Examples 2.22

(a) Let X be the space of real numbers with the trivial topology. Then for any point p in X, the space X is the only neighborhood of p. Thus for any nonempty A, $\text{cl}(A) = X$ and $\text{int}(A) = \emptyset$ unless $A = X$. If A is not all of X, then $\text{Fr}(A) = X$.

(b) Let X be the space of real numbers with the usual topology, and let $A = (0, 1]$. Then $\text{cl}(A) = [0, 1]$, $\text{int}(A) = (0, 1)$, and $\text{Fr}(A) = \{0, 1\}$.

(c) Let X be the space of real numbers with the discrete topology, and let $A = (0, 1]$. Then every subset of X is both open and closed, so $A = \text{cl}(A) = \text{int}(A)$, and $\text{Fr}(A) = \emptyset$.

(d) Let X be the complex plane with the usual topology, and let $S = A \cup B$, where $A = \{z : |z| < 1\}$, and $B = \{z = (x, 0) : x \geq 1\}$. Then $\text{cl}(S) = \{z : |z| \leq 1\} \cup B$, $\text{int}(S) = A$, and $\text{Fr}(S) = \{z : |z| = 1\} \cup B$.

(e) Let X be the space of real numbers with the usual topology, and let A be the rational numbers. Then $\text{cl}(A) = X$, $\text{int}(A) = \emptyset$, and $\text{Fr}(A) = X$.

Definition

A subset A of a topological space X is said to be **dense** if $\text{cl}(A) = X$.

Thus the set of rational numbers is a dense subset of the space of real numbers with the usual topology, and in a space with the trivial topology, every nonempty subset is dense.

Exercises

1. Let X be the space of real numbers with the usual topology, and let N be the integers. Find the derived set, the closure, the interior, and the boundary of N.

2. Let X be the set of reals, and let $\mathbf{T} = \{S \subset X : 0 \in X - S\} \cup \{X\}$. Show that $\mathbf{T}$ is a topology for X and find the closure of the interval $A = (1, 2)$ and of the interval $B = (-1, 1)$.

3. Let X be the set of positive integers. For each $n \in X$, let $S_n = \{k \in X : k \geq n\}$. Show that $\mathbf{T} = \{S_n : n \in X\} \cup \{\emptyset\}$ is a topology for X, and find the closure of the set of even integers. Find the closure of the singleton set $A = \{100\}$.

4. Let X be the space of reals with the cofinite topology (Example 2.1(d)), and let A be the positive integers and $B = \{1, 2\}$. Find the derived set, the closure, the interior, and the boundary of each of the sets A and B.

5. Let X be the space of reals with the topology $\mathbf{T} = \{X, [0, 1], \emptyset\}$ (Example 2.1(b)), and let $A = [-1, 1]$ and $B = [2, 3]$. Find the derived set, the closure, the interior, and the boundary of each of the sets A and B.

6. Let $\mathbf{S}$ and $\mathbf{T}$ be topologies for a set X. Prove that $\mathbf{S} \subset \mathbf{T}$ if and only if for every set $A \subset X$, it is true that $\text{cl}_{\mathbf{T}}A \subset \text{cl}_{\mathbf{S}}A$.

For each of the following, if the statement about a topological space is always true, prove it; otherwise, give a counterexample.

7. The derived set of every set is a closed set.

8. The boundary of every set is a closed set.

9. The boundary of every set has empty interior.

10. Every nonempty closed set with empty interior is the boundary of some set.

11. If a set has empty interior, then so does its closure.

12. The closure of a set coincides with the closure of its interior.

13. The interior of a set coincides with the interior of its closure.

14. Every set that does not meet its boundary is open.

2.2 Base for a Topology

We next introduce the idea of a base for a topology, a concept that yields a certain economy of thought and effort in defining a topology for a set.

Definition

Let $(X, \mathbf{T})$ be a topological space. A subset $\mathbf{B}$ of $\mathbf{T}$ such that every element of $\mathbf{T}$ is a union of elements of $\mathbf{B}$ is a **base** for $\mathbf{T}$.

Note that a collection of subsets of a set X cannot be a base for two different topologies.

Theorem 2.23

If (X, d) is a pseudometric space, then the collection of all cells is a base for the topology generated by d.

Proof: Proposition 1.11 tells us that each cell is an element of the topology, and Theorem 1.12 tells us that for an open U and an $x \in U$, there is a cell $C(x; r_x) \subset U$. Then we have $U = \cup\{C(x; r_x) : x \in U\}$.

Theorem 2.24

Let X be a topological space, and let $\mathbf{B}$ be a base for the topology. Then a point p is a limit point of a set S if and only if every member of $\mathbf{B}$ containing p meets S in a point other than p.

Proof: If p is a limit point of S, then every member of $\mathbf{B}$ containing p meets S in a point other than p since $\mathbf{B}$ is a subset of the topology.

Suppose every member of $\mathbf{B}$ containing p meets S in a point other than p, and let U be an open set containing p. Then $U = \cup\mathbf{C}$, where $\mathbf{C} \subset \mathbf{B}$. This means $p \in C$ for at least one $C \in \mathbf{C} \subset \mathbf{B}$, so C, and hence U, meets S in a point other than p.

Corollary 2.25

A point p is a member of the closure of a set S if and only if every element of $\mathbf{B}$ containing p meets S.

Theorem 2.26

Suppose X is a set, and $\mathbf{B}$ is a collection of subsets of X such that

(a) $X = \cup\mathbf{B}$; and

(b) if B_1 and B_2 are elements of $\mathbf{B}$ and $p \in B_1 \cap B_2$, then there is a $C \in \mathbf{B}$ such that $p \in C \subset B_1 \cap B_2$.

Then the collection of all unions of elements of $\mathbf{B}$ is a topology for X.

Proof: Evidently $\emptyset$ and X are unions of elements of $\mathbf{B}$. It is clear also that any union of unions of elements of $\mathbf{B}$ is a union of elements of $\mathbf{B}$.

Let $U = \cup\mathbf{C}$ and $V = \cup\mathbf{D}$, where $\mathbf{C}$ and $\mathbf{D}$ are subsets of $\mathbf{B}$. Then $U \cap V = \cup\{C \cap D : C \in \mathbf{C}$ and $D \in \mathbf{D}\}$. We see then that we need only show that for C and D in $\mathbf{B}$, the set $C \cap D$ is a union of elements of $\mathbf{B}$. This is easy to do. For each $x \in C \cap D$, let $C_x \in \mathbf{B}$ be such that $x \in C_x \subset C \cap D$. Then $C \cap D = \cup\{C_x : x \in C \cap D\}$.

Suppose X is a set, and $\mathbf{C}$ is a collection of subsets of X. There are circumstances under which we shall want to have a topology on X that includes $\mathbf{C}$. This is, of course, easy to accomplish, for the discrete topology includes every collection of subsets of X. The real objective usually is to find the smallest topology on X that includes the given collection $\mathbf{C}$. It is clear that the intersection of a collection of topologies for a set X is itself a topology for X, so we are led to the following definition.

Definition

Let X be a set and let $\mathbf{C}$ be a collection of subsets of X. The topology $\mathbf{T} = \cap\{\mathbf{T} : \mathbf{T}$ is a topology for X and $\mathbf{C} \subset \mathbf{T}\}$ is the topology **generated by** $\mathbf{C}$. The collection $\mathbf{C}$ is sometimes called a **subbase** for the topology $\mathbf{T}$.

Theorem 2.27

Let X be a set, and let $\mathbf{C}$ be a collection of subsets of X. Then the collection $\mathbf{B} = \{\cap\mathbf{F} : \mathbf{F}$ is a finite subset of $\mathbf{C}\}$ is a base for the topology generated by $\mathbf{C}$.

Proof: First note that $X \in \mathbf{B}$ since $\emptyset$ is a finite subset of $\mathbf{C}$ and $\cap\emptyset = X$. Next, it is clear that if $B_1 = \cap\mathbf{F}_1$ and $B_2 = \cap\mathbf{F}_2$ are elements of $\mathbf{B}$, then $B_1 \cap B_2 = \cap(\mathbf{F}_1 \cup \mathbf{F}_2)$ is also in $\mathbf{B}$, since $\mathbf{F}_1 \cup \mathbf{F}_2$ is a finite subset of $\mathbf{C}$ whenever $\mathbf{F}_1$ and $\mathbf{F}_2$ are finite subsets of $\mathbf{C}$. Thus $\mathbf{B}$ is a base for a topology $\mathbf{T_B}$, and it is clear that $\mathbf{T_B} \subset \mathbf{T}$, where $\mathbf{T}$ is the topology generated by $\mathbf{C}$. To see that $\mathbf{T} \subset \mathbf{T_B}$, observe that $\mathbf{B} \subset \mathbf{T}$, so that the union of any subcollection of $\mathbf{B}$ is an element of $\mathbf{T}$.

Examples 2.28

(a) Let X be the set of real numbers, and let $\mathbf{B} = \{(a, b) : a$ and b are reals with $a < b\}$. The intersection of any two elements of $\mathbf{B}$ is either empty or a member of $\mathbf{B}$. Thus $\mathbf{B}$ is a base for a topology $\mathbf{T}$. In fact, $\mathbf{T}$ is the usual topology since $\mathbf{B}$ consists of all cells generated by the usual pseudometric.

(b) Let X be the set of real numbers and let $\mathbf{B} = \{[a, b) : a$ and b are reals with $a < b\}$. Then $\mathbf{B}$ is a base for a topology on X. (This space is called the **Sorgenfrey line.**)

(c) Let X be the plane, and for each real x, let L_x be the vertical line passing through the point $(x, 0)$. Define $\mathbf{B} = \{L_x : x \in \mathbf{R}$, the real numbers$\}$. Since $L_x \cap L_y = \emptyset$ for $x \neq y$, $\mathbf{B}$ is a base for a topology. If $A = \{z = (x, y) : |z| < 1\}$, then in the topology for which $\mathbf{B}$ is a base, we have $\text{cl}(A) = \{(x, y) : 0 < x < 1\}$, $\text{int}(A) = \emptyset$, and $\text{Fr}(A) = \text{cl}(A)$.

(d) Let X be the plane, and for real numbers a, b, c, and d, let

$$R(a, b, c, d) = \{(x, y) : a < x < b, \text{ and } c < y < d\}.$$

Then the collection of all such "open" rectangles $R(a, b, c, d)$ is a base for a topology on X, which is, in fact, the usual topology for the plane.

(e) Let X be a set of real numbers, and let

$$\mathbf{C} = \{(-\infty, a) : a \in X\} \cup \{(a, \infty) : a \in X\}.$$

The topology generated by $\mathbf{C}$ is the usual topology for the real numbers.

(f) Let Z be the integers. For positive integers n and m, define $B(m,n) = \{km+n : k \in Z\}$. Then $\mathbf{B} = \{B(m,n) : n$ and m are positive integers$\}$ is a base for a topology on Z. It is clear that $Z = \cup\mathbf{B}$. Suppose $x \in B(m,n) \cap B(p,q)$. Then $x \in B(mp,x) \subset B(m,n) \cap B(p,q)$.

Exercises

15. Let X be the set of real numbers, and let $\mathbf{B} = \{(a,b) : a$ and b are rational$\}$. Prove that $\mathbf{B}$ is a base for the usual topology on X.

16. Let X be the plane, let d be the usual pseudometric, and let $\mathbf{U} = \{C(p;r) : p$ has rational coordinates and r is rational$\}$. Prove that $\mathbf{U}$ is a base for the usual topology on X.

17. Let X be the plane, and let $\mathbf{B}$ be the collection of all circles centered at the origin, including the origin itself (that is, the circle of radius 0). Show that $\mathbf{B}$ is a base for a topology on X, and find the closure and the interior of the square $S = \{(x,y) : -1 \le x \le 1, \text{ and } -1 \le y \le 1\}$.

18. Suppose x and y are points in a topological space, the topology of which is generated by a pseudometric. Prove that if there is a neighborhood of x that does not contain y, then there is a neighborhood of y that does not contain x.

19. Let X be any nonempty set, and let $\mathbf{B} = \{\{x\} : x \in X\}$. Show that $\mathbf{B}$ is a base for a topology on X. Describe the open sets of this topology.

20. Suppose (X,d) is a pseudometric space, and m is a positive integer. Prove that $\mathbf{B} = \{C(x;1/n) : x \in X, n = m, m+1, \ldots\}$ is a base for the topology generated by d.

21. Let (X,d) be a pseudometric space. Show there is a pseudometric d^* on X that is equivalent to d and that has the property that there is a real number M such that $d^*(x,y) < M$ for all x and y in X. (A pseudometric with this property is usually said to be **bounded**.)

22. Let X be the plane. For each positive real number a, define

$$B_a = \{(x,y) : 3x - a < y < 3x + a\}.$$

Show that the collection $\mathbf{B} = \{B_a : a > 0\}$ is a base for a topology on the plane. Find the closure of the singleton set consisting of the origin.

23. Suppose X is the set of reals, $\mathbf{U}$ is the usual topology, and $\mathbf{I}$ is the collection of all subsets of the irrational numbers. Let $\mathbf{T}$ be the topology generated by $\mathbf{U} \cup \mathbf{I}$. In the topological space $(X,\mathbf{T})$, find the closure of the interval $(0, \sqrt{2})$ and the interior of the interval $[0, \sqrt{2}]$.

