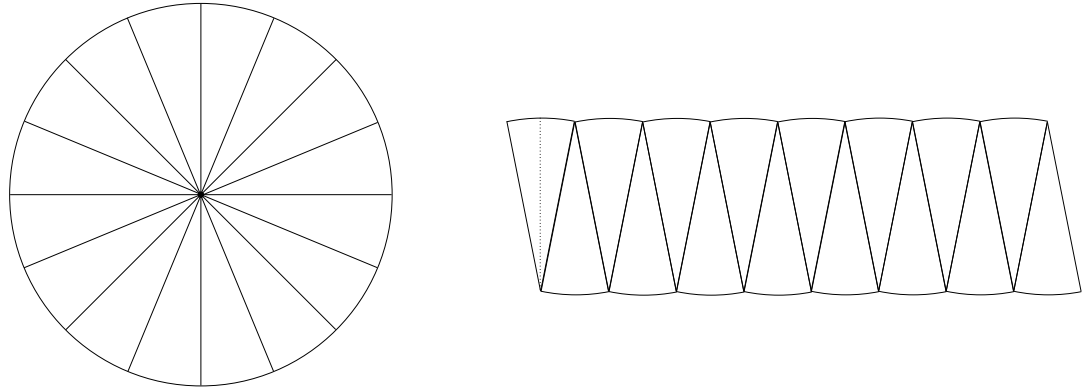
(b) Now cut each sector in half, from the tip to the half-way point of the arc, along the dotted line shown above. Rearrange them into a similar shape to the one shown above. To what well-known polygon is your rearrangement now even more similar than the one above?

(c) If one were to continue cutting sectors in half and rearranging, the new shape would get closer and closer to the polygon you mentioned above. If the radius of the original circle was r, find the area of this "limiting" polygon.

7.7 Varying Surface Areas

As you change the dimensions of a three-dimensional object, how will its surface area be affected? This question may not have an intuitive answer, as the formulas for surface area rely on the dimensions of the shape in several different ways. In this activity you will explore the relationship between the dimensions of a three-dimensional shape and its surface area.

Objectives:

- Students will better understand the relationship between surface area and the dimension of a shape.
- Students will practice computing surface area for several common three-dimensional shapes.

I. **Rectangular Prisms**
The surface area of a right-rectangular prism can be found using the formula $S = 2lw + 2lh + 2wh$ where l is the length, w the width, and h the height of the prism. Use this formula to help you answer the following questions.

(a) Suppose that a cube measures s units on a side.

    i. What will the surface area of the cube be?

    ii. If the side length of the cube is doubled, what happens to the surface area?

    iii. If the side length of the cube is multiplied by n, what happens to its surface area?

(b) Now consider a rectangular prism with length l and width and height s, so that one end is a square.

    i. What will the surface area of the prism be?

    ii. If the length l is doubled, but s remains the same, what happens to the surface area?

    iii. If the width and height s is doubled but l remains the same, what happens to the surface area?

(c) Write a paragraph explaining what happens to the surface area of a right rectangular prism as the dimensions are changed.

II. **Right Circular Cones**

A cone has three important linear measurements. The radius r, the height h and the slant height l as shown below. Cut out the circle on the provided sheet, and then cut down the radius line. By placing the cut line on top of the lettered surface of the circle and moving the edge of the cut line to one of the lettered positions, the lateral surface of a cone is formed. Moving the cut edge from letter to letter varies the height of the cone.
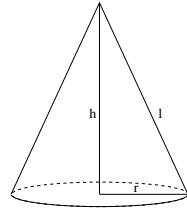
(a) Does the radius of the cone increase, decrease, or remain the same as you move from letter A to letter B to letter C?

(b) How does the slant height change as you move from A to B to C?

(c) What is the largest possible radius and at what letter does it occur?

(d) What is the greatest possible height, and at what letter does it occur?

(e) At what height is the lateral area the greatest?

(f) At what height is the area of the base the smallest?

(g) As the height of the cone decreases, what value does the area of the base approach?

(h) As the height of the cone decreases, what value does the lateral surface area approach?

(i) As the height of the cone decreases, what value does the total surface area approach?

7.8 Relating Volume and Surface Area

The surface area of a solid is the number of *square* units it takes to cover its surface. The volume is the number of *cubic* units it takes to fill the space it occupies. Are these two measurements related? In this activity, you will explore this question by building figures to various specifications.

Objectives:

- Students will develop a better understanding of surface area and volume.
- Students will explore the relationship between surface area and volume.

I. **Building Rectangular Prisms**

Use the provided centimeter cubes to build rectangular prisms to the following specifications. Include sketches of each prism as your answer to the given problem.

(a) Build all possible right rectangular prisms that can be made from 8 centimeter cubes (so that the volume is 8 cm^3).

(b) Compute the surface area of each of the prisms you constructed above.

(c) If you were given 64 centimeter cubes, how would you construct a rectangular prism with the smallest possible surface area? What would the dimensions of your prism be?

(d) Using those same 64 cubes, how would you construct a prism with the largest possible surface area? What would its dimensions be?

(e) In general, what are the dimensions of the rectangular prisms of a fixed volume V that have the largest and smallest possible surface area S?

II. **Circular Cylinders**
Consider a right cylinder whose base is a circle of radius r and having height h.

(a) What are the volume and surface area of the cylinder?

(b) Suppose the two cylinders shown below have the same volume. If the radius of the first is twice the radius of the second, how do their heights compare? Be specific.
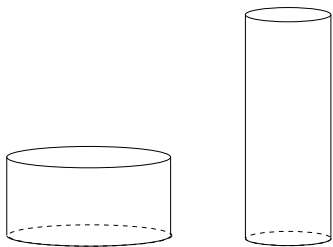
(c) How would the surfaces areas of the two cylinders above compare? Again, be specific.

(d) If two cylinders had the same surface area, must they necessarily have the same volume? Give justification or examples for your answer.

Adapted from Frank Lester, Jr. *Mathematics for Elementary Teachers via Problem Solving*, 88-93.

7.9 Varying Volume

As you change the dimensions of a three-dimensional object, how will its volume be affected? The answer to this question may not be intuitive because of the way volume is computed. In this activity, you will explore the relationship between the dimensions of a three-dimensional shape and its volume.

Objectives:

- Students will better understand the relationship between volume and the dimensions of a shape.
- Students will practice computing volume for several common three-dimensional shapes.

I. **Rectangular Prisms**
   Use the provided centimeter cubes to help visualize the various shapes mentioned below. Remember that the formula for the volume of a right-rectangular prism is $V = lwh$ where the length is l, the width w, and the height is h.

   (a) Build a rectangular prism that is 1 cm by 2 cm by 3 cm. What is the volume of your prism?

   (b) Now build a rectangle that is twice as long, twice as wide, and twice as high. What effect does doubling each of the dimensions have on the volume?

   (c) Suppose the volume of a cube is V.

      i. What is the volume (in terms of V) of a cube whose sides are four times as long?

      ii. What is the volume of a cube whose sides are half as long as the given cube?

Adapted from Frank Lester, Jr. *Mathematics for Elementary Teachers via Problem Solving*, 90.

7.10 The Volume of Irregularly Shaped Objects

There is an ancient, and mostly likely fictional, account of the Greek mathematician Archimedes's discovery of a method for measuring the volume of an irregularly shaped object. The story goes something like this.

The tyrannical king Hiero demanded that Archimedes prove his goldsmith was stealing. The king believed that the goldsmith had replaced replaced some of the gold he was given to make a crown with silver. Archimedes was threatened with death if he did not find proof.

During a trip to the public baths, as he was worrying about how to accomplish the task, Archimedes realized that the more his body sank into the water, the more water was displaced. Thus, he could measure the volume of an object by measuring the amount of water it displaced. Because gold weighs more than silver a crown of mixed gold and silver would weigh less than a crown of the same volume made of pure gold. As Archimedes realized he had discovered a solution, he leapt out of the bath and rushed home naked crying "Eureka! Eureka!" Which translates as: "I've found it! I've found it!"

In this activity, you will remain fully clothed while duplicating the rest of Archimedes experiment.

Objectives:

- Students will use one method of measuring the volume of an irregularly shaped object.
- Students will practice estimating the volume of irregularly shaped objects.

I. **Estimation of Volume**
   Your group will be provided with several regularly and irregularly shaped objects. Your first task is to either compute or estimate the volume of each object in cubic centimeters. You may use the provided tape measures as well as formulas for the volume of regular objects such as prisms, pyramids, cones, and spheres. Write the estimated volume of each object and a sentence or two describing how you got it *before* you continue on to the next page.

(a) Object: _____

(b) Object: _____

(c) Object: _____

(d) Object: _____

II. **Measurement of Volume**

You will now get a more accurate measurement of each objects volume using the provided measuring cups, water, and a copious amount of paper towels to clean up any spills.

(a) Before taking any measurements, discuss procedure with your group. Here are some questions you may wish to consider:

- How will you measure the amount of water displaced by an object?
- How will you convert your volume to cubic centimeters? (1 fluid ounce ≈ 29.6 cubic centimeters)
- What if the object floats?

(b) Now use the strategy you devised above to measure the volume of each object. If you have any difficulties or observations, record them below.

i. Object: _____   iii. Object: _____

ii. Object: _____   iv. Object: _____

(c) How good were your estimations? Compute the percent error in your estimations as follows.

Example: If you estimated 12 cubic centimeters of volume, but the actual volume was 13 cubic centimeters, your error is 1 cubic centimebers. So your percent error is $1/13 \approx 0.0769$ or 7.69%.

i. Object: _____   iii. Object: _____

ii. Object: _____   iv. Object: _____

7.11   Packing Efficiency

Many consumer products come inside of three dimensional surfaces. for example, cereal usually comes in a right rectangular prism. Soda often comes in a right circular cylinder (approximately). By finding the ratio of the volume of a particular package to its surface area, we can compute a number that indicates how efficiently the packaging is being used. In this activity, you will practice several of these computations in computation for a lab project.

> **Objectives:**
>
> - Students will gain experience computing surface areas and volumes.
> - Students will explore issues related to packing efficiency.

I. **Computing Efficiency Ratios**
In this first part of the activity, you will be given the shape and dimensions of several commercial containers. Your job is to:

- Compute the volume of the containers
- Compute the surface area of the containers
- Find the efficience ratio of the containers (volume/surface area)

Efficient containers will enclose the greatest volume within the least amount of surface area, and will therefore have a higher efficiency ratio.

(a) A Cheerios Box
A particular Honey Nut Cheerios box is a right prism with dimensions $7\frac{3}{4}$" by $11\frac{7}{8}$" by $3\frac{11}{16}$".

(b) A Toblerone Bar
A particular Toblerone chocolate bar comes in a triangular prism. The lateral sides of the box are 12 cm long and the triangular ends are equilateral triangles with sides 3 cm long.

(c) A Quaker Oats Container
A particular Quaker Oats package is a right circular cylinder with diameter $5\frac{1}{8}$" and height $9\frac{5}{8}$".

II. **Analyzing the Ratios**

Based on the computations just completed, answer the following questions about the three containers.

(a) What are the units of your efficiency ratios? Can you directly compare the ratios of the various containers, or do you need to make any conversions?

(b) After making any necessary conversions, determine which container is most efficient. Are you surprised by the results?

III. **Group Project: Analyze Cereal Packing Efficiency**

Our Math for Elementary Teachers class has decided to start a cereal company to produce and sell delicious breakfast delicacies such as "Mathy Flakes" and "Geo-Ohs." Your lab group has been given the task of determining the shape of the packages in which we sell the cereal. Write a two-page proposal including the following sections:

- a market review in which you find the dimensions and efficiency ratio of at least three competing packages. The three must be different from the packages seen in part I, and only two of the three may be rectangular prisms.

- an analysis of which simple surface (prisms, cylinders, pyramids, cones, and spheres) would provide the most efficient packaging. That is, which shape gives the most volume for a given amount of surface area.

- a concluding paragraph in which you recommend the shape for our new company to use. Discuss issues other than efficiency in making your selection. If your final choice is not the most efficient shape from your previous section, explain why.

7.12 Understanding Reflections

Reflecting an object over a line is one of the four basic rigid motions, and also can lead to beautiful symmetries. In this activity, you will develop several rules for reflecting figures over lines and hopefully gain some intuition for what reflections will look like.

Objectives:

- Students will gain intuition for reflection transformations.
- Students will discover properties of multiple reflections.

I. **Reflections and Distance**

Reflecting a figure over a line changes the position of the figure. How much the new figure moves is dependent on the placement of the line of reflection. In this first section, you will develop a general rule for predicting the displacement of a figure when it is reflected over a line.

(a) Start by sketching the reflection of the letter F below over the given lines.

i.

iii.

ii.

iv.
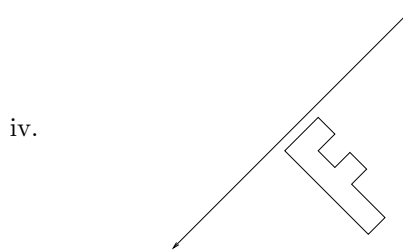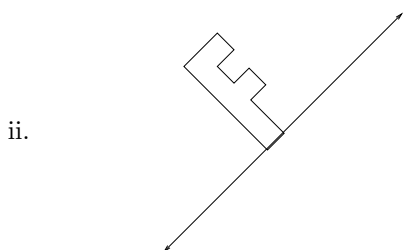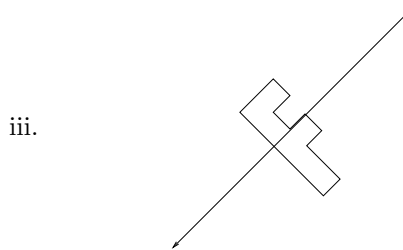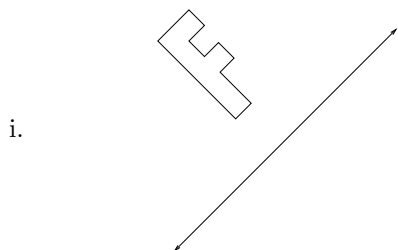
(b) Based on the sketches above, how does the line of reflection affect how much the position of a reflected figure will change? Write a sentence or two explaining this relationship.

II. **Reflections and Orientation**

The orientation of a figure is also affected by reflecting it. How the orientation changes depends on the placement of the line of reflection. Again, your task is to develop a general rule for predicting the orientation of a figure when it is reflected over various lines.

(a) Start by sketching the reflection of the kite below over the given lines.

i.

iii.

ii.

iv.
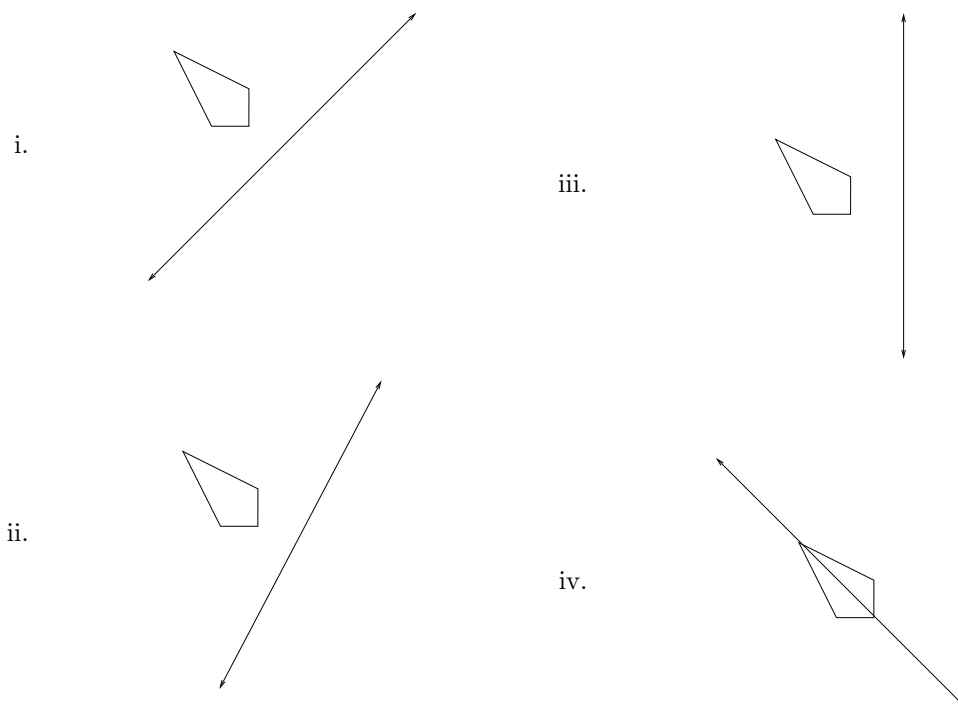
(b) Based on the sketches above, how does the orientation of a reflection change in relation to the line of reflection? Write a sentence or two describing this relationship.

(c) Are there other basic transformations that would produce the same result? For each of the figures above, either describe such a transformation or argue that none exists.

    i.   iii.

    ii.   iv.

III. **Multiple Reflections**
In the last part of this activity, you will explore what happens when a figure is reflected over two different lines one after the other. Your task is to determine what general rules exist in these situations.

(a) Start by sketching the reflection of the letter P below first over line m, and then over line l.

i.

ii.
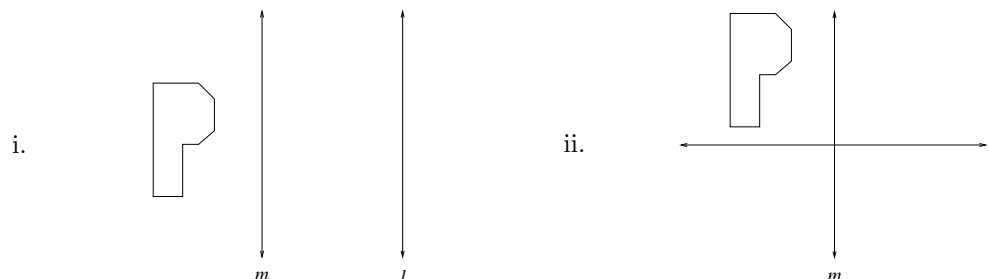
(b) Are there other basic transformations that would produce the same result? For each of the figures above, either describe such a transformation or argue that none exists.

i.   ii.

(c) Does the order in which you perform the reflections matter? In the figures below, reflect over line l first and then line m.

i.   ii.
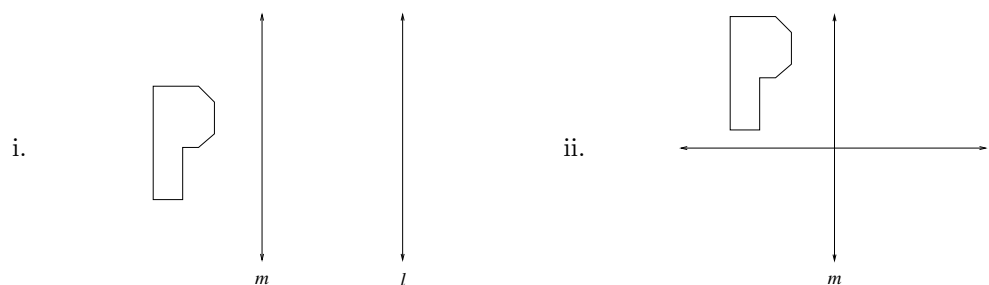
(d) Based on these sketches, describe the results of reflecting a figure over two lines, one after the other.

7.13 Name That Transformation

There are often several different transformations or sequences of transformations that can be used to go from a starting figure to a different, but congruent, figure. In this activity, you will practice identifying multiple transformations that produce the same shape.

Objectives:

- Students will gain experience with the four basic congruence transformations.
- Students will gain insight into the relationships between congruence transformations.

I. **Single Transformations**
Every congruence transformation can be performed using a exactly one of the four basic rigid motions:

- A translation (slide)
- A rotation (turn)
- A reflection (flip)
- A glide-reflection

In this first exercise, you will practice identifying different basic rigid motions that produce the same transformation.

(a) Start by identifying and describing a basic ridged motion (slide, flip, turn, or glide reflection) that would take figure A to figure B on each of the geoboards below.
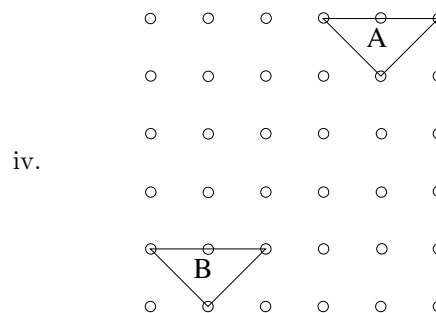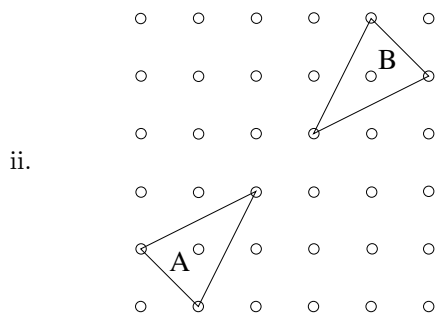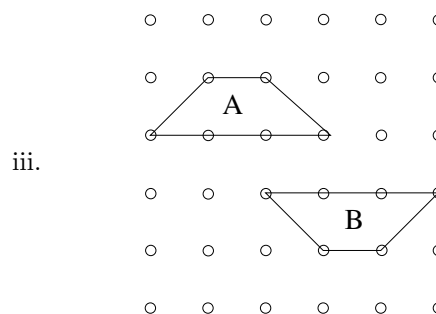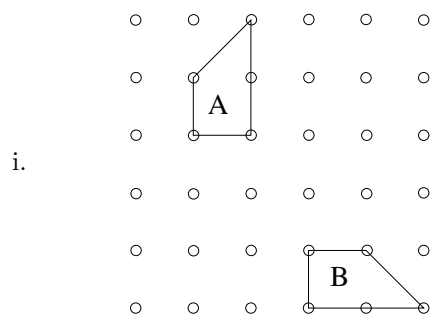
(b) Now for each of these figures, determine if there is a different single transformation that takes A to B. If so, describe it. If not, explain why not.

i.

iii.

ii.

iv.
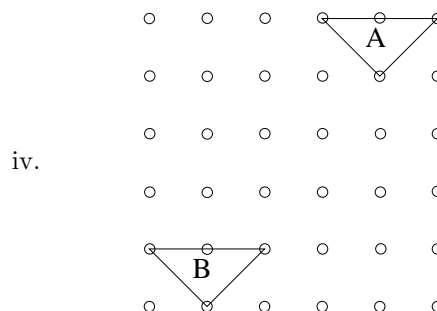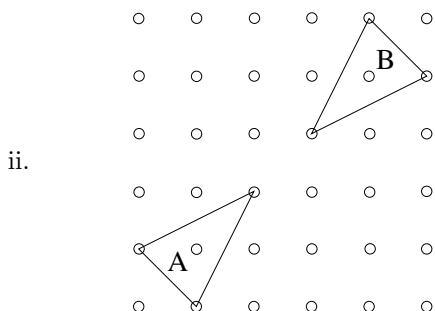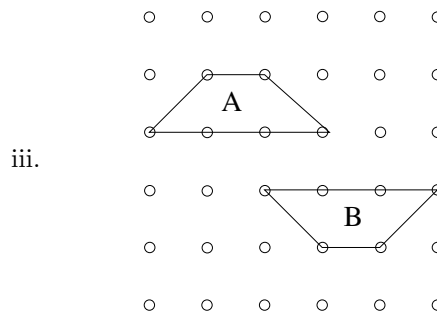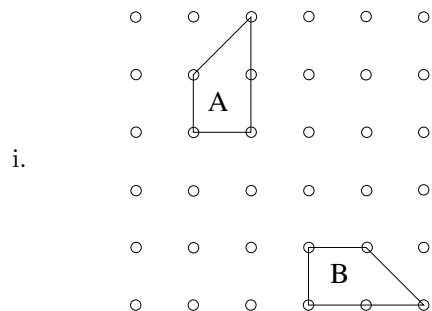
II. **Multiple Transformations**

You should have found at least one transformation above that can be done in only one way using a single rigid motion. However, when you combine transformations, there are many more possibilities.
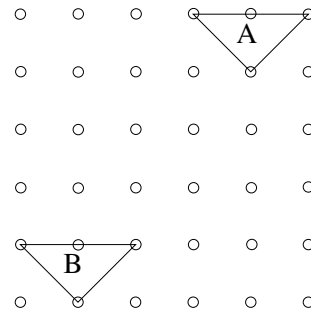
(a) Working again with these same shapes, find a sequence of two or more transformations that are non-trivial (meaning they actually move the figure) and not inverses (meaning one does not undo another) which take figure A to B.

i.

ii.