24. Let X be the plane, and let $\mathbf{T}$ be the topology generated by the set of all straight lines through the origin. In the topological space $(X,\mathbf{T})$ find the closure of each of the following sets:
    (a) $A = \{(0,0)\}$
    (b) $B = \{(1,1)\}$
    (c) $C = \{(x,1) : 0 < x < 1\}$

25. In Example 2.28(f), find the closure of each base element $B(m,n)$.

2.3 Subspaces

Theorem 2.29

Suppose $(X,\mathbf{T})$ is a topological space and $A \subset X$. Then the collection $\mathbf{T}_A = \{U \cap A : U \in \mathbf{T}\}$ is a topology for A.

Proof: That A and $\emptyset$ belong to $\mathbf{T}_A$ is clear. Suppose that $\mathbf{C}$ is a collection of elements of $\mathbf{T}_A$, and consider the set $\cup\mathbf{C}$. For each C in $\mathbf{T}_A$, there is a U_C in $\mathbf{T}$ so that $C = U_C \cap A$. Thus

$$\cup\mathbf{C} = \cup\{U_C \cap A : C \in \mathbf{C}\} = (\cup\{U_C : C \in \mathbf{C}\}) \cap A,$$

which is an element of $\mathbf{T}_A$ since a union of elements of $\mathbf{T}$ is an element of $\mathbf{T}$.

Similarly, if G and H are elements of $\mathbf{T}_A$, then we have $G = U_G \cap A$ and $H = U_H \cap A$ for some U_G and U_H in $\mathbf{T}$. Thus

$$G \cap H = (U_G \cap A) \cap (U_H \cap A) = (U_G \cap U_H) \cap A,$$

which is in $\mathbf{T}_A$ since the intersection of two elements of $\mathbf{T}$ belongs to $\mathbf{T}$.

Definition

If $(X,\mathbf{T})$ is a topological space and A is a subset of X, the topology $\mathbf{T}_A$ in the previous theorem is called the **subspace** topology, or the **relative** topology, and the topological space $(A,\mathbf{T}_A)$ is said to be a **subspace** of $(X,\mathbf{T})$.

Again, where there is no danger of confusion, we frequently omit explicit mention of the topologies and say simply that $A \subset X$ is a subspace of X. We must, however, be very careful not to refer simply to an "open" subset of A, or to the closure or interior of a subset; we must add some modifier to make it clear whether we are referring to the original topology on X or to the subspace topology on A. We shall usually do this by using the notation $\text{cl}_A(S)$ to denote the closure with respect to the subspace topology on A and $\text{int}_A(S)$ to denote the interior with respect to that topology.

If $S \subset A \subset X$, and $(X, \mathbf{T})$ is a topological space, then S can be given two different subspace topologies: one resulting from being a subspace of $(X, \mathbf{T})$ and one resulting from being a subspace of $(A, \mathbf{T}_A)$. These two topologies turn out, mercifully, to be the same, a fact that has an easy proof, which is omitted.

Proposition 2.30

Suppose X is a topological space, and A is a subspace of X. If $\mathbf{B}$ is a base for the topology of X, then the collection $\mathbf{B}_A = \{B \cap A : B \in \mathbf{B}\}$ is a base for the subspace topology on A.

Proof: We need to show that each member of the subspace topology is a union of members of $\mathbf{B}_A$. To see this, let G be a member of the subspace topology. Then $G = U \cap A$ for some U in the topology for X, and we know that $U = \cup\mathbf{C}$ for some $\mathbf{C} \subset \mathbf{B}$. Thus

$$G = (\cup\mathbf{C}) \cap A = \cup\{C \cap A : C \in \mathbf{C}\}.$$

This finishes the proof since each $C \cap A$ is an element of $\mathbf{B}_A$.

Theorem 2.31

Suppose A is a subspace of $(X, \mathbf{T})$, and x is an element of A. Then $M \subset A$ is a $\mathbf{T}_A$-neighborhood of x if and only if there is a $\mathbf{T}$-neighborhood N of x such that $M = N \cap A$.

Proof: First, suppose $M = N \cap A$ for some $\mathbf{T}$-neighborhood N of x. Then there is a $\mathbf{T}$-open set U so that $x \in U \subset N$. Thus $x \in U \cap A \subset N \cap A = M$.

Now suppose M is a $\mathbf{T}_A$-neighborhood of x. This means there is a $\mathbf{T}_A$-open set G so that $x \in G \subset M$. But $G = U \cap A$ for some $\mathbf{T}$-open set U. So $N = U \cup M$ is a $\mathbf{T}$-neighborhood of x, and $M = N \cap A$.

Theorem 2.32

Suppose A is a subspace of $(X, \mathbf{T})$. Then for any subset $S \subset A$, we have $\text{cl}_A(S) = (\text{cl}_X(S)) \cap A$.

Proof: First, suppose $x \in \text{cl}_A(S)$, and let N be a $\mathbf{T}$-neighborhood of x. Then, by the previous theorem, $N \cap A$ is a $\mathbf{T}_A$ neighborhood of x, and hence meets S. Thus N meets S, and so $x \in \text{cl}_X(S)$. This shows $\text{cl}_A(S) \subset \text{cl}_X(S)$. From the fact that $\text{cl}_A(S) \subset A$ it follows that $\text{cl}_A(S) \subset (\text{cl}_X(S)) \cap A$.

Next, suppose $x \in (\text{cl}_X(S)) \cap A$, and let M be a $\mathbf{T}_A$-neighborhood of x. Then M meets S since $M = N \cap A$ for some $\mathbf{T}$-neighborhood N of x. So $x \in \text{cl}_A(S)$, which means that $(\text{cl}_X(S)) \cap A \subset \text{cl}_A(S)$, completing the proof.

Corollary 2.33

The $\mathbf{T}_A$-derived set of a subset S of A is the intersection of the $\mathbf{T}$-derived set of S with A.

Examples 2.34

(a) Let X be the space of real numbers with the usual topology, and let $A = [0, 1)$. In the subspace A, the interval $[0, 1/2)$ is open, and the interval $[1/2, 1)$ is closed.

(b) Let X be the plane with the usual topology, and let C be the circle of radius 1 centered at the origin. For real numbers $a < b$, let $U(a, b) = \{(\cos\Theta, \sin\Theta) : a < \Theta < b\}$. (Thus the $U(a, b)$ are "open" arcs of the circle.) Then the collection of all $U(a, b)$ is a base for the subspace topology on C, since each such U is the intersection of C with a cell in the plane.

(c) Let X be the plane with the usual topology and let $H \subset X$ be the closed upper half-plane; that is, $H = \{(x, y) : y \geq 0\}$. Let $A = \{(x, y) : 0 < x < 1, \text{ and } y \geq 0\}$. Then A is an open subset of the subspace H, since $A = H \cap \{(x, y) : 0 < x < 1\}$.

(d) Let X be the plane with the usual topology, and let Y be the x-axis. The subspace topology for Y is the usual topology for the reals since every open interval of Y is the intersection of a cell in X with Y.

(e) Let X be the space of reals with the usual topology, and let Y be the integers. Then the subspace topology for Y is the discrete topology since for each integer n, $\{n\} = Y \cap (n - 1/2, n + 1/2)$.

Exercises

26. Let X be the space of real numbers with the usual topology, and let Q be the subspace of rational numbers. Let $A = \{r \in Q : 0 < r < 1\}$, $B = \{r \in Q : 0 < r \leq \sqrt{2}\}$, $C = \{r \in Q : -\sqrt{2} \leq r \leq \sqrt{2}\}$. For each of the sets A, B, and C, tell whether in the subspace topology it is open, closed, or neither.

27. Let H be the subspace of the plane described in Example 2.34(c). Let $A = \{(x, y) : 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1\}$. Find the interior and the closure of A in the subspace topology.

28. Suppose X is a topological space, the topology of which is generated by a pseudometric d. If A is a subspace of X, show that the subspace topology of A is generated by the restriction of d to $A \times A$.

29. Suppose X is a topological space, and the topology for X is generated by a collection of sets $\mathbf{C}$. Prove that if A is a subset of X, the collection $\mathbf{C}_A = \{C \cap A : C \in \mathbf{C}\}$ generates the subspace topology for A.

Suppose $(X,\mathbf{T})$ is a topological space and $(A,\mathbf{T}_A)$ is a subspace. For each of the following, if the statement is true, prove it; otherwise, give a counterexample.

30. If C is a subset of A, then $\text{int}_A(C) = A \cap \text{int}_X(C)$.
31. If C is a subset of X, then $\text{Fr}_A(C) = A \cap \text{Fr}_X(C)$.
32. If C is a $\mathbf{T}_A$-closed subset of A, then there is a $\mathbf{T}$-closed subset F of X such that $C = A \cap F$.

CHAPTER 3

Continuous Functions

We now introduce what is perhaps one of the most important concepts in all of mathematics, the idea of a continuous function. At an intuitive level, a continuous function is a function in which the image of points close to a set is close to the image of the set. What it means for a point to be close to a set is, of course, the question answered by the introduction of the notion of a limit point of a set. The definition of a topology and the subsequent development of this idea provide us with the machinery to give a precise answer to this question. Specifically, the closure of a set is the collection of all points close to the set, and so we have the following definition.

3.1 Continuity of a Function

Definition

A function $f : (X, \mathbf{T}) \to (Y, \mathbf{V})$ from one topological space to another is **continuous** if for every $S \subset X$, it is true that $f(\text{cl } S) \subset \text{cl } f(S)$.

Note that the continuity of a function depends on the topologies assigned to the domain and range, so it is a slight abuse of language to describe a function as being continuous without reference to the topologies under consideration. Strictly speaking, we should say f is continuous with respect to $\mathbf{T}$ and $\mathbf{V}$, or something similar. This is, however, unnecessary if we are careful to make clear exactly what are the domain and range topologies.

The following theorem shows that the continuity of a function actually depends only on the topology of the domain X and the topology of its image $f(X)$.

Theorem 3.1

A function $f : X \to Y$ from one topological space into another is continuous if and only if $f : X \to f(X)$ is continuous.

Proof: For any subset $S \subset X$, we have $\text{cl}_{f(X)} f(S) = \text{cl}_Y f(S) \cap f(X)$. Thus, if $f(\text{cl } S) \subset \text{cl}_{f(X)} f(S)$, then $f(\text{cl } S) \subset \text{cl}_Y f(S)$. On the other hand, if $f(\text{cl } S) \subset \text{cl}_Y f(S)$, then $f(\text{cl } S) \subset \text{cl}_{f(X)} f(S)$, since, of course, $f(\text{cl } S) \subset f(X)$.

Theorem 3.2

Suppose X is a topological space, Y is a topological space, A is a subspace of X, and $f : X \to Y$ is continuous. Then the restriction $f \mid A$ of f to A is continuous.

Proof: Let S be a subset of A. Then $(f \mid A)(\text{cl}_A S) = f(\text{cl}_A S)$. But $\text{cl}_A S \subset \text{cl}_X S$, so that $f(\text{cl}_A S) \subset f(\text{cl}_X S) \subset \text{cl } f(S)$, which is the same as $\text{cl}(f \mid A)(S)$.

Examples 3.3

(a) Suppose the set Y is endowed with the trivial topology. Then the closure of every nonempty subset of Y is all of Y, so every function from a topological space into Y is continuous.

(b) Suppose the set X is endowed with the discrete topology. Then the closure of every subset of X is the set itself, so every function from X into a topological space is continuous.

(c) Suppose $\mathbf{R}$ is the space of real numbers with the usual topology, and $f : \mathbf{R} \to \mathbf{R}$ is the function defined by setting $f(x) = 0$ for $x \le 0$, and $f(x) = 1$ for $x > 0$. Now consider the set $A = (0, 1]$. We have $f(A) = \{1\} = \text{cl } f(A)$, but $\text{cl } A = [0, 1]$. Thus f is not continuous since $f(\text{cl } A) = \{0\} \cup \{1\}$.

(d) Let X be the subset of the real numbers given by $X = \{x : |x| \ge 1\}$, together with the topology it inherits by being a subspace of the space of real numbers with the usual topology. Let $Y = \{0, 1\}$ with the discrete topology. Define $f : X \to Y$ by $f(x) = 0$ if $x \le -1$, and $f(x) = 1$ if $x \ge 1$. Then f is continuous. To see this, consider $S \subset X$; if S contains points larger than or equal to one and points less than or equal to one, then $f(S) = f(\text{cl } S) = Y$. On the other hand, if $x \ge 1$ for all x in S, then $f(S) = f(\text{cl } S) = \{1\} = \text{cl } f(S)$; and if $x \le 1$ for all x in S, then $f(S) = f(\text{cl } S) = \{0\} = \text{cl } f(S)$.

Theorem 3.4

A function from one topological space to another is continuous if and only if the inverse image of every closed set is closed.

Proof: First suppose $f : X \to Y$ is continuous and let $F \subset Y$ be closed. Then we have $f(\text{cl } f^{-1}(F)) \subset \text{cl}(f(f^{-1}(F))) \subset \text{cl}(F) = F$, since F is closed. This, of course, means that $\text{cl}(f^{-1}(F)) \subset f^{-1}(F)$, which, together with the fact that every set is included in its closure, tells us that $f^{-1}(F) = \text{cl}(f^{-1}(F))$. In other words, $f^{-1}(F)$ is closed.