190
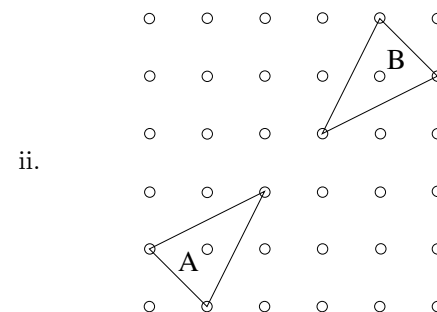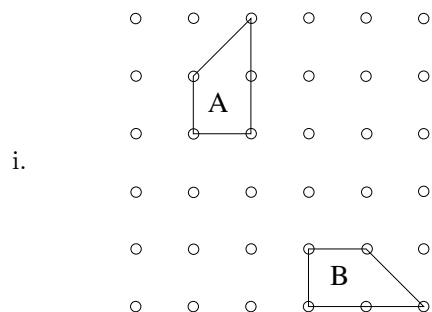
iii.
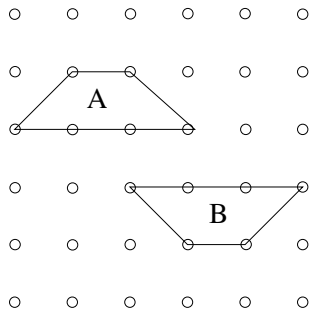
iv.

7.14 Groups of Symmetries

When a congruence transformation takes a figure and places it back in the same position with the same orientation, that transformation is called a symmetry. With non-repeating figures, the two types of symmetries that can occur are symmetries in a reflection, called lines of symmetry, and symmetries in a rotation, called points of symmetry. In this activity you will examine the symmetries of several basic polygons and discover how they can be combined.

Objectives:

- Students will gain experience working with lines and points of symmetry.
- Students will learn that transformations are geometric operations.

I. **Symmetries of the Triangle**
   The simplist regular polygon is an equilateral triangle. Even this most basic shape, however, has several different symmetries. In this first part of the activity, you will explore the symmetries of a triangle.

   To get started, construct a triangle as follows.

   1. Trace and cut out the triangle to the right.
   2. Number the vertices of one side of the triangle 1, 2, and 3 in clockwise order.
   3. Number the vertices of the other side to match (the same number on each side of a vertex).
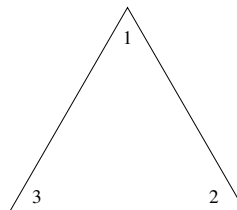
We can now distinguish between symmetries by starting with the triangle in "standard" position (1 on top, 2 at the bottom right, and 3 at the bottom left) and then observing where the numbers wind up after a flip, turn, or combination thereof.

(a) There are three distinct point symmetries, named below, that can be found by rotating the triangle around its center point. Fill in the vertex numbers to show the result of applying each symmetry.
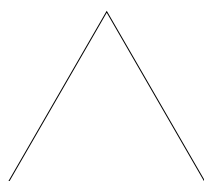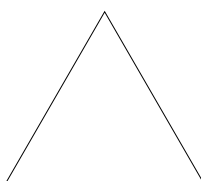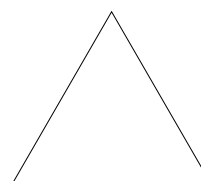
R_{120} $\qquad$ R_{240} $\qquad$ $R_{360} = R_0 = Id$

(b) There are three distinct line symmetries, named below, than can be found by flipping the triangle over a line of reflection Fill in the vertex numbers to show the result of applying each symmetry.

V $\qquad$ D_L $\qquad$ D_R

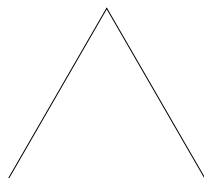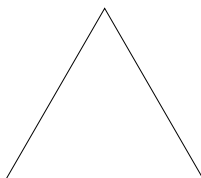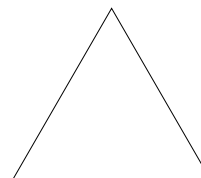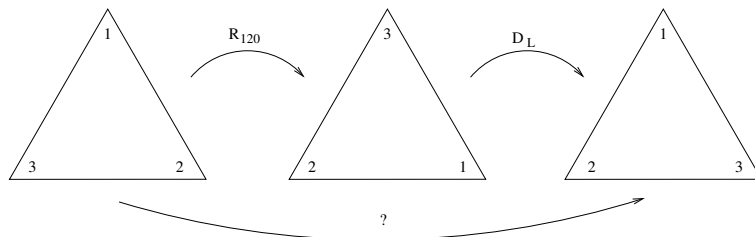(c) If we think of these symmetries as objects (like numbers), we can combine them by performing one transformation after another. Consider the picture below.

The first transformation is R_{120}, a 120° rotation clockwise. The second transformation is D_L, a flip over the diagonal line through the left bottom vertex. What single symmetry results in the same final triangle as the combination above?

(d) Performing two symmetries one-after-the-other is called composing the symmetries. The resulting symmetry is the product of the two symmetries we composed. Fill in the body of the table below with the product of the symmetries across top row (performed first) with the symmetries in the left column (performed last).

Two examples have been filled in for you. Note the order in which the symmetries are listed!

Example 1: $R_{120} \circ R_{240} = Id$

Example 2: $D_L \circ R_{120} = V$

Fill in the rest of the table!

$\circ$	Id	R_{120}	R_{240}	V	D_L	D_R
Id						
R_{120}			Id			
R_{240}						
V						
D_L		V				
D_R						

(e) If we view the table above as an operation table, we can ask questions similar to those that we asked last quarter about $+$, $-$, $\times$, and $\div$. Answer some of these questions below.

  i. Is the operation closed?

  iii. Does each symmetry have an inverse?

  ii. Does the operation have an identity?

  iv. Is the operation commutative?

(f) By carefully examining the operation table for these symmetries, we can see that we don't really need to use six symmetries. For example, we don't need V if we have R_{120} and D_L because $D_L \circ R_{120} = V$. Find a minimum set of symmetries that can be used to *generate* all of the others.

II. Symmetries of the Square

This same procedure that we just completed for the triangle can be repeated for any regular polygon. Cut out a square and label its vertices as shown to the right. Using this square, answer the following questions about its symmetries.
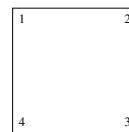
(a) How many distinct point or line symmetries does a square have?

(b) Label the vertices in the square below to match the final position of the square after each of these symmetries. Name each symmetry in a way similar to what we did for the triangle. Note that you may be given more blank squares than are needed!
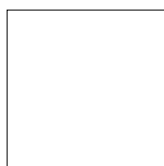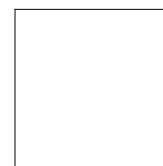
(c) Fill in the operation table for the symmetries of the square provided below.

$\circ$

(d) Find the identity in this operation table.

(e) Find the inverse of each symmetry in the table.

(f) Does the composition operation represented in this table have the associative property? In other words, is it true that $(x \circ y) \circ z = x \circ (y \circ z)$?

7.15 Tiling the Plane

Have you ever noticed that most floor tiles are squares? If so, you probably just attributed it to the fact that squares are easy to make and fit together nicely, but is there more to it than this? Covering a floor of arbitrary dimensions with a collection of polygons that are all congruent, don't overlap, and don't leave any gaps between them is just one example of the process called tiling a plane. In this activity you will explore tiling the plane with various polygons.

Objectives:

- Students will identify which regular polygons tile the plane.
- Students will be able to determine if a given shape tiles the plane.

I. **Regular Polygons**

A square is an example of a regular polygon, in which all sides have equal length and all angles have equal measure. Below are diagrams of the first few regular polygons, including the square.
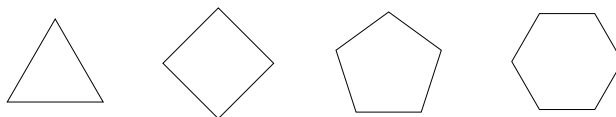
Trace several copies of these figures on a blank piece of paper and then cut them out. Use these manipulatives to help you answer the following questions.

(a) Determine if each regular polygon above can be used to tile the plane. If it can, sketch an example tiling. If it can not, explain why.

    i. Equilateral Triangle   iii. Regular Pentagon

    ii. Square   iv. Regular Hexagon

(b) You should have seen above that the regular pentagon does not tile the plane, but the other regular polygons do. Develop a test to determine if a regular polygon will tile the plane based only on its number of sides. Here are some useful facts to remember:

- There are $360°$ in a full rotation.
- The sum of the measures of the interior angles of a regular n-gon is $180°(n-2)$.

II. **Irregular Polygons**

While there are only three regular tilings of the plane, if one uses irregular polygons, an infinite number of tilings is possible. In this section you will create your own tilings, and also create polygons which do not tile the plane.

(a) Use the geoboard grids provided below to create two distinct non-regular polygons that will tile the plane. Think about the strategies you use to ensure that your polygon will tile.
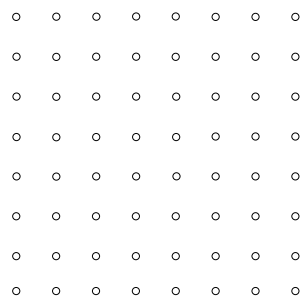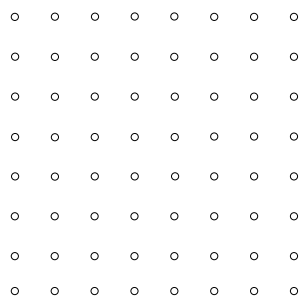
(b) Finally, create two distinct non-regular polygons that will not tile the plane and sketch them on the grids provided below.
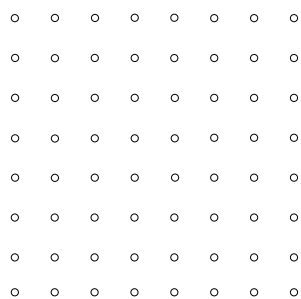
7.16   Escher Tilings

Dutch artist M.C. Escher is famous for his many mathematically based works of art, an example of which is shown to the right. Many more examples of Escher's work can be found at the website http://www.mcescher.com/. Using what you have learned about translations and rotations, you are now in a position to be able to understand how Escher made his tilings, also called tessellations. In this activity you will learn to make what we will call Escher tilings.
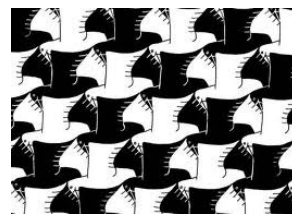
Objectives:

- Students will understand how tessellations can be created with transformations.
- Students will create their own tessellation using transformations.

I. **Transformations and Escher Tiling**

Escher created his tessellations by starting with a simple polygon that would itself tessellate and then using a combination of the following transformations.

- **Translations**

  Recall that a translation moves a figure from one position to another. We start with a tessellating polygon, such as the rectangle shown in step 1 below. Then modify one side of the polygon as shown in step 2. Finally, translate that modification to the opposite side as in step 3. The resulting figure is no longer a rectangle, but will still tessellate.

  Step 1   Step 2   Step 3
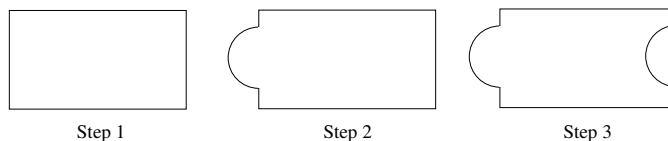
- **Rotations**

 The next transformation is a rotation, in which we rotate a figure about a center point. We start again with a tessellating polygon, such as the triangle shown in step 1. Next we modify a side as seen in step 2, and then finally we rotate that modification to another side about a center of rotation (which can be either a vertex or a midpoint) resulting in the figure seen in step 3.

  center of rotation

  Step 1   Step 2   Step 3
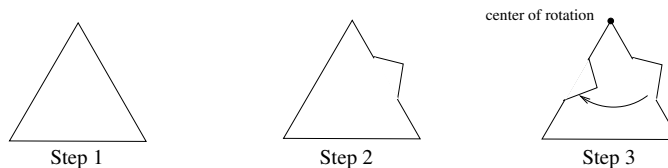
Below is an example of how transformations can be combined to create a complex tessellating figure.

  Step 1   Step 2   Step31

  Step 4   Step 5   Step 6   Tessellating Figure

198
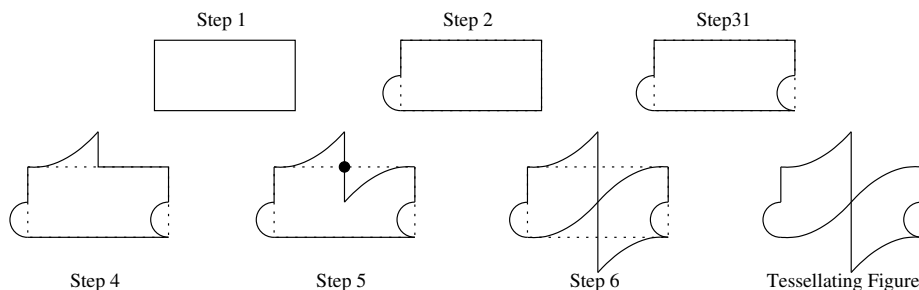
II. **Identifying Escher Tilings**

Now that you know some of the basic methods for creating Escher style tessellations, it is time to work backwards. Each of the tessellating figures below was created from a basic tessellating figure by a sequence of transformations. Determine what that starting polygon was and what transformations were used to change it into the figure given.

(a)

(c)

(b)

(d)
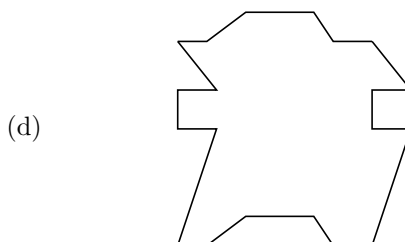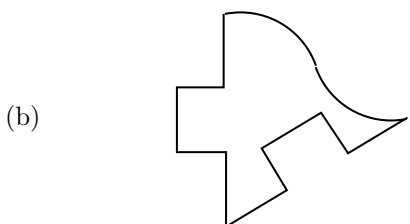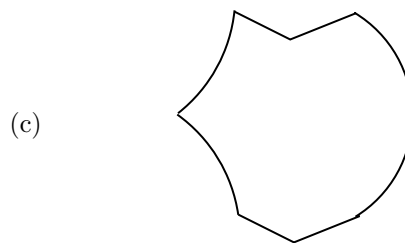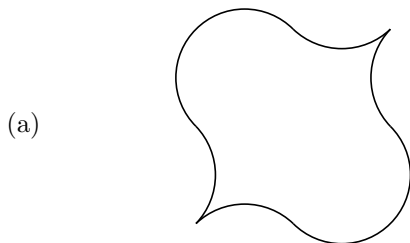
III. **Group Project: Create an Escher Tessellation**

As your group project for this lab, create an Escher-like tessellation of your own. The following criteria need to be followed.

- You must use at least three transformations, with a minimum of one translation and one rotation, to create your tessellating figure.

- You must then create a poster showing at least six copies of your figure tessellated.

- Use colors or other details inside your figure as appropriate (for an example see the bird tessellation on the first page of this lab activity).

- You must provided step-by-step instructions for creating your figure starting with a basic polygon.

Bring your project poster with you to the next lab. We will begin the lab period with an art show exhibiting the tessellations created by various groups. Be creative!

Adapted from Bassarear *Mathematics for Elementary Teachers Explorations*, 274-276.

7.17 Toothpick Triangles

How many different triangles can be constructed with a given number of toothpicks by connecting them only at their ends? While this may seem like a trivial question, in the process of solving it one comes to a better understanding of congruent triangles. In this activity, you will work through this problem for several different numbers of toothpicks.

Objectives:

- Students will gain experience building triangles with given side lengths.
- Students will understand the triangle inequality.

I. **Triangles by Length of Sides**

One of the tests for triangle congruence is the side-side-side test. This test states that if two triangles have the same three side lengths, then the triangles are congruent. In this activity, you will be asked to find the number of different (non-congruent) triangles that can be made out of toothpicks with certain side length restrictions. Give each triangle as a list of side lengths (i.e. 1-1-1, 1-1-2, etc.).

(a) How many triangles can be made where there is a side one toothpick long, and no other side is more than one toothpick long?

(b) How many triangles can be made where there is a side two toothpicks long, and no other side is more than two toothpicks long?

(c) How many triangles can be made where there is a side three toothpicks long, and no other side is more than three toothpicks long?

(d) How many triangles can be made where there is a side four toothpicks long, and no other side is more than four toothpicks long?

(e) How many triangles can be made where there is a side five toothpicks long, and no other side is more than five toothpicks long?

II. **Triangles by Number of Toothpicks**
   We will now change the parameters just a little. Instead of telling you the number of toothpicks you can have on a side, we will give you a set number of toothpicks. You must determine the total number of different (non-congruent) triangles that can be made using exactly that number of toothpicks. Again, specify each triangle by listing the side lengths (i.e. 1-1-1, 1-1-2, etc.).

(a) Determine the number of possible triangles that can be made with each number of toothpicks. Enter this information in the middle column of the table below, leaving the last column of the table blank for now. The first one is done for you.

Number of Toothpicks	Possible Triangles	Types of Triangle
3	1-1-1	
4		
5		
6		
7		
8		
9		
10		
11		
12		

(b) Now go back and fill in the final column of the table above with any restrictions on the types of triangles that can be built (i.e. only isosceles, or only acute equilateral, etc.).

(c) Finally, suppose you divide n toothpicks into three piles with a, b, and c toothpicks in each pile, so that $a + b + c = n$. What must be true about a, b, and c in order to be able to construct a triangle with these side lengths?

7.18   Triangle Constructions

Is there a relationship between the lengths of the sides of a triangle and its interior angle measures? While this may seem like a simple question, it deserves careful consideration. For example, we can construct an equilateral triangle with any side length, and the angles will always be 60°. But we can construct isosceles triangles that have an angle ranging from 1° to 179°. In this activity you will attempt to build triangles with given side lengths and angle measures to help you understand this relationship. When performing classical geometric constructions, we are not allowed to measure angles. To make this activity run more smoothly, you may use the provided protractor to measure angles in your "constructions."

Objectives:

- Students will explore the relationship between triangle side length and angle measure.
- Students will explore potential SAS and ASA congruence rules for triangles.

I. **Building Side-Angle-Side Triangles**
   Suppose you were given two side lengths and the included angle. As long as the angle was reasonable (less than 180°), would you be able to construct a triangle with the given properties? In this first section of the activity, you will explore this question.

   (a) In the space provided below, construct a triangle with side lengths 3 cm and 4 cm, and with the given angle measure between them.

       i. 30° angle   iii. 90° angle

       ii. 60° angle   iv. 120° angle

   (b) Can you always construct a triangle given two sides and a reasonable angle measure between them? If there are any restrictions, explain them.

(c) How many different triangles can have a given pair of side lengths and included angle measure?

(d) What can be said of triangles with a pair of congruent sides and a congruent angle between them?

(e) Construct a triangle with 4 cm and 6 cm sides and a 60° angle that is **not** between the given sides. How many such triangles are there?

II. **Building Angle-Side-Angle Triangles**
Now suppose you are given two angle measures and an included side length. Will the same rules apply as did in the SAS constructions you just completed? Explore this question by doing the following activities.

(a) In the space provided below, construct a triangle with a 6 cm base and with angles of the given measure at each end of that base. In each case, determine how many triangles can be constructed with the given angle-side-angle information.

    i. 30° and 60° angles   iii. 120° and 30° angles

    ii. 40° and 90° angles   iv. 90° and 90° angles

(b) Is it always possible to construct a triangle given two angle measures and the included side length? If there are any restrictions, explain them.

(c) How many different triangles can have a pair of angle measures and included side length?

(d) What can be said of triangles with a pair of congruent angles and a congruent side between them?

(e) Construct a triangle with 40° and 90° angles and a 4 cm side that is **not** between the given angles. How many different triangles can be constructed with these specifications?

7.19 Similarity with Pattern Blocks

Similarity is often a harder concept to grasp than Congruence. Especially when it comes to justifying why two figures are similar (vs. congruent). In this activity, you will work as a group to build similar figures out of pattern blocks.

Objectives:

- Students will gain experience building similar figures.
- Students will learn to show that figures are similar.

I. **Similar Triangles**
We begin our exploration of similarity with the simplist block, the triangle. Use pattern blocks to complete each of the following tasks.

(a) Use pattern blocks to build two non-congruent shapes that are similar to the small green triangle. Sketch those shapes below.

(b) Can you build a triangle that is not similar to the small green triangle using pattern blocks? Justify your answer.

II. **Squares**
Continuing on to the next basic shape represented in our set of pattern blocks, answer the following questions regarding squares.

(a) Use pattern blocks to build two non-congruent shapes that are similar to the small orange square. Sketch those shapes below.

(b) Can you build a square that is not similar to the small orange square using pattern blocks? Justify your answer.

III. **Parallelograms**

We now examine parallelograms. As you hopefully discovered above, all equilateral triangles are similar, and all squares are also similar. But the same is not true of parallelograms!

(a) Use pattern blocks to build two non-congruent shapes that are similar to the small blue parallelogram. Sketch those shapes below.

(b) Can you build a parallelogram that is not similar to the small blue parallelogram? Justify your answer.

IV. **Trapezoids**

The final shape that we will examine is the trapezoid. Answer the following questions about similar shapes to the small red trapezoid.

(a) Use pattern blocks to build two non-congruent shapes that are similar to the small red trapezoid Sketch those shapes below.
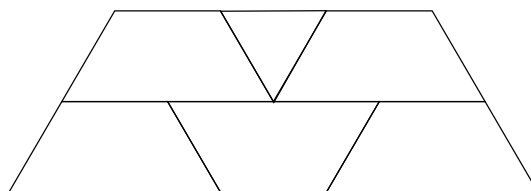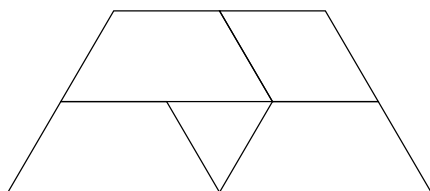
(b) Which of the trapezoids shown below is similar to the small red trapezoid? Justify your answer.

7.20 Exploring Similarity

Congruent figures must have the same size and shape. Similar figures, however, must have the same shape, but may have different sizes. In this activity, you will explore similarity between triangles and other shapes.

Objectives:

- Students will gain experience building similar figures.
- Students will learn to recognize similar figures.

I. **Exploring in Triangles**
   We begin our exploration of similarity by building and investigating several right triangles on a geoboard. Work as a group to complete each of the following tasks.

(a) On the geoboard grid shown to the right, draw a right triangle with legs of lengths 1 and 2 units. Label your triangle vertices A, B, and C.

(b) Now draw two more non-congruent triangles that are enlargements of $\triangle ABC$. Label them $\triangle DEF$ and $\triangle GHI$.

(c) Give the side lengths of your two new triangles.

   i. $\triangle DEF$

   ii. $\triangle GHI$

(d) Determine the scaling factor of each triangle with respect to $\triangle ABC$. Justify your conclusions.

(e) Now place $\triangle ABC$, $\triangle DEF$, and $\triangle GHI$ on the grid below so that one vertex of each triangle is on the same peg and two of the sides overlap.

(f) What do you notice about the corresponding angles of the three triangles?