Next suppose inverse images of closed sets are closed, and let $A \subset X$. Then the set $f^{-1}(\text{cl } f(A))$ is closed, and it includes A, so $\text{cl } A \subset f^{-1}(\text{cl } f(A))$. In other words, $f(\text{cl } A) \subset \text{cl}(f(A))$.

Theorem 3.5

A function from one topological space to another is continuous if and only if the inverse image of every open set is open.

Proof: This follows directly from the previous theorem and the observation that for any function $f : X \to Y$ it is true that $f^{-1}(G) = X - f^{-1}(Y - G)$ for every subset G of Y.

Every open set is the union of elements of a base for the open sets, so in the previous theorem we need only require that inverse images of all elements of a base be open to insure continuity of a function.

Theorem 3.6

Suppose $f : X \to Y$ is a function from one topological space to another, and $\mathbf{B}$ is a base for the topology of Y. If $f^{-1}(B)$ is open whenever B is an element of $\mathbf{B}$, then f is continuous.

Proof: If G is an open subset of Y, then $G = \cup\mathbf{C}$, where $\mathbf{C}$ is a subcollection of the base $\mathbf{B}$. Thus $f^{-1}(G) = \cup\{f^{-1}(C) : C \in \mathbf{C}\}$, which, being a union of open sets, is open.

Recall that if $f : X \to Y$ is a function and $\mathbf{K}$ is a collection of subsets of Y, then the set $\{f^{-1}(S) : S \in \mathbf{K}\}$ is denoted $f^{-1}(\mathbf{K})$. Thus if $f : (X, \mathbf{T}) \to (Y, \mathbf{V})$ is a function from one topological space into another, it is continuous if and only if $f^{-1}(\mathbf{V}) \subset \mathbf{T}$.

Theorem 3.7

Suppose X, Y, and Z are topological spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous. Then the composition $g \circ f : X \to Z$ is continuous.

Proof: If U is an open subset of Z, then $g^{-1}(U)$ is an open set in Y, and so $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. Thus according to Theorem 3.5, $g \circ f$ is continuous.

Examples 3.8

(a) Suppose X is the space of real numbers with the usual topology, and $f : X \to X$ is the function defined by setting $f(x) = x^2$. The collection of all intervals (a, b) is a base for this topology, so we consider $f^{-1}((a, b))$. If $a < b \leq 0$, then the inverse image of (a, b) is empty. If $0 \leq a < b$, then $f^{-1}((a,b)) = (-b, a) \cup (a, b)$. Finally, if $a < 0 < b$, then $f^{-1}((a, b)) = (-b, b)$. In each case $f^{-1}((a,b))$ is open, so f is continuous.

(b) Let X be the set of reals, let $\mathbf{T}$ be the topology generated by the base $\mathbf{B} = \{[a, b) : a \text{ and } b \text{ are real numbers with } a < b\}$, and let $\mathbf{U}$ be the usual topology. Let $f : (X, \mathbf{T}) \to (X, \mathbf{U})$ be the function given by $f(x) = x$, for $x < 0$ and $f(x) = x + 1$ for $x \geq 0$. Then f is continuous. As before, the collection of all intervals (a, b) is a base for $\mathbf{U}$, so consider all sets $f^{-1}((a, b))$. If $a \geq 1$, then $f^{-1}(a, b) = (a - 1, b - 1)$, which is an element of $\mathbf{T}$. If $0 \leq a < 1$, then $f^{-1}((a, b)) = [0, b-1)$ if $b > 1$, or is empty if $b \leq 1$; in either case it is in $\mathbf{T}$. If $a < 0$, then $f^{-1}((a, b)) = (a, b - 1)$ if $b > 1$. If $0 \leq b \leq 1$, then $f^{-1}((a, b)) = (a, 0)$. If $b < 0$, then $f^{-1}((a, b)) = (a, b)$. In any case, $f^{-1}((a, b))$ is an element of the topology $\mathbf{T}$. Thus f is continuous.

(c) Let $X, \mathbf{T}, \mathbf{U}$, and $f : (X, \mathbf{U}) \to (X, \mathbf{T})$ be defined as in the previous example. Here f is not continuous, for $f^{-1}([-2, -3)) = [-2, -3)$, which is not an element of the usual topology $\mathbf{U}$.

(d) Let X be the space of reals with the usual topology, let Y be the space of reals with the cofinite topology (Example 2.1(d)), and let p be any real nonconstant polynomial. Then $p : X \to Y$ is continuous. To see this, let F be a closed subset of Y. This means F is finite, say $F = \{y_1, y_2, \ldots, y_k\}$. Then $p^{-1}(F) = \cup\{p^{-1}(y_i) : i = 1, 2, \ldots, k\}$, which is finite since each set $p^{-1}(y_i)$ is finite. Thus $p^{-1}(F)$ is closed.

(e) Suppose $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are continuous functions from a topological space X into the space of reals R with the usual topology. Then the function $(f - g) : X \to \mathbf{R}$ defined by $(f - g)(x) = f(x) - g(x)$ is also continuous. To prove this, we shall show that for an open interval $I = (a, b) \subset \mathbf{R}$, the set $S = (f - g)^{-1}(I)$ is a neighborhood of each of its points, and hence open. Let $x_0 \in S$; then $f(x_0) - g(x_0) \in (a, b)$, $f(x_0) \in I_1 = (g(x_0) + a, g(x_0) + b)$ and $g(x_0) \in I_2 = (f(x_0) - b, f(x_0) - a)$. Now f and g are continuous, so $f^{-1}(I_1)$ and $g^{-1}(I_2)$ are both open neighborhoods of x_0. It follows that their intersection $N = f^{-1}(I_1) \cap g^{-1}(I_2)$ is an open neighborhood of x_0. To see that N is a subset of S, consider $f(x) - g(x)$ for x in N. Then $g(x_0) + a < f(x) < g(x_0) + b$ since $f(x) \in I_1$, and $f(x_0) - b < g(x) < f(x_0) - a$, since $g(x) \in I_2$. It follows that $a < f(x) - g(x) < b$, or in other words, $f(x) - g(x) \in S$.

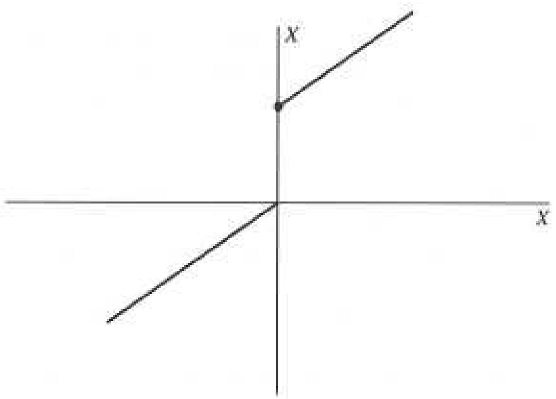


Figure 3.1 Examples 3.8(b) and (c)

The next theorem shows that our concept of a continuous function from one topological space to another is consistent with the perhaps more familiar notion of a continuous function from one pseudometric space to another.

Theorem 3.9

Suppose $f : (X, d) \to (Y, \rho)$ is a function from one pseudometric space into another. Then f is continuous if and only if it is true that for each x in X, given an $\epsilon > 0$, there is a $\delta > 0$ so that $\rho(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$.

Proof: Suppose f is continuous. The cell $C(f(x); \epsilon)$ is open, and so its inverse image is open and contains x. Thus there is a cell $C(x; \delta)$ included in the inverse image of $C(f(x); \epsilon)$. This, of course, means that $f(y) \in C(f(x); \epsilon)$ for every y in $C(x, \delta)$.

Now assume that for each x in X, given an $\epsilon > 0$, there is a $\delta > 0$ so that $\rho(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. The collection of all cells is a base for the topology for Y, so consider the set $G = f^{-1}(C(y; r))$, where $r > 0$. To show this is open, let $x \in G$. (If G is empty, then it is certainly open.) Now $f(x) \in C(y; r)$, so there is an $\epsilon > 0$ such that $z \in C(y; r)$ whenever $\rho(f(x), z) < \epsilon$. Choose δ so that $\rho(f(x), f(v)) < \epsilon$ when $d(x, v) < \delta$. Then $C(x; \delta) \subset G$, establishing that G is open, and that f is continuous.

Example 3.10

For the set of real numbers, the pseudometric $d(x, y) = |x - y|$ generates the usual topology. So for a real valued function f defined on the set of real numbers, continuity with respect to the usual topology on both the domain and the range is equivalent to the traditional "$\epsilon - \delta$" definition of elementary calculus: For each x, given an $\epsilon > 0$, there is a $\delta > 0$ so that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

Exercises

1. Let X be the space of real numbers with the usual topology, and let $f : X \to X$ be given by $f(x) = x^3$. Show that f is continuous.

2. Let X be the set of real numbers, let **U** be the usual topology, and let **D** be the discrete topology. Suppose $f : X \to X$ is given by $f(x) = x^3$. Which of the following are continuous? Explain.

   (a) $f : (X,\mathbf{U}) \to (X,\mathbf{D})$.

   (b) $f : (X,\mathbf{D}) \to (X,\mathbf{U})$.

3. Let $f : X \to Y$ be a function from one topological space into another. Suppose that **B** is a base for the topology of X and **C** is a base for the topology of Y. Prove that the following two statements are equivalent.

   (i) f is continuous.

   (ii) For each $x \in X$, if C is an element of **C** with $f(x) \in C$, then there is an element B of **B** such that $x \in B$ and $f(B) \subset C$.

4. Suppose $i : (X,\mathbf{T}) \to (X,\mathbf{S})$ is the identity function; that is, $i(x) = x$ for every $x \in X$. Prove that i is continuous if and only if $\mathbf{S} \subset \mathbf{T}$.

5. Suppose $f : (X, d) \to (Y, \rho)$ is a function of one pseudometric space into another having the property that there is a constant k so that $\rho(f(x), f(y)) \leq kd(x, y)$ for all x and y in X. (Such a function is sometimes called a **Lipschitz** function.) Prove that f is continuous.

6. Let a be a point in a pseudometric space (X, d), and define the function f_a from X into the reals with the usual pseudometric topology by setting $f_a(x) = d(a, x)$. Prove that f_a is continuous.

7. Suppose $f : (X,\mathbf{T}) \to (Y,\mathbf{S})$ is a continuous function from one topological space into another. Prove the following.

   (a) If **U** is a topology for Y such that $\mathbf{U} \subset \mathbf{S}$, then $f : (X,\mathbf{T}) \to (Y,\mathbf{U})$ is continuous.

   (b) If **V** is a topology for X such that $\mathbf{T} \subset \mathbf{V}$, then $f : (X,\mathbf{V}) \to (Y,\mathbf{S})$ is continuous.

8. Suppose $f : X \to Y$ is a continuous function from a topological space X into a topological space Y, and let $Z \subset Y$ be any subspace of Y with $f(X) \subset Z$. Prove that $f : X \to Z$ is continuous.

9. Suppose X is a topological space, and F_1 and F_2 are closed subsets of X such that $X = F_1 \cup F_2$. Prove that if $f : X \to Y$ is a function of X into a topological space Y, and if the restrictions $f \mid F_1$ and $f \mid F_2$ are continuous, then f is continuous.

10. Suppose $f : X \to Y$ is a function from one topological space into another, and suppose the topology for Y is generated by a collection of sets **C**. Prove that f is continuous if and only if $f^{-1}(C)$ is open for each $C \in \mathbf{C}$.

11. Suppose $f : X \to Y$ is a function from one topological space into another, and $x \in X$. Then f is said to be **continuous at** x if the inverse image of every neighborhood of $f(x)$ is a neighborhood of x. Prove that f is continuous if and only if f is continuous at every $x \in X$.

12. Let $(X,\mathbf{T})$ be a topological space and let $\{f_n : n \in F\}$ be a finite collection of continuous functions from X into the space of real numbers with the usual topology. Let $f : X \to \mathbf{R}$ be defined by setting $f(x) = \min\{f_n(x) : n \in F\}$. Prove that f is continuous.

13. Suppose $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are continuous functions from a topological space X into the space of reals $\mathbf{R}$ with the usual topology.

    (a) Prove that the function $(fg) : X \to \mathbf{R}$ defined by $(fg)(x) = f(x)g(x)$ is continuous.

    (b) Suppose that $g(x) \neq 0$ for all x in X. Prove that the function $(f/g) : X \to \mathbf{R}$ defined by $(f/g)(x) = f(x)/g(x)$ is continuous.

3.2 Homeomorphisms

If there is a one-to-one correspondence between two sets, then insofar as purely set theoretic notions are concerned, these two sets are indistinguishable. If, in addition, each of the sets is endowed with a topology, and if the one-to-one correspondence between the two sets also provides a one-to-one correspondence between the topologies, then the two topological spaces are topologically indistinguishable. Any property of one that is entirely expressed in terms of the topology must also be a property of the other. This idea is formalized in the definition of a homeomorphism.

Definition

A function $f : X \to Y$ from one topological space into another is a **closed function** if $f(F)$ is a closed set in the subspace $f(X)$ whenever F is a closed set in X. The function f is an **open function** if $f(U)$ is an open set in the subspace $f(X)$ whenever U is an open set in X.

Theorem 3.11

Suppose $f : X \to f(X) = Y$ is a function from one topological space onto another. Then f is a closed function if and only if it is true that for each y in Y, if U is a neighborhood of the set $f^{-1}(y)$, there is a neighborhood V of y such that $f^{-1}(V) \subset U$.