(g) What do you conclude about similar triangles?

207
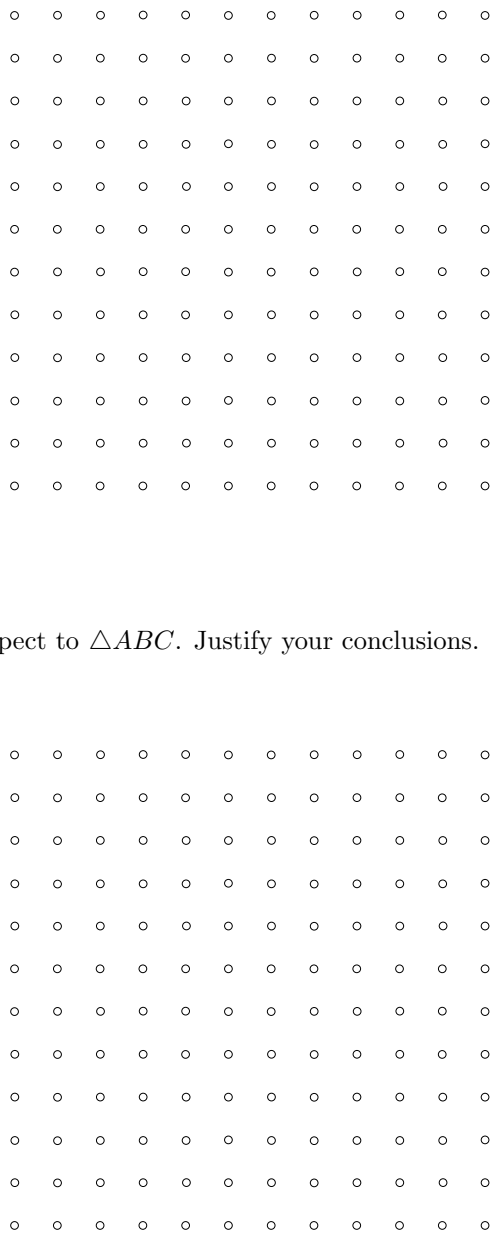
II. **Recognizing Similarity**
When two shapes have different sizes, it can be difficult to tell if they are still similar. Careful measurements and comparisons are often the only way to tell for sure.

(a) Determine if the two shapes in each geoboard grid shown below are similar. Explain your reasoning.

i.

ii.

iii.

iv.
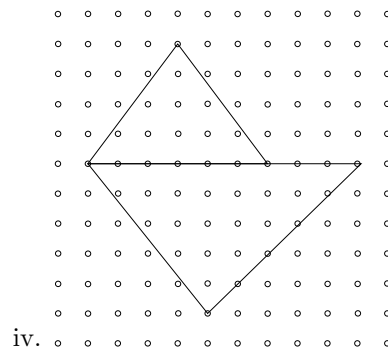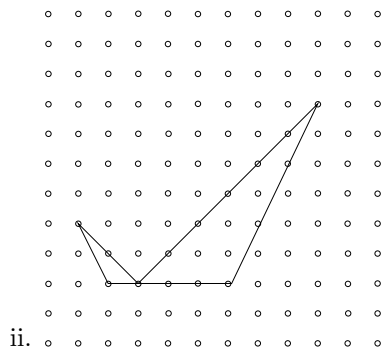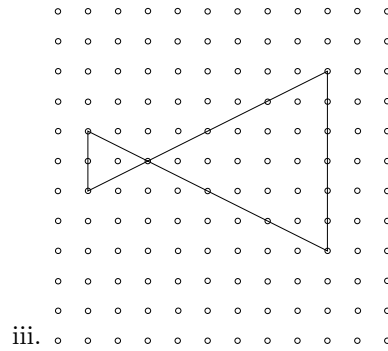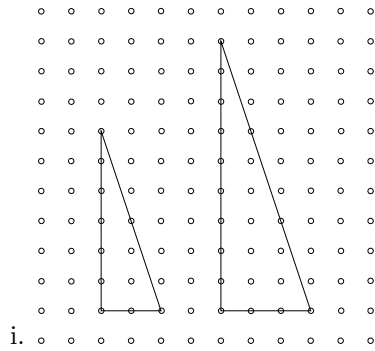
v. Which of the following four house shapes are similar? Explain your reasoning.
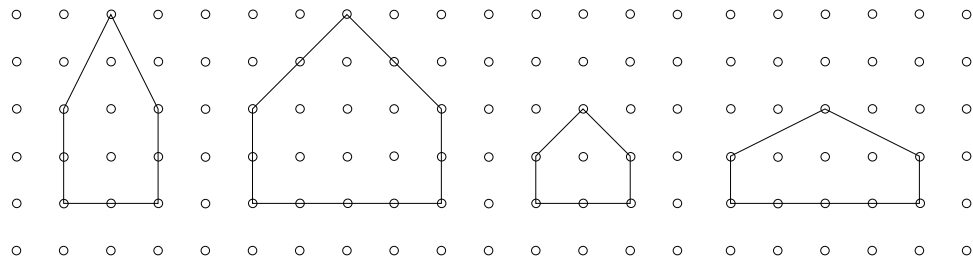
7.21 Congruence and Similarity With Paper Folding

It is amazing how much geometry can be demonstrated and understood by just folding paper. In this activity, you will work in your group to explore similarity and congruence by folding several different sizes of paper.

Objectives:

- Students will gain a better understanding of congruent and similar triangles.
- Students will learn that congruence and symmetry are common in every-day life.

I. **Paper Folding**
   Your group will be provided with a protractor to measure angles, a ruler to measure side lengths, and two sets each of three different sizes of paper. Your paper sizes are letter (8.5×11), legal (8.5×14) and an 8.5×8.5 square. Each member of your group (where possible) should pick a different size sheet of paper with which to complete the following tasks.

   (a) Choose a random point along the top edge of your page and fold the paper so that the bottom right corner touches the point you chose, as shown in the figure to the right.

   (b) Do you think any of the triangles in your figure are congruent or similar? If so, write down your conjecture below.
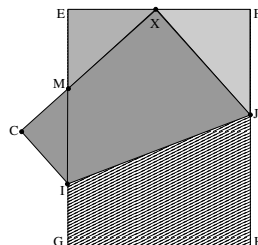
   (c) Now measure the angles and side lengths of each triangle to determine if any pairs are congruent or similar. Fill in the table below with this information.

Triangle	Angle Measures	Side Lengths	Similar/ Congruent To

   (d) Now compare your results from the previous section with your group. What do you notice?

209

(e) If you pick a different point X along $\overline{EF}$ to which to fold the paper, will you get the same pairs of similar triangles? First make a conjecture, and then try it out with at least two other points.

II. **Further Folding**
In this activity, you will work as a group to fold your pages to make specific shapes. Keep in mind that you must be able to prove that you get the shapes required. Again, have each member of your group (where possible) work with a different size sheet of paper.

(a) Find a point X along $\overline{EF}$ so that after folding $\triangle XFJ$ is isosceles. Explain how you can verify that this is an isosceles triangle, and sketch your fold below.

(b) If $\triangle XFJ$ is isosceles, what other triangles must also be isosceles? Make a conjecture and verify it using your protractor and ruler.

(c) Find a point X along $\overline{EF}$ so that the fold will produce congruent triangles. Sketch your fold below, name the congruent triangles, and verify that they are congruent.

(d) Compare where you placed the point X with where other members of your group placed X in the folds above. What do you notice?

7.22 Triangle Proofs and Problems

You should now have a pretty good understanding of congruent and similar triangles. It is time to put that understanding into practice. In this activity you will prove several claims and solve several word problems using the concepts and theorems related to congruent and similar triangles.

> **Objectives:**
>
> - Students will be able to recognize and apply properties of congruent triangles.
> - Students will be able to recognize and apply properties of similar triangles.

I. **Proofs**

Your first selection of problems ask you to apply properties of congruent and similar triangles to find values or to complete proofs. Work through each activity as a group, making sure that each group member understands how the problem was solved.

(a) The measure of an exterior angle of a triangle is 108°. The two non-adjacent interior angles are congruent. Sketch a picture of this situation and find the measures of the interior angles of the triangle.

(b) If $\overline{AC}$ and $\overline{BD}$ are parallel in the figure below, are $\triangle ABC$ and $\triangle DBC$ congruent? If so, name the property that proves this.
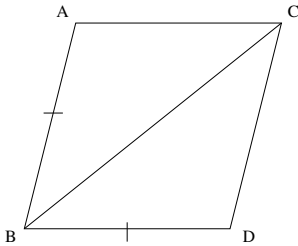
(c) In the figure below, if $\angle 1 \cong \angle B$ and $\measuredangle A : \measuredangle B : \measuredangle C = 1 : 2 : 1$, what is $\measuredangle 2$?
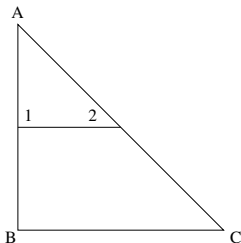
(d) Suppose that that B is the midpoint of $\overline{FC}$ in the figure below, and that $\overline{AB}$ and $\overline{FD}$ bisect each other. Prove that if $\overline{AD} \cong \overline{BC}$ then $\angle ADF \cong \angle F$.
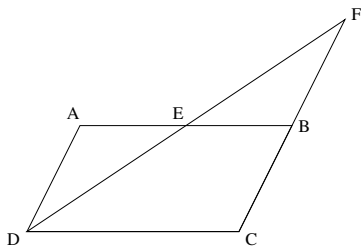
II. **Problems**

The following questions ask you to use properties of congruent and similar triangles to solve word problems. In each case, it will be helpful to sketch a picture of the situation described. Again, complete this activity as a group and make certain that each group member understands every solution.

(a) A girl who is 3 feet 8 inches tall stands 10 feet from a lamp post at night. She casts a shadow in the lamp-light that is 2 feet long. How tall is the lamp post?

(b) Sam runs a cat circus and wants to build a ramp so that his herd of cats can prance up onto a platform. Sam wants the ramp to rise 10 cm for every 30 cm it runs horizontally. The height of the cat platform is 180 cm. Sam calculates that he will need to have space for a ramp that is 550 cm long. Is he correct?

(c) A student wishes to estimate the height of a dormitory. She stands 100 feet back from the dormitory and faces it. A friend places a mirror flat on the ground 10 feet in front of her towards the dorm. The 5 foot 4 inch student is able to see the top of the building in the center of the mirror. How tall is the dorm?

Chapter 8

Probability and Statistics

> *The excitement that a gambler feels when making a bet is equal to the amount he might win times the probability of winning it.*
>
> *– Blaise Pascal*

> *I would like to throttle the man who wrote this book.*
>
> *– Hermann Klaus Hugo Weyl*

Contents

8.1 Organizing Data

As we begin our study of statistics, we need to be able to organize the data we may collect from its raw form, as a list of values, into tables or graphs which will help us understand the variable we are studying. In this activity you will learn to organize data in tables, to construct graphs based on those tables, and to interpret those graphs as a first step in analyzing data.

Objectives:

- Students will be able to construct a frequency table or frequency distribution.
- Students will be able to construct bar graphs, pie charts, and histograms and to interpret those graphs.

I. **Constructing Tables**

One of the first steps in organizing data is to take the long list of raw data values and summarize them into a form that is easier to read, such as a table. In this first part of the activity, you will learn how to construct various tables to summarize both qualitative (non-numeric) and quantitative (numeric) data.

(a) A qualitative variable is one whose values are not numbers that we can add, subtract, or average. For example, the state in which one was born, or even the area code of a phone number (note that even though an area code is a number, it represents a category, not a numeric value, and it wouldn't make sense to average a set of area codes). The example below shows how we would construct a table to summarize such a set of data.

Area Code	Frequency
206	4
253	3
360	1
425	3
503	3
509	1
541	3
971	2

Area Codes			
206	503	360	541
971	253	541	425
253	253	503	971
503	425	206	206
206	509	541	425

As you can see, we create a row for each value of the variable and then tally the number of times that value appears in our data set. This type of table is called a **frequency table** since we are recording the frequency with which each value appears. Construct a frequency table for the following data.