Proof: First suppose f is a closed function. Let y be any point of Y, and let U be a neighborhood of the set $f^{-1}(y)$. Then $X - \text{int}\, U$ is a closed set, and so $f(X - \text{int}\, U)$ is closed. Thus $V = Y - f(X - \text{int}\, U)$ is open. But $f^{-1}(y)$ is a subset of int U, so $y \in V$. Thus V is a neighborhood of y. That $f^{-1}(V) \subset U$ follows from the fact that $V = Y - f(X - \text{int}\, U) \subset \text{int}\, U \subset U$.

To prove the converse, let F be a closed subset of X. We shall show that $Y - f(F)$ is a neighborhood of each of its points, thus making it open and $f(F)$ closed. To see this, let y be any point of $Y - f(F)$. Then $f^{-1}(y) \subset X - F$, which is open and hence a neighborhood of $f^{-1}(y)$. So from our hypothesis, there is a neighborhood V of y such that $f^{-1}(V) \subset X - F$. In other words, $V \subset Y - f(F)$, making $Y - f(F)$ a neighborhood of y.

Corollary 3.12

If $f : X \to f(X) = Y$ is a one-to-one function, then the function $f^{-1} : Y \to X$ is continuous if and only if f is a closed function.

Definition

A one-to-one function from one topological space onto another is a **homeomorphism** if both it and its inverse function are continuous. Two topological spaces are said to be **homeomorphic** if there is a homeomorphism of one onto the other.

Theorem 3.13

A one-to-one function $f : X \to Y$ from a topological space X onto a space Y is a homeomorphism if and only if it is continuous and closed.

Proof: Recall that a function is continuous if and only if the inverse image of every closed set is a closed set. So the function f^{-1} is continuous if and only if $(f^{-1})^{-1}(F) = f(F)$ is closed whenever F is.

Of course, a function is continuous if and only if the inverse image of every open set is open, so we have a companion theorem.

Theorem 3.14

A one-to-one function $f : X \to Y$ from a topological space X onto a space Y is a homeomorphism if and only if it is continuous and open.

If the two topological spaces $(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are homeomorphic, a homeomorphism $h : X \to Y$ provides not only a one-to-one correspondence between the sets X and Y but also between the topologies $\mathbf{T}$ and $\mathbf{S}$: if U is a member of $\mathbf{T}$, then $h(U)$ is a member of $\mathbf{S}$, and if V is a member of $\mathbf{S}$, then $h^{-1}(V)$ is a member of $\mathbf{T}$. This, of course, means that the two spaces are indistinguishable as far as any properties that are defined in terms of a topology. Homeomorphic spaces are thus sometimes said to be **topologically equivalent**.

Examples 3.15

(a) Let X be a topological space, and let Y be any discrete space (that is, any set together with the discrete topology). Then every function from X into Y is both open and closed.

(b) Let X be any set together with the trivial topology, and let Y be a topological space. Then every function from X into Y is both open and closed since X is the only nonempty open set and the only nonempty closed set of X.

(c) Let X be the interval $[0, 2\pi)$ with the topology it inherits from the space of real numbers with the usual topology, and let Y be the plane with the usual topology. Let $f : X \to Y$ be the function given by $f(t) = (\cos t, \sin t)$. Then f is one-to-one and continuous but is not closed. To see that it is not closed, consider $F = [\pi, 2\pi)$. This is a closed subset of X, but $f(F)$ is not closed, because it does not include (0,0), which is a limit point of $f(F)$. We know that f is not open, for if it were, it would be a homeomorphism and hence closed. To see this directly, consider $U = [0, \pi)$. This is an open subset of X, but $f(U)$ is not open in Y because (0,0) is a member of $f(U)$ that is not in its interior.

(d) Suppose X is the space of reals with the usual topology, and define $f : X \to X$ by $f(x) = x + 3/2$, for $x \leq 1/2$, and $f(x) = 1/x$ for $x > 1/2$. Then f is continuous (see Example 3.10) but not closed. To see that it is not closed, note that the interval $(-2, -1)$ is a neighborhood of $\{-3/2\} = f^{-1}(0)$. From the graph, we see that for every neighborhood N of 0, we have $f^{-1}(N) \cap (0, +\infty) \neq \emptyset$. Thus there is no neighborhood N of 0 so that $f^{-1}(N) \subset (-2, -1)$.

Exercises

14. Suppose the topology of X is generated by a pseudometric d having the property that $d(x, y) = 0$ only if $x = y$, and suppose Y has the cofinite topology. Prove that every finite-to-one function $f : X \to Y$ is continuous. (A function is called finite-to-one if for every y, the set $f^{-1}(y)$ is finite.)

15. Suppose $f : X \to Y$ is an open function from one topological space into another, and suppose $S \subset X$ is such that $S = f^{-1}(T)$ for some $T \subset f(X)$. Prove that the restriction $f \mid S$ is an open function from the subspace S into Y.

16. Suppose $f : X \to Y$ is a closed function from one topological space into another, and suppose that $S \subset X$ is a closed subset. Prove the restriction $f \mid S$ is a closed function of the subspace S into Y.

17. Suppose $f : X \to f(X) = Y$ is a continuous function from one topological space onto another, and suppose that $\mathbf{B}$ is a base for the topology of X. Then the collection $f(\mathbf{B}) = \{f(B) : B \in \mathbf{B}\}$ is a base for the topology of Y if f is an open function.

18. Show that the space of reals with the usual topology and the interval (0,1) with the subspace topology are homeomorphic.

19. Suppose $a < b$, $c < d$ are real numbers. Show that, with the subspace topologies inherited from the space of real numbers with the usual topology, the spaces $[a, b]$ and $[c, d]$ are homeomorphic.

20. Let X be the plane with the usual topology, and let $Y = \{(x, y) \in X : x^2 + y^2 < 1\}$, with the subspace topology. Show that X and Y are homeomorphic.

21. Let X be the set of real numbers, let $\mathbf{S}$ be the topology generated by the base $\{[a, b) : a, b \in X, a < b\}$, and let $\mathbf{T}$ be the topology generated by the base $\{(a, b] : a, b \in X, a < b\}$. Is the identity function $i : (X, \mathbf{S}) \to (X, \mathbf{T})$ a homeomorphism? Are $(X, \mathbf{S})$ and $(X, \mathbf{T})$ homeomorphic? Explain.

In each of the following, if the statement is true, prove it; otherwise, give a counterexample.

22. If $f : X \to Y$ is one-to-one, onto, and continuous, then $f^{-1} : Y \to X$ is a closed function.

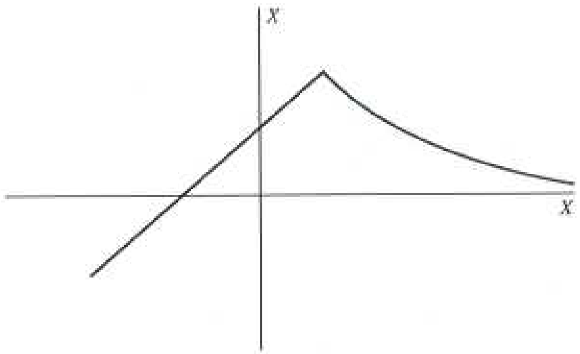


Figure 3.2 Example 3.15(d)

23. Any two discrete spaces of the same cardinality are homeomorphic.

24. If X, Y, and Z are topological spaces, X and Y are homeomorphic, and Y and Z are homeomorphic, then X and Z are homeomorphic.

25. If X is a set with the cofinite topology, and Y is the space of real numbers with the usual topology, then every function $f : X \to Y$ is closed.

3.3 The Weak Topology by a Collection of Functions

Suppose $f : X \to Y$ is a function from a set X into a topological space Y. If X is given the discrete topology, then, of course, f is continuous, since every subset of X is open. We shall see that it is interesting and useful to consider the problem of finding the smallest topology for X that makes each element of a given collection of functions continuous.

Definition

If X is a set, $\{(Y_a, \mathbf{T}_a) : a \in A\}$ is a collection of topological spaces, and for each $a \in A$, $f_a : X \to Y_a$ is a function from X into Y_a, then the **weak topology by F**, where $\mathbf{F} = \{f_a : a \in A\}$, is the topology $\mathbf{T_C}$ for X generated by the collection $\mathbf{C} = \cup\{f_a^{-1}(\mathbf{T}_a) : a \in A\}$. The weak topology for X by $\mathbf{F}$ is usually denoted $w(X, \mathbf{F})$. (Recall that $f_a^{-1}(\mathbf{T}_a) = \{f_a^{-1}(U) : U \in \mathbf{T}_a\}$.)

Theorem 3.16

If $(X, \mathbf{T})$ is a topological space, $\{Y_a : a \in A\}$ is a collection of topological spaces, and $\mathbf{F} = \{f_a : a \in A\}$ is a collection of functions $f_a : X \to Y_a$, then each f_a is continuous if and only if $\mathbf{T}$ includes the weak topology by $\mathbf{F}$.

Proof: First suppose every f_a is continuous. Then $\mathbf{C} = \cup\{f_a^{-1}(\mathbf{T}_a) : a \in A\}$ is a subset of $\mathbf{T}$, so $\mathbf{T_C} \subset \mathbf{T}$. Conversely, suppose $\mathbf{T_C} \subset \mathbf{T}$. Then each $f_a^{-1}(\mathbf{T_a}) \subset \mathbf{T}$; or in other words, each f_a is continuous.

The next proposition often provides a somewhat simpler description of the weak topology.

Proposition 3.17

Suppose X is a set, $\{(Y_a, \mathbf{T}_a) : a \in A\}$ is a collection of topological spaces, $\mathbf{F} = \{f_a : a \in A\}$ is a collection of functions $f_a : X \to Y_a$, and for each $a \in A$, $\mathbf{B}_a$ is a base for the topology of Y_a. Let $\mathbf{B} = \cup\{f_a^{-1}(\mathbf{B}_a) : a \in A\}$. Then the topology $\mathbf{T_B}$ generated by $\mathbf{B}$ is the weak topology by $\mathbf{F}$.

CHAPTER 4

Connected Spaces

We come now to another old and honorable mathematical concept: the idea of a connected set. What does it mean for a set to be connected? A set is connected if at any time it is split into two disjoint nonempty subsets, one of them must contain points close to the other. We see again that the idea of a point being close to a set is the fundamental notion in providing a sound logical basis for an important mathematical concept.

4.1 Connected Spaces

Definition

A topological space X is **disconnected** if there are nonempty subsets A and B such that $X = A \cup B$, with $A \cap \text{cl } B = \emptyset$, and $B \cap \text{cl } A = \emptyset$. A space that is not disconnected is said to be **connected**.

Proposition 4.1

A topological space is disconnected if and only if it has a nonempty proper subset that is both open and closed.

Proof: Suppose X is a disconnected space, and let $X = A \cup B$, where A and B are nonempty, $A \cap \text{cl } B = \emptyset$, and $B \cap \text{cl } A = \emptyset$. Then A and B are disjoint proper subsets of X, and $A = X - B$. Thus cl $A = A$, since $B \cap \text{cl } A = \emptyset$ implies that $\text{cl } A \subset X - B = A$. This makes A closed; and in the same way, we see that $B = X - A$ is closed. Hence both A and B are also open.

If we suppose, conversely, that X has a proper subset A that is both open and closed, then letting $B = X - A$, we have $X = A \cup B$, and B is also nonempty and both open and closed. Then $A \cap \text{cl } B = A \cap B = \emptyset$, and $B \cap \text{cl } A = B \cap A = \emptyset$.

Examples 4.2

(a) Let X be any set with the trivial topology $\mathbf{T} = \{\emptyset, X\}$. Then X is connected because it has no proper subset that is open.

(b) Let X be any set that has at least two points, with the discrete topology. Then X is disconnected, since every subset of X is both open and closed.

(c) Let $X = [0,1] \cup [2,3]$ have the topology it inherits as a subspace of the set of real numbers with the usual topology. Then with $A = [0,1]$ and $B = [2,3]$, we see that X is disconnected.

(d) The set of rational numbers $\mathbf{Q}$ with the topology inherited as a subspace of the space of real numbers with the usual topology is disconnected, for the set $S = \{r \in \mathbf{Q} : -\sqrt{2} < r < \sqrt{2}\}$ is both open and closed in $\mathbf{Q}$.

Theorem 4.3

With the usual topology, every closed interval $[a,b] = \{x \in \mathbf{R} : a \leq x \leq b\}$ is connected.

Proof: If $[a,b]$ is disconnected, it is the union of two nonempty disjoint open and closed sets, A and B. Let A be the one containing a. Since A is open, there is an $r > 0$ so that $[a, a+r) \subset A$. If we let c denote the greatest lower bound of $B = [a,b] - A$, then $a < a + r \leq c$. It cannot be that $c = b$, for if it were, then we would have $B = \{b\}$, which is clearly not open. Thus $a < c < b$. If $c \in A$, then there is an $s > 0$ so that $(c-s, c+s) \subset A$, since A is open. But in this case $[a, c+s) \subset A$, and $c+s$ would be a lower bound for B that is larger than c. Hence, we must have $c \in B$. Since B is open, there is an $s > 0$ so that $(c-s, c+s) \subset B$. But this means c is not a lower bound for B. Our original assumption that $[a,b]$ is disconnected leads to a contradiction, so $[a,b]$ is connected.