Birth State				
OR	WA	NE	TN	WA
MT	AZ	MI	MT	AK
WA	OR	TN	MT	ID
CA	MT	MT	CA	NE
ID	MT	AZ	AK	NE
OR	WA	WA	CA	ID

(b) A quantitative variable takes on numerical values that can be ordered, averaged, etc. It is likely that data for a qualitative variable, such as a tomato plant's height, will have too many values to summarize with a frequency table. Instead, we group values into ranges called classes and construct what is called a frequency distribution.

Height (in)	Frequency
10-14.9	2
15-19.9	4
20-24.9	5
25-29.9	5
30-34.9	3
35-39.9	1

Plant Heights (in)			
25.6	14.0	13.3	31.0
18.7	36.7	18.5	29.3
20.1	28.4	16.4	22.7
24.5	32.7	31.2	23.0
28.6	16.7	24.9	26.6

As you can see, we created classes for tomato plant heights that were all the same width (upper limit minus lower limit) and did not overlap. It is important that these conditions are met and that the classes are listed in ascending order. The data below gives student scores on a 100-point midterm exam. Construct a frequency distribution for this data using five equally-sized classes.

Exam Score				
75	15	97	87	87
56	29	62	73	42
72	76	68	62	82
66	62	68	85	61
21	89	51	42	84
30	98	44	66	11

II. **Creating Graphs**

Once we have a table to summarize our data, the next step is to give a graphical representation. This will enable users to understand how the values of our data set are spread with just a quick glance. In this part of the activity, you will create appropriate graphs to represent the tables you just made.

(a) The first type of graph we will use is a bar graph. A bar graph is a good way to visually summarize qualitative data that has been organized in a frequency table. In a bar graph each category is marked off along the horizontal axis (the order does not matter) and then a bar with height equal to the frequency of that category is drawn above it.

The example below shows a bar graph for the area code data. In the space on the right sketch a bar graph from the frequency table you constructed for the birth state data on the previous page.

Area Code Frequency
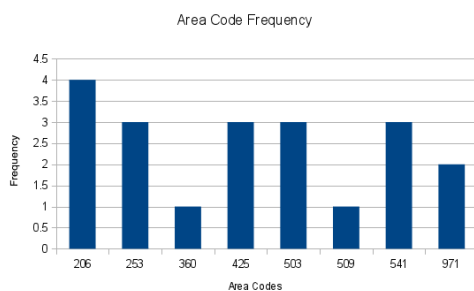
Birth State

(b) The next type of graph we will create is especially useful for summarizing how much of the whole set of data is contained in each category. It is called a pie graph. To construct a pie graph we first must build a relative frequency table. This is done by taking the frequency table and adding one more column – relative frequency. The relative frequency of a category is the frequency of the category divided by the total number of data values and expressed as a percent. This has been done for the area code data below where, for example, the first entry has a frequency of $\frac{4}{20} = 0.20 = 20\%$.

Area Code	Frequency	Relative
206	4	20%
253	3	15%
360	1	5%
425	3	15%
503	3	15%
509	1	5%
541	3	15%
971	2	10%
Total	20	100%

Area Codes
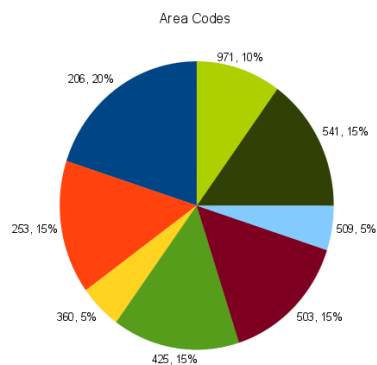
From the relative frequency table, we construct a pie graph by dividing a circle into sectors with relative sizes matching each category. The graph above gives an example of this. Construct a relative frequency table and pie graph for the birth state data in the space provided below.

(c) Our final graph is a histogram. It is similar to a bar graph, but used to represent quantitative variables. There are a few subtle differences between a histogram and a bar graph. First, the "categories" for a histogram are the classes in a frequency distribution and they must appear in increasing order. Second, since these classes don't have gaps between them, the bars in a histogram touch each other.

Below is a histogram for the tomato plant height data seen earlier. In the space provided to the right construct a histogram for the exam score data you worked with on the previous page.

Tomato Plant Height
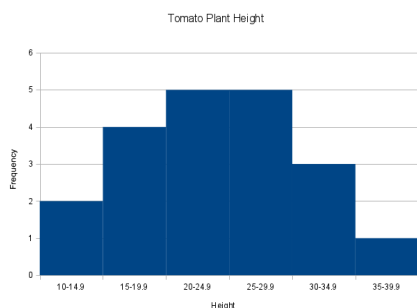
8.2 Lying With Statistics

In his 1954 book *How to Lie with Statistics*, Darrell Huff presents an introduction to the ways in which statistics are commonly used, and misused. The introduction ends with the quote "The crooks already know these tricks; honest men must learn them in self-defense." In this activity, you will learn some of these tricks both in self-defense, and to help prevent you from inadvertently making them yourself.

Objectives:

- Students will be able to identify common mistakes or deceptions found in statistics.
- Students will be able to avoid such mistakes themselves.

I. **Formulating the Question**

An unethical survey writer can easily create questions that will skew responses to the pre-determined "correct" answer. In this activity you will learn to recognize such *loaded questions* and avoid writing them yourselves.

(a) Suppose that you were asked to conduct a survey to see what soup what type of soup should be added to the cafeteria menu. Critique each of the following possible questions for this survey.

   i. If you had to pick a soup to eat right now, what would it be?

   ii. What is your favorite soup?

   iii. Vote for one of the following soups you like to eat: tomato, vegetable, potato, other.

   iv. Circle all of the soups you like to eat: tomato, vegetable, potato

(b) A local school district needs to determine public support for their upcoming school bond. Which of the following possible questions are loaded questions?

- Our school district is desperately in need of a new building due to overpopulation. Do you support a school bond to provide for that building?

- Do you support the upcoming school bond to continue wasteful spending in our school district?

- Do you support the upcoming school bond to benefit our children?

- Yet another school bond is on the ballot this year. Do you plan to vote for the bond?

II. **Gathering Data**

By carefully selecting the group of individuals to which you administer your survey, you can also skew the results. Such selection results in a *biased sample* that does not represent the population as a whole. As with question writing, this can happen unintentionally. The following activities will test your ability to recognize, and avoid, biased samples.

(a) The Collegian is doing an article about chapel attendance. The writer of the article wishes to select 20 students to participate in a survey. Do you think the sampling plans described below will produce a representative sample? If not, what bias would you expect?

   i. The article writer asks her 20 closest friends.

   ii. The writer asks the first 20 people to come out of the boy's dorm.

   iii. The writer asks the first 10 people to come out of the boys' dorm, and the first 10 to come out of the girls' dorm

   iv. The writer asks the first 20 people to come out of the cafeteria at 11:30 on Tuesday morning.

   v. The writer sends out an email to all students and surveys the first 20 who respond.

   vi. The writer generates a list of 20 random student ID numbers and contacts those students.

(b) In the 1970's newspaper columnist Ann Landers conducted a now famous poll. She asked her readers who had had children to write in and tell her if they would do so again if the could go back in time. Over 70% of those who responded said that if they could start over, they would not have children again. Do you believe this statistic is accurate? Critique the sample.

III. **Presenting Your Findings**

Even assuming that you've collected accurate data from an unbiased sample, it is still possible to mislead people by the way you choose to present the data. In the activities below you will see several examples to be avoided.

(a) If you are told that the average number of students at a small private school who had discipline problems during a week was half that of the near-by large public school, what can you conclude?

(b) Suppose you were presented with the graph below showing the number of discipline incidents involving girls and boys. What would you conclude?
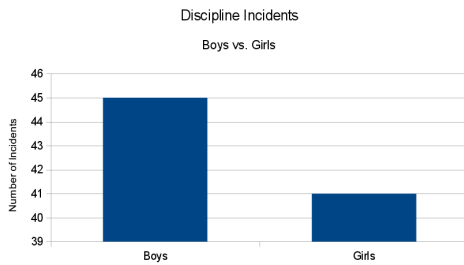
(c) The same information is presented in the pie graph below. What would you conclude?
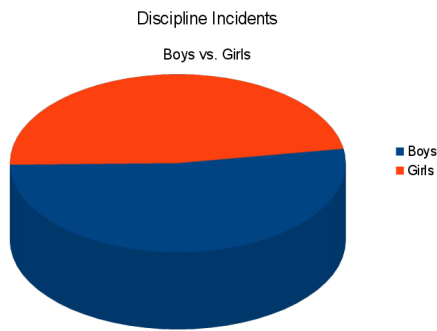
IV. **Drawing A Conclusion**

When a well-crafted statistic study shows a correlation between A and B, that does not mean that A causes B, or that B causes A. Drawing conclusions from statistics is tricky and must be done with the utmost care. Critique each of the following conclusions. Assume the statistics involved are correct.

(a) There is a close relationship between the salaries of Presbyterian ministers in Massachusetts and the price of rum in Havana. We therefore conclude that Presbyterian ministers are large consumers of Cuban rum.

(b) There are more weddings in June, and there are more suicides in June. Which causes which?

(c) A study shows that in central Africa, malnutrition is much less of a problem in households with a television than it is in households without a television. We therefore conclude that watching TV improves nutrition.

8.3 Understanding Measures of Center

What does it mean to ask for the "typical" value of a data point? In the case of exam scores, it probably means the average or mean score. But in the case of favorite pizza toppings, the mean just doesn't work. In this activity, you will explore three different statistics that can be used to describe the typical value of a variable.

Objectives:

- Students will develop an understanding of the mean, median, and mode.
- Students will be understand the relationship between the mean, median, and mode

I. **Understanding the Mode**

While there are several different ways to compute the typical value in a set of data, the mode is perhaps the most intuitive. The mode of a set of data is simply the value that appears most often. Answer the following questions involving the mode.

(a) Suppose that ten students were asked how many credits they are taking this quarter and their responses are shown below. Compute the mode of this set of data.

$$16, 12, 17, 15, 15, 16, 18, 16, 16, 15$$

(b) In statistics, an *outlier* is a single data point that is significantly different from the rest of the data. Suppose that an eleventh student is found to be taking only 6 credits this quarter. Does the mode change when this student is included? Is the mode "sensitive to outliers?"

(c) Suppose that the eleventh student was instead taking 15 credits? Does the mode change? Explain.

II. **Understanding the Median**

Our next measure of center is the median. This numerical value separates the higher half of the set of data from the lower half and, as such, it only works with numeric data. If there are an odd number of values, the median will be the middle value when they are lined up in increasing order. If there are an even number of values, it will be the mean (average) of the middle two values.

(a) Find the median of the student credit hour data seen in the last activity.

$$16, 12, 17, 15, 15, 16, 18, 16, 16, 15$$

(b) Suppose that the outliner student with 6 credits is again added to the data set. Does the median change? If so, how? Is the median "sensitive to outliers?"

(c) Finally, consider the following set of ages of students. Compute the median and the mode and then determine which you believe more accurately represents the typical student age.

$$18, 20, 21, 15, 23, 21, 15, 19, 20, 15$$

III. **Understanding the Mean**
   The most common measure of center for numeric data is probably the mean. This is often referred to as the average, although the term *average* is more like *typical* and could imply any measure of center. In this activity, you will look at not just the formula for finding the mean, but also develop an understanding of what the mean means.

(a) Six students were asked how many schools they have attended since kindergarten. Find the mean of this data. If you don't remember how to compute the mean, ask your instructor.

$$3, 7, 5, 3, 2, 4$$

(b) The Mean as a Level
You should have found that the mean in part (a) was 4 schools. To help see why this is the case, complete the following tasks.
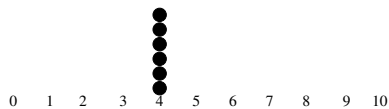
    i. Use the provided centimeter cubes to build a rod for each student where the length of the rod is the number of schools that student attended. Sketch these rods below.

    ii. Move one cube from the tallest rod to the shortest rod until they are all the same length. What is that length? How does this relate to the mean?
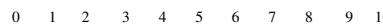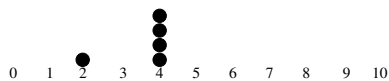
(c) The Mean as a Balance Point
Suppose that another group of six students also had attended an average of four schools each. Consider the following scenarios.

    i. While not likely, it is possible that each student attended exactly four schools as shown below.

    ii. What if we knew five of the numbers as shown? What would the sixth number have to be to make the mean 4?

    iii. What if we knew four of the numbers as shown?

    iv. Suppose that the mean is four and we know four of the numbers as shown below. What could the other two be?

    v. Make a scenario in which the mean is four but none of the six values is 4.

    vi. Make a scenario in which the mean is 4 but two of the six students were home-schooled (attended 0 schools).
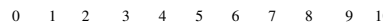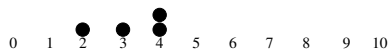
IV. **Comparing the Mean, Median, and Mode**
There are times when one of the three measures of center we've studied is preferable to another. But comparing all three measures of center can also tell us how data is distributed, or spread out. The following exercises will help you to better understand these relationships.

(a) You survey four groups of eleven students, asking how many hours each student spent watching TV over a recent week-long break. Find the mean, median, and mode of each data set.

   i. 0,0,2,3,5,6,7,15,17,20,35   iii. 3,5,5,5,7,9,12,13,15,15,17

   ii. 0,0,0,0,0,0,15,20,20,25,30   iv. 0,3,4,4,10,10,10,15,15,17,22

(b) Based on your computations above, what can you say about the relationship between the mean, median, and mode?

(c) You are thinking of applying for a job in a school district where the average teacher makes $52,000.

   i. What does this number not tell you?

   ii. If you could ask for one other piece of salary information, what would it be? Why?

   iii. Typical income is often reported using a median instead of a mean. Explain why the median would be a better choice than the mean in this type of data set.

Adapted from Bassarear *Mathematics for Elementary Teachers Explorations*, 146-148.

8.4 Passing a Hand-squeeze

If a line of people a mile long were to pass a hand-squeeze from one end to the other, how long would it take? In this activity you will work as a class to collect data to help you answer this question. Then you will work in your groups to analyze that data and come up with a reasonable prediction to answer this question.

Objectives:

- Students will collect and analyze data to help make a prediction.
- Student will see that planning is an important prerequisite to data collection.

I. **Planning the Experiment**
This is the classes activity–your instructor will not tell you what to do or answer questions. However, to help you get started in planning your data collection, you may find the following procedures helpful.

1. **Leadership Roles**
Select two individuals to fill the following roles:

- Primary Investigator – the person responsible for keeping the data collection on track, leading discussions, and guiding decision making.
- Timekeeper – the person responsible for keeping track of times for individual trials.

2. **Defining a Trial**
Each time you conduct the experiment and record a time, you are running a trial. These trials should follow the general procedures outlined below.

(a) The individuals involved should form a line or circle and hold hands.

(b) When timekeeper says "Go" and starts the stopwatch, the first individual in line squeezes the hand of the second individual, who squeezes the hand of the third, and so on.

(c) When the hand-squeeze reaches the last individual, he or she says "Stop" and the timekeeper stops the clock and records the time on the board.

3. **Planning the Experiment**
To get the best prediction possible, you will need to measure times from several different groups and people. Below are some questions you should discuss and answer before you begin collecting data.

- Will the size of the group make a difference in your accuracy?
- How many different group sizes will you measure?
- Could the individuals involved make a difference in your accuracy?
- How will you address any variation in times caused by the individuals who participate?
- If you repeated a trial with the same group of people, could you get different results?
- How will you get the best possible time for a given group of individuals?
- Will you ever need to redo a trial?
- What will you do if a person becomes distracted during a trial?

II. **Collecting the Data**
   Now carry out your experimental plan. Once the experiment has been completed, each individual should copy the times written on the board into the table below.

Number of People					

III. **Analyzing the Data**
   Now working in your lab groups, analyze the data collected above. Remember that you goal is to come up with a number that will enable you to make predictions about how long it will take to pass a hand-squeeze through a long line of individuals. In some cases, a group may decide that more data is needed, or that some of the trials should be repeated. Refer all such suggestions to your primary investigator. If the class decides to collect further data, explain why that decision was made.

IV. **Making your Prediction**
   On the bases if your data analysis above, make the following predictions. Include any assumptions you make as a part of your prediction.

   (a) How long will it take to pass a hand-squeeze through a line of 100 people?

   (b) How long will it take to pass a hand-squeeze through a mile-long line of people?

8.5 A Sampling Exercise

In descriptive statistics, we concentrate on analyzing and organizing a given set of data into a few simple values or graphs. In inferential statics, on the other hand, we wish to make claims about an entire population based on a sample of data that we collect. In this activity, you will use sampling to make a prediction of the number of "fish" swimming in a "lake."

Objectives:

- Students will understand the idea and applicability of sampling.
- Students will be able to explain factors that can influence the accuracy of a sample.

I. **Planning Your Sample**
Your instructor will supply you with a "lake" containing an unknown number of "fish." Your task in this first part of the activity is to work as a group to develop a process for estimating the number of fish in the lake without emptying them all out (draining the lake, and killing all of the fish).

(a) Working as a group, brainstorm possible ways to estimate the number of fish in the lake.

(b) Pick one of the methods above, and more fully describe how it would be accomplished. Explain why you believe this method would produce a good estimate of the number of fish in the lake.

(c) One method commonly used by scientists is outlined below.

- A "reasonable" number of fish are caught, tagged, and then released back into the lake.
- A second sample of fish is caught at a later date. The scientists record the total number of fish caught, and the number which are tagged.
- The scientists then have enough information to estimate the number of fish in the lake.

After discussing this process with your group, write a sentence explaining how the total number of fish in the lake can be estimated from the data collected.

II. **Taking Your Sample**

Now that you understand how a sample can be used to estimate the number of fish in a lake, carry out this procedure using the fish and lake provided by your instructor. Your instructor should also provide you with some "tagged fish" to help in this process.

(a) Before taking your sample(s), decide on the following.

- How many fish will you tag? Provide a justification for your choice.

- How many fish will you take in your sample after tagging? Again, justify your choice.

- Will you take more than one sample? Justify your choice!

(b) Now tag your fish, return them to the lake, and take your sample(s). Record the size of each sample and the number of tagged fish found in the sample.

Sample	# Caught	# Tagged
1		
2		
3		

III. **Analyzing Your Data**

Finally, use the data you collected above to answer the following questions. Work together as a group and make sure that you show any computations you use.

(a) How many fish does your data indicate are in the lake?

(b) If this process was done in a real lake with real fish, what issues might lead to an inaccurate estimate? How could biologists deal with these issues?

Adapted from Bassarear *Mathematics for Elementary Teachers Explorations*, 167.

8.6 The Typical University Student

As you have hopefully discovered, the meaning of the word *typical* is not as straight forward as it may seem. It may mean average, or it may mean most common, or it may even mean something completely different. In this activity, you will use the methods you learned in this lab and in class to design, conduct, and present the findings of a survey to help describe the typical student.

Objectives:

- Students will design and conduct a survey with a representative sample from a large population.
- Students will compute summary statistics and create graphs to present the results.

I. **Designing Your Survey**
Each group will be asked to explore different aspects of the typical university student. Your task in this first part of the activity is to design survey questions that will illicit accurate responses from the students you survey.

(a) Your group will be assigned one of the following blocks of characteristics. As a group, select one additional characteristic about which you wish to collect data. Be sure to get your additional characteristic approved by your instructor.

- Block A
  Height
  Class Standing
  # Hours per week spent on homework

- Block C
  Age
  Hair Color
  # meals eaten at the cafeteria last week

- Block B
  Hair Length (in inches)
  Eye Color
  # of siblings

- Block D
  Distance from home
  Wears glasses, contacts, or neither
  # credits currently taking

Additional Characteristic:

(b) Now write the questions you will ask your fellow students to collect data for each of the four characteristics selected above. Make sure that your question is clear (i.e. should you ask "How many brothers or sisters do you have?" How might that be misunderstood?).

- Question 1:

- Question 2:

- Question 3:

- Question 4:

(c) The final step before you gather your data is to determine how you will collect your sample. Answer the following questions to help you do this.

    i. Describe at least one biased sampling method (when, where, and how students will be surveyed) and explain why it would be biased.

    ii. Describe a sampling method that you believe will best minimizes bias.

    iii. Explain why the method outlined above will produce a representative sample.

II. **Group Project: Measure the Typical University Student**

As your group project for this lab, you will collect and analyze data for your assigned characteristics and then create a power point presentation to summarize your results. Make sure that each member of your group participates in both the data collection and analysis. Follow the guidelines given below as you complete the project.

- Follow the plan you created in part I to gather the data for your characteristics. Enter this data in a spreadsheet and **upload that spreadsheet to the course website**.

- Compute a measure of center for each of the characteristics. You may use the mean, median, or mode as appropriate, but you should use at least two of these measures in some way.

- Construct a graph for each of your characteristics. You may use a pie graph, a bar graph, or any other appropriate graph.

- Create a power point presentation of your findings and bring it with you to the lab. Your presentation should include the graph for each characteristic in your survey, along with a the summary statistics which best summarizes the characteristics of the "typical" student.

8.7   Comparing Data Sets

Determining how two data sets from similar sources compare with each other is not an easy task. It is, however, an important task as it allows us to determine if two groups of values are actually different from each other. In this activity, you will use graphs and descriptive statistics to help you compare two related data sets.

Objectives:

- Students will be able to compare data sets using histograms, box-plots, and descriptive statistics.
- Students will understand the relationship between the shape of a histogram and the shape of a box-plot.

I. **Comparing With Statistics**

Twenty-four 100 square foot patches of potatoes were treated with fertilizer. The first twelve used fertilizer A, and the second twelve used fertilizer B. The number of pounds of potatoes yielded by each patch, in kilograms, is recorded below in two separate data sets–one for each fertilizer type.

Fertilizer A	16, 18, 19, 21, 21, 21, 22, 22, 22, 25, 26, 30
Fertilizer B	18, 19, 20, 21, 22, 23, 23, 24, 26, 27, 27, 28

Note that the data is given in increasing order. Compute the following statistics for each data set.

(a) Find the mean and standard deviation of each data set.

(b) Find the five-number summary (min, Q_1, Median, Q_3, Max) for each data set.

(c) Based on the statistics above, describe how the two crop yields compare.

II. Comparing With Graphs

While statistics can provide the ground work for comparing two data sets, it is often much easier to see similarities or differences by comparing graphs. Complete the graphs described below.

(a) Create a back-to-back stem-and-leaf plot for the two data sets using the framework shown below.

Fertilizer A	Stem	Fertilizer B
	1	
	1	
	2	
	2	
	3	
	3	

(b) Next, create a frequency distribution for the two sets of data using the table shown below.

Yield (kg)	Fertilizer A	Fertilizer B
16-18		
19-21		
22-24		
25-27		
28-30		

(c) Next, use the frequency distribution from (b) above to sketch a histogram on the provided axis system. Sketch one histogram for each set of data.

Fertilizer A

Fertilizer B
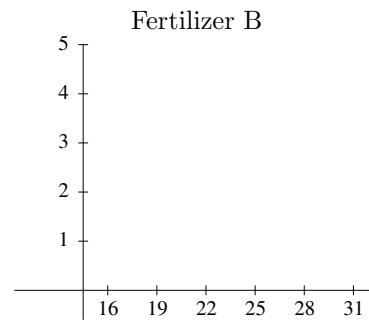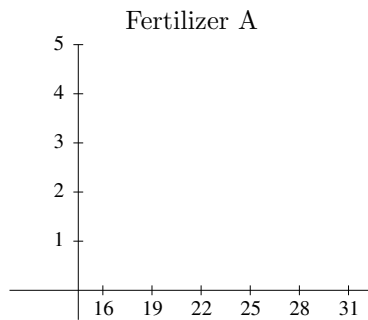
(d) Below each histogram, using the same scale, construct a box-plot for each set of data. How is the shape of the box-plot related to the shape of the histogram?