Proposition 4.4

A space X is disconnected if and only if there is a continuous function $F : X \to F(X) = D$ from X onto the two-point discrete space $D = \{0,1\}$.

Proof: First, if X is disconnected and $X = A \cup B$, with A and B nonempty, disjoint, and open and closed, then setting $F(x) = 0$ if $x \in A$ and $F(x) = 1$ if $x \in B$ provides a continuous function F from X onto D.

If, conversely, $F : X \to F(X) = D$ is a continuous function from X onto D, then $A = f^{-1}(0)$ and $B = f^{-1}(1)$ are nonempty, disjoint, and open and closed, making X disconnected.

Equivalent to Proposition 4.4 is the statement that X is connected if and only if every continuous function $F : X \to D$ from X to D is constant.

When we speak of a connected or disconnected subset of a topological space, we, of course, mean that the set with the subspace topology is a connected or disconnected topological space.

Theorem 4.5

Suppose S is a connected subset of a space X, and let K be a set such that $S \subset K \subset \text{cl } S$. Then K is connected.

Proof: Let $F : K \to D = \{a,b\}$ be any continuous function from K into the two-point discrete space D. Now S is connected, and the restriction of F to S is continuous, so $F|S$ cannot map S onto D. Thus $F(S)$ is a singleton, say $\{a\}$. From the continuity of F, we know that $F(\text{cl}_K S) \subset \text{cl } F(S)$, and this implies $F(K) = \{a\}$, since $K = \text{cl}_K S = \text{cl } S \cap K$ and $\text{cl } F(S) = \{a\}$. Thus every continuous $F : K \to D$ is constant, so K is connected.

Theorem 4.6

If $f : X \to Y$ is a continuous function from a topological space X into a topological space Y, and if C is a connected subset of X, then the image $f(C)$ is connected.

Proof: Let $F : f(C) \to D$ be a continuous function from $f(C)$ into the two-point discrete space D. Then the composition $F \circ f$ is a continuous function of C into D, and hence is constant since C is connected. This means F is constant, so $f(C)$ is connected.

Examples 4.7

(a) In the plane with the usual topology, the circle $C = \{(x,y) : x^2 + y^2 = 1\}$ is connected, since it is a continuous image of the interval [0,1]. Let $f : [0,1] \to C$ be defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $C = f([0,1])$.

(b) The graph of any continuous function $f : [a,b] \to \mathbf{R}$ from an interval $[a,b]$ into the space of real numbers, both with the usual topology, is a connected subset of the plane with the usual topology. Setting $g(t) = (t, f(t))$ defines a continuous function from $[a,b]$ into the plane, and $g([a,b])$ is the graph of f.

Theorem 4.8

Suppose $\mathbf{C}$ is a collection of nonempty connected subsets of a topological space. If there is a $C^* \in \mathbf{C}$ that meets each C in $\mathbf{C}$, then $\cup\mathbf{C}$ is connected.

Proof: We show that $\cup \mathbf{C}$ is connected by showing that every continuous function from it into the two-point discrete space D is constant. Let $F : \cup \mathbf{C} \to D$ be continuous. Now the sets $F(C)$ must be singletons since they are connected, and they are all the same singleton since they each meet $F(C^*)$. Thus F is constant.

Corollary 4.9

Suppose $\mathbf{C}$ is a collection of connected subsets of a topological space that has a nonempty intersection. Then $\cup \mathbf{C}$ is connected.

Proof: Let x be a point in $\cap \mathbf{C}$, and let $C^* = \{x\}$. Then the collection $\mathbf{C} \cup \{C^*\}$ satisfies the hypothesis of the theorem.

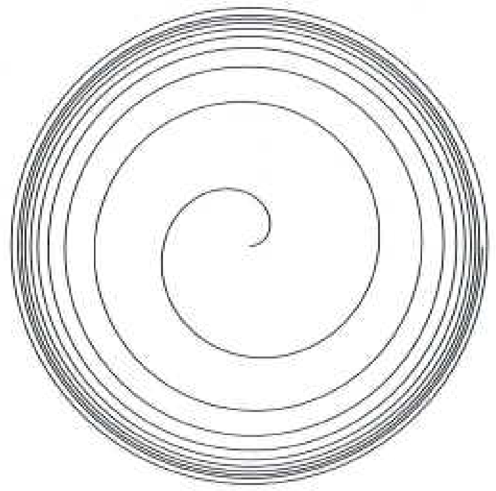
Examples 4.10

(a) The set of real numbers with the usual topology is connected since it is the union of $\mathbf{C} = \{[-n, n] : n = 1, 2, \ldots\}$. The semi-infinite interval $[0, +\infty)$ is also connected since $[0, +\infty) = \cup\{[0, n] : n = 1, 2, \ldots\}$.

(b) Any interval of the space of real numbers with the usual topology is connected. Any such interval is the union of a collection of closed intervals having a nonempty intersection. For example, $[a, b) = \cup\{[a, t] : t \in [a, b)\}$.

(c) The plane with the usual topology is connected. Each straight line through the origin is connected because it is homeomorphic to the space of reals with the usual topology, and the plane is the union of all such lines.

(d) In the plane with the usual topology, let $S = \{(r \cos t, r \sin t) : r = 1 - \frac{1}{t}, t \geq 1\}$, and let $C = \{(\cos t, \sin t) : t \in \mathbf{R}\}$. Then $X = S \cup C$ is connected, for S is a continuous image of the connected space $[1, +\infty)$, and every point of C is a limit point of S, making $X = \text{cl } S$.



$X = S \cup C$

Figure 4.1 Example 4.10(d)

Exercises

1. Let X be the set of real numbers with the topology $\mathbf{T} = \{X, \emptyset, [0, 1]\}$. Is $(X, \mathbf{T})$ connected? Explain.
2. Is the Sorgenfrey Line (Example 2.28(b)) a connected space? Explain.
3. Let K be a connected subset of the set of reals with the usual topology and suppose that a and b are in K and $a < b$. Prove that the interval $[a, b] \subset K$.
4. Suppose $[a, b]$ is an interval of real numbers with the usual topology, and let $f : [a, b] \to \mathbf{R}$, again with the usual topology, be continuous. Prove that if c is a real number between $f(a)$ and $f(b)$, then there is an $x \in [a, b]$ such that $f(x) = c$.
5. Suppose (X, d) is a pseudometric space and suppose a and b are points in X with $d(a, b) > 0$. Suppose further there is an r, $0 < r < d(a, b)$ such that $d(a, x) \neq r$ for every x in X. Prove that X is disconnected.
6. Let (X, d) be a pseudometric space, and let $S \subset X$ be such that $d(a, b) = 0$ for all a and b in S. Prove that S is connected.
7. Let $f : [0, 1] \to \mathbf{R}$, the reals, be defined by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that the graph $G = \{(x, f(x)) : x \in [0, 1]\}$ is a connected subset of the plane with the usual topology.
8. Prove or give a counterexample: A function f from a closed interval of real numbers with the usual topology into the set of real numbers, also with the usual topology, that has a connected graph is continuous.
9. Suppose A is a connected subset of a topological space. For each of the following, if the set is always connected, prove it; if not, give an example showing that it is not necessarily connected.

   (a) cl A (b) int A (c) Fr A (d) A'
10. Let X be the set of real numbers with the topology $\mathbf{T} = \{U \subset X : 0 \in U\} \cup \{\emptyset\}$. Is the space $(X, \mathbf{T})$ connected? How about the subspace $X - \{0\}$? Explain.

4.2 Components of a Space

Proposition 4.11

Let X be a topological space and let $R \subset X \times X$ be the relation $\{(x, y) :$ there is a connected subset of X that contains x and $y\}$. Then R is an equivalence relation on X.

Proof: The relation R is reflexive since singleton sets are always connected, and it is obviously symmetric. To see that it is also transitive, let (x, y) and (y, z) be in R. Then there are connected sets K_1 and K_2 with x and y in K_1 and y and z in K_2. Then according to Corollary 4.9, $K_1 \cup K_2$ is a connected set. It contains x and z, so $(x, z) \in R$.

Definition

Let X be a topological space, and let R be the equivalence relation in the previous proposition. The equivalence classes determined by R are called the **components** of X.

Theorem 4.12

If K is a connected subset of a topological space X, and C is a component of X such that K meets C, then $K \subset C$.

Proof: If x is a point in $K \cap C$, then obviously each point of K is related to x, and since $x \in C$, each point of K is contained in C.

Theorem 4.13

A component of a topological space is a connected set.

Proof: Let C be a component of a space, and let $x \in C$. Then for each $y \in C$, there is a connected subset of X, say K_y, that contains x and y. Clearly each point of K_y is related to x, so $K_y \subset C$. From this we see that $C = \cup\{K_y : y \in C\}$, and this set is connected since it is a union of a collection of connected sets having a nonempty intersection. (The point x is in every K_y.)

The last two theorems show that the components of a space are maximal connected subsets in the sense that they are connected and are not included in any larger connected subset. It is clear that a space is connected if and only if it has exactly one component.

Theorem 4.14

A component of a space is a closed set.

Proof: One need only note that the closure of a component C is a connected superset of C, and so must be C.

Examples 4.15

(a) Let $X = [0, 1] \cup [2, 3]$ have the topology it inherits as a subspace of the space of real numbers with the usual topology. Then X has two components, $[0, 1]$ and $[2, 3]$.

(b) The components of $\mathbf{Q}$, the rational numbers with the topology inherited from the space of reals with the usual topology, are the singleton subsets of $\mathbf{Q}$. Suppose C is a connected subset of $\mathbf{Q}$ containing two points, say $s < t$. Then there is an irrational z such that $s < z < t$. Thus $A = \{r \in C : r < z\}$ is a proper nonempty subset of C that is both open and closed, contradicting the connectedness of C. This shows that a connected subset of $\mathbf{Q}$ contains at most one point.

(c) Let X be the plane with the usual topology, and for each positive integer n let $C_n = \{(x, y) \in X : x^2 + y^2 = n^2\}$. Then the C_n are the components of the set $S = \cup\{C_n : n \in Z_+\}$.

Exercises

11. Let I be the set of irrational numbers with the usual topology (that is, the topology it inherits as a subspace of the real numbers with the usual topology). Describe the components of I.

12. Let X be the plane with the usual topology, and let S be the subspace consisting of all points of X both of whose coordinates are rational. Describe the components of S.

13. Suppose X is a topological space having only a finite number of components. Prove that each component of X is open.

14. Let X be the set of reals with the usual topology, and let $S = \{1/n : n \in Z_+\} \cup \{0\}$. Which components of S are open? Explain.

15. Give an example of a space having no open components, or prove there is no such space.

16. Define the equivalence relation R on a pseudometric space (X, d) by $R = \{(x, y) \in X \times X : d(x, y) = 0\}$. Prove that if X is finite, the equivalence classes are the components of X.

17. Let $f : X \to f(X) = Y$ be a continuous function from a topological space X onto a topological space Y having n components. Prove that X must have at least n components.

18. Let $f : X \to f(X) = Y$ be a continuous function, and suppose F is a subset of Y such that $Y - F$ has at least n components. Prove that $X - f^{-1}(F)$ has at least n components.

19. Are the set of reals and the plane with their usual topologies homeomorphic spaces? Explain.

20. Let X be the set of reals with the usual topology, and let Y be the two coordinate axes in the plane with the usual topology inherited from the plane. (That is, $Y = \{(x,0) : x \in X\} \cup \{(0,y) : y \in X\}$.) Suppose $f : X \to f(X) = Y$ is continuous. Prove that $f^{-1}((0,0))$ must contain at least three points.

21. Let C be the circle $C = \{(x,y) : x^2 + y^2 = 1\}$ with the usual topology inherited from the plane, and let Y be the interval $[a,b]$ with the usual topology. Suppose $f : C \to f(C) = Y$ is continuous. Prove that for every c, $a < c < b$, the set $f^{-1}(c)$ contains more than one point.

4.3 Path-Connected Spaces

One might also think of a connected space as being one in which every two points can be joined by a "continuous curve." We formalize this notion here.

Definition

If a and b are points in a topological space X, a **path** in X from a to b is a continuous function $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$, where the interval $[0,1]$ has the usual topology.

Definition

A topological space is **path-connected** if for every a and b in X there is a path in X from a to b.

Theorem 4.16

Every path-connected space is connected.

Proof: For every pair a, b in X, there is a path f in X from a to b, so $f([0,1])$ is a connected subset of X containing a and b. In other words, every pair of points of X is contained in a connected subset of X. This means X has just one component, and is thus connected.

Examples 4.17

(a) Any set X with the trivial topology is path-connected, for any function from $[0,1]$ into X is continuous.

(b) Let $X = S \cup C$ be the connected space described in Example 4.10(d). Suppose $f : [0,1] \to X$ is a path in X. Then the set $f^{-1}(C)$ is both open and closed. It is closed since it is the inverse image under a continuous function of the closed set C. To see that $f^{-1}(C)$ is open, let $t \in f^{-1}(C)$, and let U be a neighborhood of $f(t)$ in X small enough to insure that U is not a connected subset of X. For example, U can be the intersection of the cell centered at $f(t)$ of radius 0.1 with X. Now f is continuous, so there is an interval $I \subset [0,1]$ containing t so that $f(I) \subset U$. But $f(I)$ is connected, so $f(I) \subset C$ because $f(t) \in f(I) \cap C$. In other words, $I \subset f^{-1}(C)$, making $f^{-1}(C)$ open.
It now follows that if f were a path from a point in S to a point in C, then we would have $f^{-1}(C) \subset [0,1]$ both open and closed, nonempty, and not all of [0,1], contradicting the connectivity of [0,1]. Thus there is no such path, and X is not path-connected.

(c) Let X be the plane with the usual pseudometric. Then every cell $C(p;r) = \{z \in X : |p - z| < r\}$ is path-connected. This is easy to see, for if z and w are in $C(p;r)$, then $f(t) = tz + (1-t)w$ defines a path in $C(p;r)$ from w to z. Clearly $f(0) = w$ and $f(1) = z$ and $|p - f(t)| = |t(p-z) + (1-t)(p-w)| \le t|p-z| + (1-t)|p-w| < tr + (1-t)r = r$, making f a path in $C(p;r)$ from w to z.

Exercises

22. Suppose x, y, and z are points in a topological space X. Suppose there is a path in X from x to y and a path in X from y to z. Prove there is a path in X from x to z.

23. Prove that every connected subset of the reals with the usual topology is path-connected.

24. Let X be an open subset of the plane with the usual topology, and let $p \in X$. Let $S = \{x \in X :$ There is a path in X from p to $x\}$.
    (a) Prove that S is an open subset of X.
    (b) Prove that S is a closed subset of X.
    (c) Prove that every open connected subset of the plane with the usual topology is path-connected.

For each of the following, if the statement is true, prove it; otherwise, give a counterexample.

25. If X is a path-connected space and $f : X \to f(X) = Y$ is a continuous function from X onto a topological space Y, then Y is path-connected.

26. If S is a path-connected subset of a space X, and $S \subset K \subset \text{cl } S$, then K is path-connected.

27. If $\mathbf{C}$ is a collection of path-connected subsets of a space, and if there is a C^* in $\mathbf{C}$ that meets each C in $\mathbf{C}$, then $\cup\mathbf{C}$ is path-connected.

4.4 Locally Connected and Locally Path-Connected Spaces

Definition

A topological space X is **locally connected at x** in X if every neighborhood of x includes a connected open neighborhood of x. If X is locally connected at each of its points, it is said to be **locally connected**.

Definition

A topological space X is **locally path-connected at x** in X if every neighborhood of X includes a path-connected open neighborhood of x. If X is locally path-connected at each of its points, it is said to be **locally path-connected**.

Clearly each locally path-connected space is locally connected. The following proposition is essentially a rephrasing of the definitions of a locally connected space and of a locally path-connected space.

Proposition 4.18

(a) A space is locally connected if and only if there is a base for the topology consisting of connected sets.

(b) A space is locally path-connected if and only if there is a base for the topology consisting of path-connected sets.

Examples 4.19

(a) The space $X = S \cup C$ of Example 4.10(d) is connected but not locally connected. Let x be a point of C; then every sufficiently small open neighborhood of x is disconnected.

(b) Let X be the set of reals with the usual topology, and let A be the subspace of X given by $A = (0,1) \cup (2,3)$. Then A is not connected, but it is locally path-connected since it has a base consisting of open intervals of reals.

(c) For each positive integer n, define the following subset $L(n)$ of the plane: $L(n) = \{(\frac{1}{n}, y) : 0 \leq y \leq 1\}$. Let $M = \{(0, \frac{1}{k}) : k = 1, 2 \ldots\}$. Let $Y = \cup\{L(n) : n = 1, 2, \ldots\}$, and let I be the interval $I = \{(x, 0) : 0 \leq x \leq 1\}$. Then the subspace of the plane $X = M \cup Y \cup I$ is locally connected but not locally path-connected at $(0,0)$.

First note that $Y \cup I$ is connected as it is the union of a collection of connected sets, one of which meets every set in the collection (Theorem 4.8). Now X is connected since $Y \cup I \subset X \subset \text{cl } (Y \cup I)$ in the plane. Notice that every neighborhood of the origin (0,0) includes a neighborhood of the origin that is homeomorphic to X, thus making X locally connected at the origin. To see that no neighborhood of the origin is path-connected, notice that every neighborhood of the origin contains a point $p = (0, \frac{1}{k})$ for some $k \geq 1$. If f were a path from the origin to p, then $f^{-1}(p)$ would be a proper open and closed subset of the connected space [0,1], giving a contradiction. The set $f^{-1}(p)$ is closed since $\{p\}$ is closed, and the argument to show it is open is similar to the one used in showing that the space in Example 4.17(b) is not path-connected.

We know that the components of a space are closed but not necessarily open. The next theorem shows that a characterization of spaces in which components are open is provided by local connectedness.

Theorem 4.20

A space X is locally connected if and only if for every open $U \subset X$, each component of U is open.

Proof: First suppose X is locally connected, and U is an open subset of X. Let C be a component of U, and let $x \in C$. Then there is an open connected neighborhood N of x such that $N \subset U$, since X is locally connected. Since N meets C and is connected, $N \subset C$. This shows that C is a neighborhood of x; thus C is open because it is a neighborhood of each of its points.

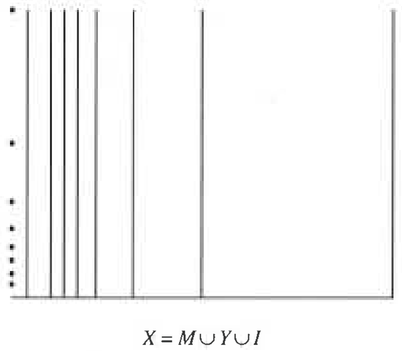


Figure 4.2 Example 4.19(c)

To prove the converse, suppose x is a point of X, let N be a neighborhood of x, and let $U = \text{int } N$. Let C be the component of U that contains x. Then, by assumption, C is open, and hence is a neighborhood of x included in U. Of course, C is connected, so X is locally connected.

One answer to the question of which connected spaces are also path-connected is provided by the next theorem.

Theorem 4.21

A connected space that is locally path-connected is path-connected.

Proof: Let X be connected and locally path-connected, and let a be a point of X. Define $S = \{y \in X : \text{There is a path in } X \text{ from } a \text{ to } y\}$. Then, of course, S is not empty since $a \in S$. We shall show that X is path-connected by showing that S is both open and closed, and hence all of X.

To see that S is open, let x be a point of S, and let U be a path-connected open neighborhood of x. Then for any y in U, there is a path f_1 in U from x to y, since U is path-connected. There is a path f_2 in X from a to x since x is in S. Define $f : [0,1] \to X$ by

$$f(t) = \begin{cases} f_2(2t), & \text{for } 0 \le t \le 1/2, \\ f_1(2t-1), & \text{for } 1/2 \le t \le 1. \end{cases}$$

Then f is clearly a path in X from a to y, so $y \in S$. Thus $U \subset S$, making S open.

We now show that S is closed. To this end, let y be a limit point of S, and let U be a path-connected open neighborhood of y. Then there is a point $x \in U \cap S$. This means there is a path in X from a to x and a path in U from x to y. As in the previous paragraph, there is thus a path in X from a to y, placing y in S and making S closed.

The space X is connected, so $X = S$. This shows there is a path in X from a to any point in X. In other words, X is path-connected.

Example 4.22

Every open connected subset of the plane with the usual topology is path-connected. We have seen that each cell $C(p;r)$ is path-connected (Example 4.17(c)), so any open subset of the plane is locally path-connected. Thus from Theorem 4.21, any such open connected set is path-connected.

Exercises

28. A set S in the plane is called **star-shaped** if there is a point x_0 in S so that if $x \in S$, then the line segment joining x and x_0 is a subset of S. Prove that every star-shaped subset of the plane with the usual topology is path-connected.

29. Let S be the set of points in the plane given by $S = \{(x,0) : x \text{ is rational}\}$. Let X be the union of all line segments joining the point $(0,1)$ to points in S. Assume X has the topology inherited as a subspace of the plane with the usual topology.
    (a) Prove that X is path-connected.
    (b) Are there any points at which X is locally path-connected? Explain.
30. Give an example of a continuous function from a locally connected space onto a space that is not locally connected, or prove there is no such example.
31. Suppose $f : X \to f(X) = Y$ is a continuous open function from one topological space onto another. Prove that Y is locally connected if X is.
32. Let U be an open subset of the space of real numbers with the usual topology. Prove that $U = \cup \mathbf{I}$, where $\mathbf{I}$ is a collection of open intervals such that if I_1 and I_2 are in $\mathbf{I}$, and $I_1 \cap I_2 \neq \emptyset$, then $I_1 = I_2$.

CHAPTER 5

Compact Spaces

Compactness is a property lacking the immediate intuitive content of continuity and connectedness. Unlike these older notions, it is a relatively recent idea that has resulted from the distillation of the essence of several older notions; it is one of the most important ideas in all of analysis and related mathematical areas. The importance of compactness will become clear only as the development unfolds.

5.1 Compact Spaces

Definition

A collection $\mathbf{C}$ of subsets of a space X such that $X = \cup\mathbf{C}$ is called a **cover** of X. A cover of a space that consists of open sets is called an **open cover**.

Definition

A topological space X is **compact** if every open cover of X includes a finite subcollection that is also a cover of X.

Examples 5.1

(a) Every finite space is compact.

(b) The set of real numbers $\mathbf{R}$ with the usual topology is not compact; for $\mathbf{C} = \{(-r, r) : r \in \mathbf{R}\}$ is an open cover, and every finite subset of $\mathbf{C}$ fails to be a cover of $\mathbf{R}$.

(c) Let X be the set of real numbers with the cofinite topology. Let $\mathbf{C}$ be an open cover of X, and let $C \in \mathbf{C}$. Then $X - C$ is finite, and so it is a subset of the union of a finite subcollection $\mathbf{F} \subset \mathbf{C}$. Thus $\mathbf{F} \cup \{C\}$ is a finite subcollection of $\mathbf{C}$ that is a cover for X, showing that X is compact.

Theorem 5.2

With the usual topology, every closed interval $[a, b] \subset \mathbf{R}$ is compact.

Proof: Let $\mathbf{C}$ be an open cover of $[a, b]$. Define the set $S = \{x \in [a, b] : [a, x] \subset \cup\mathbf{F}$ for some finite $\mathbf{F} \subset \mathbf{C}\}$. We shall show that S is both open and closed.

To see that S is open, let $x \in S$. Then there is a finite $\mathbf{F} \subset \mathbf{C}$ so that $[a, x] \subset \cup\mathbf{F}$. Now $\mathbf{C}$ is a cover for $[a, b]$, so $x \in C$ for some $C \in \mathbf{C}$. Let $I = (x - \delta, x + \delta)$ be an interval such that $I \cap [a, b] \subset C$. Then $\mathbf{F}^* = \mathbf{F} \cup \{C\}$ is a finite subset of $\mathbf{C}$, and I is obviously included in its union; thus we have $I \cap [a, b] \subset S$, making S a neighborhood of each of its points x. In other words, S is open.

Now to see that S is closed, let y be a limit point of S. Then $y \in C$ for some $C \in \mathbf{C}$. The set C is a neighborhood of y, so there is an interval $J = (y - \gamma, y + \gamma)$ so that $J \cap [a, b] \subset C$. Now y is a limit point of S, so there is an $x \in J \cap S$. There is a finite $\mathbf{F}^* \subset \mathbf{C}$ with $[a, x] \subset \cup\mathbf{F}^*$, so $\mathbf{F} = \mathbf{F}^* \cup \{C\}$ is a finite subset of $\mathbf{C}$, and $[a, y] \subset \cup\mathbf{F}$, thus making $y \in S$. So S is closed since it contains all its limit points.

Now S is an open and closed subset of the connected space $[a, b]$ so it must be either empty or all of $[a, b]$. But $a \in S$, so $S = [a, b]$, which shows that $[a, b]$ is indeed compact.

Proposition 5.3

Suppose A is a subspace of a topological space X. Then the subspace A is compact if and only if it is true that for every collection $\mathbf{C}$ of open subsets of X with $A \subset \cup\mathbf{C}$, there is a finite subcollection $\mathbf{F} \subset \mathbf{C}$ such that $A \subset \cup\mathbf{F}$.

Proof: Suppose first that A is compact, and let $\mathbf{C}$ be a collection of open subsets of X with $A \subset \cup\mathbf{C}$. Then $\mathbf{C_A} = \{C \cap A : C \in \mathbf{C}\}$ is an open cover of the compact space A, and so there is a finite subcollection $\mathbf{F_A} \subset \mathbf{C_A}$ that is a cover of A. It follows that $\mathbf{F} = \{C \in \mathbf{C} : C \cap A \in \mathbf{F_A}\}$ is finite and $A \subset \cup\mathbf{F}$.

For the converse, let $\mathbf{C_A}$ be an open cover of the space A. For each $C_A \in \mathbf{C_A}$, we have $C_A = C \cap A$, where C is an open subset of X. Then $A \subset \cup\mathbf{C}$, where $\mathbf{C} = \{C : C \cap A \in \mathbf{C_A}\}$. From the hypothesis, we know A is included in the union of a finite subcollection $\mathbf{F}$ of $\mathbf{C}$. Now, $\mathbf{F_A} = \{C \cap A : C \in \mathbf{F}\}$ is a finite subcollection of $\mathbf{C_A}$ that is a cover of A. This shows A is compact.