(e) Now that you have a graphical representation of the distribution of data from each potato crop, describe how they compare.

8.8   Sampling Distributions

If we draw a representative sample from a population, how can we use that sample to estimate the mean of the population? In this activity, you will explore taking samples from a small population, computing the sample means, and then discover what the mean of those sample means will be.

Objectives:

- Students will see why the sample mean is the best estimate of the population mean.
- Students will experience the central limit theorem.

I. **The Sampling Distribution of $\overline{X}$**

Your group has been given a "population" of eight pennies. To better understand why the sample mean is a good estimate for the population mean, you will compare the mean date of the population of pennies with the mean date of the possible samples of pennies.

(a) First, record the date of each penny below, and then find the average date of all eight pennies.

(b) Now, suppose that you were to draw a sample of two pennies at random from your population of eight. How many different possible samples of two are there?

(c) After checking with your instructor to see if you have the right number of samples, find each sample and record its mean date.

_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____
_____ , _____ = _____	_____ , _____ = _____	_____ , _____ = _____

(d) Finally, find the mean of the means of the samples that you found above.

(e) Based on this activity, explain why a sample mean is a good estimate of the population mean.

II. **Sampling and the Shape of a Distribution**

One of the most important theorems in statistics is called the "Central Limit Theorem." It states that no matter what the initial disribution of data is, if you draw samples from that distribution and then look at the sample means, those sample means will have a mound-shaped normal distribution.

(a) Fill in the frequency distribution below with the number of pennies in your population from each decade. Then sketch a histogram next to the distribution.

Decade	# Pennies
1970s	
1980s	
1990s	
2000s	

(b) How would you describe the shape of this distribution?

(c) Now take ten random samples of size three from your population of pennies, replacing them before each new sample. Record the mean date for each sample below.

(d) Create a frequency distribution with four equal date ranges using the table provided below. Then sketch a histogram for this distribution.

Date Range	# Pennies

(e) How would you describe the shape of your new distribution?

8.9 Counting License Plates

Counting may seem like an elementary topic, but even simple every-day life questions can call for advanced counting techniques. One area in which this is certainly true is in designing license plate configurations. In this activity you will explore the history of license plate configurations in Washington state, in particular as it relates to the number of possible plates in a given configuration.

Objectives:

- Students will learn a variety of counting techniques.
- Students will appreciate a real-life application of counting.

I. **Washington State License Plates**

The license plate configurations used even just in Washington have changed over time. As more and more vehicles are registered, it has become necessary to change the licensing scheme to accommodate larger numbers of plates. Below are several configurations that have been used by Washington state over the last 100+ years. Determine the number of possible license plates that could be made under each configuration.

(a) The first vehicle licensed in Washington was a 30 HP Pope-Toledo Touring car licensed to Mr. S.A. Perkins of Tacoma on 2 May 1905. His license number was **B-1**

(b) In the 1920's vehicles were becoming more common, with more than 137,000 vehicles registered. License plates during this era consisted of up to six digits separated into groups of three with any leading zeros left off. For example, **12-542** or **201-782**.

(c) In 1939 a golden jubilee plate was issued celebrating Washington's 50th anniversary. It consisted of a letter followed by five digits, such as **D-16340**.

(d) In the late fifties the size of the plate changed to the now-standard 6" by 12" plate. In 1958, a new type of plate was issued with a combination of three letters followed by three numbers, such as **NXA 842**.

(e) By the late 2000's, the state needed a new scheme to accommodate all of the cars on the road. The new format has three letters followed by four digits, as in **ABC 2345**

II. **Modifying Plate Configurations**
Sometimes not all of the possibilities in a plate configuration are allowed. At other times, we may need to expand on a given system without making the older plates obsolete. Answer each of the questions below which have to do with modifying plate configurations.

(a) It is expected that in the future Washington will allow for two letters followed by five digits. For that reason, the letters **I**, **O**, and **Q** are not allowed in the third position. How many plates does this prohibit?

(b) Because of the potential for offending people, certain three-letter combinations are not allowed on plates, such as **APE**, **BUT**, and **HOE**. Suppose that there are 30 such three-letter words that are not allowed in the new Washington plate scheme (three letters followed by four digits). How many plates does this prohibit?

(c) As mentioned above, in the future Washington may allow two letters followed by five digits. Another possibility is four letters followed by three digits. Which of these expansions would produce the most new plates? What other considerations might go into deciding how to expand on the current configuration?

8.10 Probability Games

The study of probability has many real-world applications, from predicting the weather to quantum chemistry. However, the mathematical field of probability was originally developed to help people understand and win games of chance. In this activity, you will explore several games involving probability.

Objectives:

- Students will develop an understanding of experimental probability.
- Students will gain experience with several probability manipulatives.

I. **Probability With Dice**
Dice have been used since before recorded history. The oldest known dice were excavated as part of a 5000-year-old backgammon set in Iran. Dice are certainly one of the earliest known devices used in games of chance. In this first part of the activity, you will explore several probabilities involving dice.

(a) **Experiment One:** You throw a single die and note the number that comes up.

    i. What are the possible outcomes of this experiment?

    ii. Make a guess of the probability for each outcome.

    iii. Now roll a die 30 times and note the number that comes up each time. Find the experimental probability of each outcome by forming the fraction $\frac{x}{30}$ where x is the number of times that outcome appeared.

(b) **Experiment Two:** You throw two dice and note the sum of the numbers that come up.

    i. What are the possible outcomes of this experiment?

    ii. Make a guess of the probability for each outcome.

    iii. Now roll two dice 30 times and note the sum that comes up each time. Find the experimental probability of each outcome by forming the fraction $\frac{x}{30}$ where x is the number of times that outcome appeared.

II. **Probability With Spinners**
    Colored and/or numbered spinners are often used in elementary school classrooms to teach concepts about probability. Consider the following experiments.

(a) **Experiment One:** You spin a single spinner and note the color on which you land.

    i. Find the possible outcomes for this experiment and predict their probabilities.

    ii. Now spin your spinner 30 times and find the experimental probability of each outcome.

(b) **Experiment Two:** You spin two spinners and determine if the sum of the numbers on which you land is even or odd.

    i. Find the possible outcomes for this experiment and predict their probabilities.

    ii. Now spin your spinners 30 times and find the experimental probability of each outcome.

(c) **Experiment Three:** You spin two spinners and note the colors and the sum of the digits on which you land.

    i. Estimate the probability of each outcome

    - You land on at least one red or you get an even sum.

    - Both spinners land on either yellow or blue, and the sum is less than 5.

    ii. Now spin your spinners 30 times and find the experimental probability of each outcome.

III. **Probability With Marbles in an "Urn"**
     An urn is a vase which is opaque so that the contents can not be seen. In probability we often represent a
     real-world problem involving randomly selected objects by representing them as colored marbles drawn
     from an urn, with the goal of determining the probability of drawing one color or another. Follow the
     directions to complete each of the urn experiments below.

   (a) **Experiment 1:** An urn contains 6 marbles: 2 of one color and 4 of another. You draw two marbles
       from the urn without replacing them and note their color.

       i. Your outcomes from this experiment will be pairs of colors, one for the first marble drawn and
          one for the second. Represent these possibilities using a *tree diagram* where each branching of
          the tree represents drawing one marble.

       ii. Use your tree diagram to help make an educated guess as to the probability of drawing two
           marbles with different colors.

       iii. Now perform the experiment 30 times and determine the experimental probability of drawing
            two marbles with different colors.

   (b) **Experiment 2:** Repeat the experiment above, except this time put your first marble back before
       drawing your second marble.

       i. Will this change your probability of drawing two marbles with different colors? Explain.

       ii. Now repeat the experiment another 30 times with replacement to see if you are correct.

8.11 Expected Value

In many cases where probability is used, we wish to determine the value of some randomly generated number, called a random variable. For example, we may buy a lottery ticket that could be worth nothing, $1, $5, or $100. If we repeat this process over and over, we will find that the lottery ticket has an average value that is less than what we paid for it. This average value is called the expected value of the random variable. In this activity, you will model and explore the expected value of several different random variables.

Objectives:

- Students will understand what the expected value of a random variable is.
- Students will be able to compute the expected value both experimentally and using theoretical probability for simple probability distributions.

I. **Pocket Change**

Suppose that you had two quarters and three dimes in your pocket. If you reach in and pull out a coin, you will either have $0.25 or $0.10. But if you were to repeat this over-and-over and then look at the average value of the coin you pulled out, what would this be?

(a) Design an experiment using the tools you have on hand (dice, pennies, spinners, etc) to simulate this process. Describe your simulation below.

(b) Conduct the simulation at least 20 times and find the average value of the "coin" you "draw.".

(c) Now could we figure out the expected value without conducting a simulation? Complete the following steps to do this.

   i. Find the probability of each possible outcome (drawing a quarter and drawing a dime).

   ii. Determine the value of the two possible outcomes.

   iii. Multiply the probability of each outcome by its value and add the products together.

(d) This is the expected value of the experiment. Is it close to your experimental average?

II. **Drawing Two Coins**
What happens if you draw two coins from that same pocket containing two quarters and three dimes? While there are now more options, you should still be able to determine the expected value of your combined draws.

(a) How will your simulation need to change?

(b) Conduct the simulation at least 20 times and find the average value of the "coin" you "draw.".

(c) Again, complete the following steps to determine the expected value.

   i. Find the probability of each possible outcome. Drawing a tree diagram may help!

   ii. Determine the value of each possible outcome.

   iii. Multiply the probability of each outcome by its value and add the products together.

(d) Was your expected value close to your experimental average value?

8.12 Conditional Probability

Can the probability of an event taking place be changed just by providing extra information? Even though it may not seem reasonable at first, the answer is yes. This type of probability is called a conditional probability. In this activity, you will explore several problems involving conditional probability that have stumped many individuals.

Objectives:

- Students will be compute conditional probabilities.
- Students will see that intuition is not always correct in situations involving chance.

I. **It's A Boy!**

Marilyn vos Savant is an author and magazine columnist famed for having the Guinness world record for the highest IQ. In her column *Ask Marilyn* in Parade Magazine, she answers questions from readers on a wide range of mostly academic subjects. In the 19 October 1997 issue of Parade Magazine, Vos Savant answered the following reader question.

> *Say that a woman and a man (who are unrelated) each has two children. We know that at least one of the woman's children is a boy and that the man's oldest child is a boy. Can you explain why the chances that the woman has two boys do not equal the chances that the man has two boys?*

Even after Marilyn's explanation, many readers wrote back to say that the chance that the man has two boys is in fact the same as the chance that the woman has two boys. One reader insisted that "I will send $1000 to your favorite charity if you can prove me wrong." Who is correct?

(a) Determine the chance that the man, whose oldest child is a boy, has two boys. Explain.

(b) Determine the chance that the woman, who has at least one boy, actually has two boys. It may be helpful to draw a tree diagram.

(c) What do you conclude? Did the man have to send $1000 to the American Heart Association, vos Savant's favorite charity?

241

II. **It's A Goat!**

In the 9 September 1990 issue of her column, vos Savant answered what is perhaps her most famous, and controversial, question.

> *Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host who knows what's behind the doors opens another door, say number 3, which has a goat. He then says to you "Do you want to pick door number 2?" Is it to your advantage to switch your choice?*

This became known as the Monty Hall problem because of its similarity to the game show *Let's Make a Deal*, hosted by Monty Hall. Below are two possible answers, one of which is correct. Vos Savant provided the correct one, can you?

Answer A: Yes, you should switch because the first door has a 1/3 chance of winning, but the second door has a 2/3 chance of winning.

Answer B: There is no need to switch because after the host opens a door, there are only two doors left, so each door has the same 50% chance of concealing the car.

(a) Devise a simulation for this activity. Have one of your group members act as the game show host, and another as the contestant.

    i. Describe your simulation below.

    ii. Run ten trials of your simulation in which the contestant does not switch doors. In how many of those did he win the car?

    iii. Now run ten trials of your simulation in which the contestant switches doors. In how many of those did she win the car?

(b) Based on your simulations above, which answer do you believe is correct? Can you explain why?

Appendix A

Activities Containing Projects

Appendix B

Required Manipulatives