Theorem 5.4

Suppose $f : X \to Y$ is a continuous function from one topological space into another. If A is a compact subset of X, then $f(A)$ is compact.

Proof: If $\mathbf{C}$ is a collection of open sets in Y with $f(A) \subset \cup\mathbf{C}$, then A is included in the union of the collection $f^{-1}(\mathbf{C}) = \{f^{-1}(C) : C \in \mathbf{C}\}$ of open sets in X, and is thus included in the union of some finite subcollection $\mathbf{F}$ of $f^{-1}(\mathbf{C})$. Thus $\{C \in \mathbf{C} : f^{-1}(C) \in \mathbf{F}\}$ is finite and includes $f(A)$ in its union.

Example 5.5

In the plane X with the usual topology, the unit circle $C = \{(x, y) \in X : x^2 + y^2 = 1\}$ is compact, since it is $f([0, 2\pi])$, where f is the continuous function from $[0, 2\pi]$ into X given by $f(t) = (\cos t, \sin t)$.

Theorem 5.6

Every infinite subset of a compact space has a limit point.

Proof: Let A be a subset of a compact space X, and suppose no point of X is a limit point of A. Then there is a neighborhood $N(x)$ of each x in X that contains no points of A except possibly x itself. The collection $\mathbf{C} = \{\operatorname{int} N(x) : x \in X\}$ is an open cover of X. But X is compact, and so $X = \operatorname{int} N(x_1) \cup \operatorname{int} N(x_2) \cup \ldots \cup \operatorname{int} N(x_k)$ for some finite subset $\{x_1, x_2, \ldots, x_k\}$ of X. Thus $A \subset \{x_1, x_2, \ldots, x_k\}$. We have proved that every subset that has no limit point is finite; or in other words, every infinite set must have a limit point.

Definition

A collection $\mathbf{K}$ of subsets of a set has the **finite intersection property** if the intersection of any finite subcollection of $\mathbf{K}$ is nonempty.

The following theorem characterizes compactness in terms of collections of closed sets with the finite intersection property. It is a straightforward application of the De Morgan laws.

Theorem 5.7

A space X is compact if and only if every collection of closed subsets with the finite intersection property has a nonempty intersection.

Proof: First assume that X is compact, and let $\mathbf{K}$ be a collection of closed sets having the finite intersection property. If $\cap\mathbf{K}$ is empty, then $\mathbf{C} = \{X - K : K \in \mathbf{K}\}$ is an open cover of X, since $\cup\mathbf{C} = X - \cap\mathbf{K} = X$. But X is compact, so $X = \cup\mathbf{F}$, where $\mathbf{F}$ is a finite subcollection of $\mathbf{C}$. This, however, contradicts the fact that $\mathbf{K}$ has the finite intersection property, since the collection of complements of the elements of $\mathbf{F}$ is a finite subcollection of $\mathbf{K}$ with an empty intersection.

Now suppose that every collection of closed sets with the finite intersection property has a nonempty intersection, and let $\mathbf{C}$ be an open cover for X. Then the collection $\mathbf{K}$ of complements of elements of $\mathbf{C}$ has empty intersection, again from the De Morgan laws. Therefore $\mathbf{K}$ does not have the finite intersection property; in other words, there is a finite subcollection $\{K_1, K_2, \ldots, K_k\} \subset \mathbf{K}$ that has an empty intersection. Thus $\{X - K_1, X - K_2, \ldots, X - K_k\} \subset \mathbf{C}$ is a cover for X.

Examples 5.8

(a) In the space of real numbers with the usual topology, the subspace $[0,1)$ is not compact. The collection $\mathbf{K} = \{[1-1/n, 1) : n = 1, 2, \ldots\}$ is a collection of closed subsets of $[0,1)$ that has the finite intersection property but has empty intersection.

(b) In the plane Y with the usual topology, the subspace $X = Y - \{(0,0)\}$ is not compact. For each positive integer n, let $K_n = \{(x,y) \in X : x^2 + y^2 \leq 1/n\}$; then $\{K_n : n = 1, 2, \ldots\}$ is a collection of closed sets in X that has the finite intersection property and has empty intersection.

Theorem 5.9

Every closed subset of a compact space is compact.

Proof: Let K be a closed subset of the compact space X, and suppose $\mathbf{C}$ is a collection of open subsets of X such that $K \subset \cup\mathbf{C}$. Then $\mathbf{C}^* = \mathbf{C} \cup \{X - K\}$ is an open cover of X, so there is a finite subcollection $\mathbf{F}^*$ of $\mathbf{C}^*$ that is a cover for X. It is clear that $K \subset \cup\mathbf{F}$, where $\mathbf{F} = \mathbf{F}^* - \{X - K\}$, thus making K compact.

Example 5.10

Let X be the set of real numbers with the cofinite topology. Then by an argument essentially the same as used in Example 5.1(c) to show that X is compact, every subset of X is compact. Only the finite subsets are closed.

The example shows that the converse of Theorem 5.9 is not true. It is, however, true for topological spaces in which points can be separated by open sets. This is an important class of spaces, called Hausdorff spaces.

Definition

A topological space X is a **Hausdorff** space if for every x and y in X, with $x \neq y$, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

The familiar idea of a metric space provides a rich supply of Hausdorff spaces.

Definition

A pseudometric space (X, d) in which $d(x,y) = 0$ only if $x = y$ is called a **metric space**.

Proposition 5.11

A pseudometric space is a Hausdorff space if and only if it is a metric space.

Proof: First suppose (X, d) is a metric space. If $x \neq y$, then we have $d(x,y) = r > 0$, and the cells $C(x; r/2)$ and $C(y; r/2)$ are disjoint open sets containing x and y.

Now suppose (X, d) is Hausdorff, and let x and y be different points in X. Then there is an open U containing x that does not contain y. The fact that $x \in U$ means there is an $r > 0$ such that the cell $C(x; r) \subset U$. From this it follows that $d(x,y) \geq r > 0$; for if $d(x,y) < r$, then y would be in $C(x; r)$, and hence in U.

Theorem 5.12

Suppose K is a compact subset of a Hausdorff space X, and suppose p is a point in the complement of K. Then there are disjoint open sets U and V with $p \in V$ and $K \subset U$.

Proof: The point p is in $X - K$, and X is Hausdorff; so for each $x \in K$, there are an open neighborhood $U(x)$ of x and an open neighborhood $V(x)$ of p such that $U(x) \cap V(x) = \emptyset$. Then $K \subset \cup\{U(x) : x \in K\}$, and so there is a finite subcollection $\{U(x_i) : i = 1, 2, \ldots, k\}$ with $K \subset U = \cup\{U(x_i) : i = 1, 2, \ldots, k\}$. Now then, $V = \cap\{V(x_i) : i = 1, 2, \ldots, k\}$ is an open neighborhood of p that does not meet U.

Corollary 5.13

Every compact subset of a Hausdorff space is closed.

Proof: The theorem tells us that every point of the complement of a compact set K has an open neighborhood that does not meet K. In other words, no point of the complement of K is a limit point of K, so K is closed.

Theorem 5.14

Suppose K and M are disjoint compact subsets of a Hausdorff space. Then there are disjoint open sets U and V with $K \subset U$ and $M \subset V$.

Proof: For each $x \in K$ there are disjoint open sets $U(x)$ and $V(x)$ so that $x \in U(x)$ and $M \subset V(x)$. Now $K \subset \cup\{U(x) : x \in K\}$, and so $K \subset \cup\{U(x_i) : i = 1, 2, \ldots, k\}$, a finite subcollection of $\{U(x) : x \in K\}$. Let $U = \cup\{U(x_i) : i = 1, 2, \ldots, k\}$ and $V = \cap\{V(x_i) : i = 1, 2, \ldots, k\}$. Then U and V are disjoint, $K \subset U$, and $M \subset V$.

Theorem 5.15

If $f : X \to Y$ is a continuous function from a compact space into a Hausdorff space, then f is a closed function.

Proof: If F is a closed subset of X, then it is compact. Consequently, $f(F)$ is compact, and hence a closed subset of the Hausdorff space Y. Thus $f(F) = f(F) \cap f(X)$ is a closed set in the subspace $f(X)$.

Corollary 5.16

If $f : X \to f(X) = Y$ is a one-to-one continuous function from a compact space X onto a Hausdorff space Y, then f is a homeomorphism.

Definition

A subset B of a pseudometric space (X, d) is **bounded** if there is a real number M such that $d(x, y) \leq M$ for all x and y in B. The **diameter** of a bounded set B is the real number $D(B) = \sup\{d(x, y) : x, y \in B\}$.

Note that a subset B of a pseudometric space is bounded if and only if it is a subset of a cell $C(x; r)$ with $r > 0$.

Example 5.17

It is important to notice that the property of being bounded is not a topological property. If X is the set of real numbers, then both metrics $d(x, y) = |x - y|$, and $\rho(x, y) = \min\{|x - y|, 1\}$ generate the usual topology for X, but (X, d) is not bounded, while (X, ρ) is.

Proposition 5.18

Every compact subset of a pseudometric space is bounded.

Proof: Let B be a compact subset of the pseudometric space X, and let z be any point in X. Then B is a subset of the union of the collection of cells $\{C(z; n) : n \in Z_+\}$, and thus is a subset of some finite subcollection of these cells. But this means that B is a subset of $C(z; m)$, where m is the largest of the radii of the cells in this finite collection.

Theorem 5.19

A subset of the space of real numbers with the usual pseudometric is compact if and only if it is closed and bounded.

Proof: We have just proved that every compact set K in a pseudometric space is bounded. Such a set K is closed because the usual pseudometric is a metric, making $\mathbf{R}$ a Hausdorff space.

Suppose K is a closed and bounded subset of the space of real numbers. Being bounded, it is a subset of some interval $[a, b]$. Thus K is a closed subset of the compact space $[a, b]$, and hence is compact.

Once again, notice that the metric is crucial. Although not compact, the set $\{r \in \mathbf{R} : r \geq 0\}$ is a closed and bounded subset of the reals with the metric ρ of Example 5.17.

Examples 5.20

(a) If a subset of S of the real numbers with the usual topology is compact and connected, then it is a closed interval $[a, b]$. If S is compact, it is bounded, and so has both a least upper bound b and a greatest lower bound a. Since S is closed, we have $a \in S$ and $b \in S$. Since S is connected, every c such that $a < c < b$ is also in S. In other words, $S = [a, b]$.

(b) Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function, where both $[a, b]$ and $\mathbf{R}$, the set of real numbers, have the usual topology. Then there are points x_n and x_m in $[a, b]$ such that $f(x_n) \leq f(x) \leq f(x_m)$ for all x in $[a, b]$. This is an easy consequence of the fact that $f([a, b])$ must be compact and connected because $[a, b]$ is. Thus $f([a, b])$ is a closed interval, and $f(x_n)$ and $f(x_m)$ are simply the endpoints of $f([a, b])$.

Exercises

1. Show that the plane with the usual topology is not compact.
2. Prove that every subset of the set of reals with the cofinite topology is compact.
3. Prove that the union of two compact sets is compact.
4. Prove that a discrete space is compact if and only if it is finite.
5. Suppose X is a space and $\mathbf{T_1}$ and $\mathbf{T_2}$ are topologies for X with $\mathbf{T_1} \subset \mathbf{T_2}$.

Prove the following or give a counterexample.

(a) If $(X, \mathbf{T_1})$ is compact, then $(X, \mathbf{T_2})$ is compact.

(b) If $(X, \mathbf{T_2})$ is compact, then $(X, \mathbf{T_1})$ is compact.

6. Suppose $\mathbf{B}$ is a base for the topology of a space X, and suppose that every cover of X consisting of elements of $\mathbf{B}$ has a finite subcollection that is also a cover of X. Prove that X is compact.

7. Prove that if (X, d) is a compact pseudometric space, then there is an M so that $d(x, y) < M$ for every x and y in X.

8. Let (X, d) be a compact pseudometric space, and let $\epsilon > 0$. Prove there is a finite subset F of X such that for every x in X, there is a point y in F so that $d(x, y) < \epsilon$.

9. Prove that every bounded infinite subset of the reals with the usual topology has a limit point.

10. Prove or give a counterexample:

    (a) The intersection of a collection of compact subsets of a space is compact.

    (b) The intersection of a collection of compact subsets of a Hausdorff space is compact.

11. Prove that if d and ρ are equivalent pseudometrics for a set X, then d is a metric if and only if ρ is a metric.

12. Let $\{C_i : i = 1, 2, \ldots\}$ be a countable collection of closed compact subsets of a space X such that $C_{i+1} \subset C_i$ for each $i = 1, 2, \ldots$. Prove that $\cap\{C_i : i = 1, 2, \ldots\}$ is not empty.

13. Suppose $f : X \to X$ is a continuous function from a compact Hausdorff space into itself. Prove there is a subspace $A \subset X$ such that $f(A) = A$.

14. Let $\{C_i : i = 1, 2, \ldots\}$ be a countable collection of compact subsets of a Hausdorff space X such that $C_{i+1} \subset C_i$ for each $i = 1, 2, \ldots$. Let U be an open neighborhood of $\cap\{C_i : i = 1, 2, \ldots\}$. Prove there is an integer N so that $C_i \subset U$ for all $i \geq N$.

15. Let $\{C_i : i = 1, 2, \ldots\}$ be a countable collection of compact connected subsets of a Hausdorff space X such that $C_{i+1} \subset C_i$ for each $i = 1, 2, \ldots$. Prove that $A = \cap\{C_i : i = 1, 2, \ldots\}$ is nonempty, compact, and connected.

16. Prove that a pseudometric space is a metric space if and only if each singleton set is closed.

17. Suppose $f : X \to f(X) = Y$ is a continuous function from one compact Hausdorff space onto another. Suppose further that for each y in Y, $f^{-1}(y)$ is connected. Prove that if K is compact and connected, then $f^{-1}(K)$ is compact and connected.

18. Let $f : X \to f(X) = Y$ be a continuous function from one topological space onto another. Prove that if X is compact and Hausdorff, then so also is Y.

19. Suppose $(X, \mathbf{T})$ is a compact Hausdorff space. Prove that if $\mathbf{S}$ is any Hausdorff topology for X such that $\mathbf{S} \subset \mathbf{T}$, then $\mathbf{S} = \mathbf{T}$.

20. Suppose X is a compact Hausdorff space, $x \in X$, and N is a neighborhood of x. Prove that there is an open set U such that $x \in U \subset \text{cl } U \subset N$.

21. Let B be a bounded subset of a pseudometric space. Prove that $D(B) = D(\text{cl } B)$. ($D(B)$ is the diameter of B.)

5.2 The One-Point Compactification

In the sequence of results to follow, we show that for any noncompact topological space $(X, \mathbf{T})$, we can adjoin exactly one point p to X and define a topology $\mathbf{T}^*$ for $X^* = X \cup \{p\}$ in such a way as to make $(X^*, \mathbf{T}^*)$ a compact space with $(X, \mathbf{T})$ as a dense subspace.

Proposition 5.21

Suppose $(X, \mathbf{T})$ is a topological space, and p is a point not in X. Let $X^* = X \cup \{p\}$, and let $\mathbf{Q} = \{U \cup \{p\} : U \in \mathbf{T} \text{ and } X - U \text{ is compact}\}$. Then $\mathbf{T}^* = \mathbf{T} \cup \mathbf{Q}$ is a topology for X^*, $(X, \mathbf{T})$ is a subspace of $(X^*, \mathbf{T}^*)$, and $(X^*, \mathbf{T}^*)$ is compact.

Proof: We first show that $\mathbf{T}^*$ is a topology. Since $\emptyset \in \mathbf{T}$, and since $X^* \in \mathbf{Q}$ because $\emptyset$ is compact, it is clear that $\mathbf{T}^*$ contains both X^* and the empty set. To prove that the union of a subcollection of $\mathbf{T}^*$ is an element of $\mathbf{T}^*$, let $\mathbf{C} \subset \mathbf{T}^*$. Then $\mathbf{C} = (\mathbf{C} \cap \mathbf{T}) \cup (\mathbf{C} \cap \mathbf{Q})$. Let $\mathbf{R} = \mathbf{C} \cap \mathbf{T}$, and let $\mathbf{S} \subset \mathbf{T}$ be such that $\mathbf{C} \cap \mathbf{Q} = \{S \cup \{p\} : S \in \mathbf{S} \text{ and } X - S \text{ is compact}\}$. There are two cases to consider. First, if $\mathbf{C} \cap \mathbf{Q} = \emptyset$, then $\mathbf{C} \subset \mathbf{T}$, and so $\cup\mathbf{C} \in \mathbf{T} \subset \mathbf{T}^*$. Second, if $\mathbf{C} \cap \mathbf{Q} \neq \emptyset$, then $\cup\mathbf{C} = (\cup\mathbf{R}) \cup (\cup\mathbf{S}) \cup \{p\} = \cup\{\mathbf{R} \cup \mathbf{S}\} \cup \{p\}$. We know that $\cup\{\mathbf{R} \cup \mathbf{S}\}$ is an element of $\mathbf{T}$ since $\mathbf{R}$ and $\mathbf{S}$ are subcollections of $\mathbf{T}$, so we now need only observe that its complement in X is compact since $X - \cup\{\mathbf{R} \cup \mathbf{S}\} = \cap\{X - U : U \in \mathbf{R} \cup \mathbf{S}\}$ is a closed subset of $X - S$ for some element of $\mathbf{S}$.

Now suppose U and V are elements of $\mathbf{T}^*$. We need to show that $U \cap V \in \mathbf{T}^*$. If $U, V \in \mathbf{T}$, then, of course, $U \cap V \in \mathbf{T} \subset \mathbf{T}^*$. If $U, V \in \mathbf{Q}$, then $U = W_u \cup \{p\}$ and $V = W_v \cup \{p\}$, where W_u and W_v are elements of $\mathbf{T}$, with $X - W_u$ and $X - W_v$ compact. Then $U \cap V = (W_u \cap W_v) \cup \{p\}$, which is an element of $\mathbf{Q}$. Finally, suppose $U \in \mathbf{T}$ and $V \in \mathbf{Q}$. Then $V = W \cup \{p\}$, where $W \in \mathbf{T}$, and $X - W$ is compact. In this case, $U \cap V = U \cap W \in \mathbf{T} \subset \mathbf{T}^*$.

Next, we shall show that X^* is compact. Let $\mathbf{C}$ be an open cover of X^*. Then p is an element of some W in $\mathbf{C}$. The set $W \in \mathbf{T}^*$, and so

$W = U \cup \{p\}$ for some $U \in \mathbf{T}$ such that $X - U$ is compact. Since

$$X - U \subset X = X \cap X^* = \cup\{V \cap X : V \in \mathbf{C}\},$$

there is a finite $\mathbf{F} \subset \mathbf{C}$ so that $X - U \subset \cup\{V \cap X : V \in \mathbf{F}\}$. Now then, $\mathbf{F} \cup \{W\}$ is a finite subcollection of $\mathbf{C}$ that is a cover for X^*.

Finally, X is a subspace of X^* since $\mathbf{T} = \{W \cap X : W \in \mathbf{T}^*\}$.

Proposition 5.22

If X is not compact, then X is a dense subspace of X^*.

Proof: If X is not a dense subspace of X^*, then p is not a limit point of X. The only way in which the adjoined point p can fail to be a limit point of X is for $\{p\} = \emptyset \cup \{p\}$ to be a member of the topology $\mathbf{T}^*$, and this means that $X = X - \emptyset$ is compact.

Proposition 5.23

Suppose X is Hausdorff, and suppose that every point of X has a compact neighborhood. Then X^* is Hausdorff.

Proof: The topology of X is a subcollection of the topology of X^*, and X is Hausdorff, so if x and y are distinct points of X, then they obviously have disjoint X^* open neighborhoods. Suppose then that x is a point of X, and let K be a compact neighborhood of x. Then K is a closed subset of X, and so $(X - K) \cup \{p\} \in \mathbf{T}^*$ is an open neighborhood of p. Thus the interior of K and $(X - K) \cup \{p\}$ are disjoint open neighborhoods of x and p, respectively.

The property in the previous proposition of each point's having a compact neighborhood has a name.

Definition

A **locally compact** topological space is one in which every point has a compact neighborhood.

Every compact space is, of course, locally compact. The space of real numbers with the usual topology is an example of a space that is locally compact but not compact.

Proposition 5.24

Suppose Z is a compact Hausdorff space, z is a point of Z, and $X = Z - \{z\}$. Then X^* and Z are homeomorphic.

Proof: Let $h : X^* \to Z$ be defined by setting $h(x) = x$ for x in X, $h(p) = z$. Then h is clearly one-to-one and onto. It remains only to show that h is continuous, since the inverse of a continuous one-to-one function with a compact domain is necessarily continuous (Corollary 5.16).

Let U be an open subset of Z. If $z \in Z - U$, then $U \subset X$ and $h^{-1}(U) = U$ is open in X and hence in X^*. If $z \in U$, then $h^{-1}(U) = (U \cap X) \cup \{p\}$. But $U \cap X$ is an open subset of X, and its complement $X - (U \cap X) = (Z - U) \cap X = Z - U$, being a closed subset of the compact space Z, is compact. Thus $h^{-1}(U) \in \mathbf{T}^*$.

We summarize the results of the preceding four propositions by collecting them into a single theorem.

Theorem 5.25

Suppose X is a locally compact Hausdorff space that is not compact. Then there is a unique compact Hausdorff space X^* that includes X as a dense subspace and that is such that $X^* - X$ consists of exactly one point.

Definition

The space X^* of the previous theorem is called the **one-point compactification** of X.

Examples 5.26

(a) The one-point compactification of the space of real numbers with the usual topology is the circle (with the usual topology, of course). To see this, observe that removing one point from the circle leaves a dense subspace homeomorphic with the reals.

(b) Let X be the positive integers with the discrete topology. Then X^* is homeomorphic with the subspace of the reals with the usual topology $Z = \{1/n : n = 1, 2, \ldots\} \cup \{0\}$. The function $h : X^* \to Z$ given by $h(n) = 1/n$, and $h(p) = 0$, is a homeomorphism of X^* onto Z.

Exercises

22. Let $X = [0, 1)$ have the topology inherited from the space of real numbers with the usual topology. Describe the one-point compactification X^*.

23. Let $X = (0, 1) \cup (2, 3)$ have the topology inherited from the space of reals with the usual topology. Describe the one-point compactification X^*.

24. Prove that if X is a dense subspace of X^*, then X is not compact.
25. Suppose X is a Hausdorff space that is not compact. Prove that if X^* is Hausdorff, then X is locally compact.
26. Prove that every open subspace of a locally compact space is locally compact.
27. Suppose X is a dense subspace of a compact Hausdorff space Z. Prove that X is locally compact if and only if it is an open subspace of Z.
28. (a) Let x be a point in a locally compact Hausdorff space X, and suppose U is a neighborhood of x. Prove there is a compact neighborhood V of x such that $V \subset U$.

    (b) Prove that every locally compact Hausdorff space has a base $\mathbf{B}$ such that for each B in $\mathbf{B}$, cl B is compact.
29. Let A be a subset of a locally compact Hausdorff space X, and let $\mathbf{X}^*$ be the one-point compactification of X. Prove that $\text{cl}_X A = \text{cl}_{X^*} A$ if and only if A is included in a compact subset of X.
30. Let $f : X \to f(X) = Y$ be a continuous function from one locally compact Hausdorff space onto another. Define the function $F : X^* \to Y^*$ from the one-point compactification of X onto the one-point compactification of Y by setting $F(x) = f(x)$ for x in X and $F(p) = q$, where $X^* - X = \{p\}$, and $Y^* - Y = \{q\}$. Prove that F is continuous if and only if it is true that $f^{-1}(K)$ is compact for every compact $K \subset Y$.

CHAPTER 6

Product Spaces

The Cartesian product of a collection of sets is one of the most important and widely used ideas in mathematics. In case each of the sets in the collection is endowed with a topology, there is, as we shall see, a useful and natural topology for the product.

6.1 Products of Spaces

Before turning our attention to topological questions, we look first at the purely set-theoretic aspects of products of collections of sets.

Definition

Let $\mathbf{X} = \{X_a : a \in A\}$ be a collection of sets indexed by a set A. The **product** of $\mathbf{X}$, denoted $\prod \mathbf{X}$, is the set of all functions $x : A \to \cup\mathbf{X}$ such that $x(a) \in X_a$.

Examples 6.1

(a) Suppose $\mathbf{X} = \{X_1, X_2\}$, so that the indexing set A consists of two points, $A = \{1, 2\}$. The product of the collection $\mathbf{X} = \{X_1, X_2\}$ thus consists of all functions $x : \{1,2\} \to X_1 \cup X_2$, with $x(1) \in X_1$ and $x(2) \in X_2$. Note that each such function x corresponds to an ordered pair $(x(1), x(2))$, and each ordered pair (a, b) with $a \in X_1$ and $b \in X_2$ defines a function $x : \{1, 2\} \to X_1 \cup X_2$ by letting $x(1) = a$ and $x(2) = b$. In other words, the product of two sets X_1 and X_2 is precisely the set of all ordered pairs (a, b) with $a \in X_1$ and $b \in X_2$, and our new definition of a product is consistent with our earlier definition of the product of two spaces.

(b) Now suppose that $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$, so that the indexing set $A = \{1, 2, \ldots, n\}$. The set of all functions $x : A \to \cup\{X_1, X_2, \ldots, X_n\}$ such that $x(i) \in X_i$ is precisely the set of all n-tuples, or finite sequences, $(x_1, x_2, \ldots, x_n)$, with $x_i \in X_i$.

(c) If the collection of sets $\mathbf{X} = \{X_n : n \in Z_+\}$ is countably infinite, then the product of $\mathbf{X}$ is just the set of all sequences (x_n), where $x_n \in X_n$.

In case $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$, it is common to denote $\prod \mathbf{X}$ by $X_1 \times X_2 \times \ldots \times X_n$.

Proposition 6.2

If $X_a \subset Y_a$ for each $a \in A$, then $\prod\{X_a : a \in A\} \subset \prod\{Y_a : a \in A\}$, and for each $a \in A$, $x(a) \in X_a \subset Y_a$.

Proof: Clearly, any function $x : A \to \cup\{X_a : a \in A\}$ is also a function from A into $\cup\{Y_a : a \in A\}$.

Definition

Let $\mathbf{X} = \{X_a : a \in A\}$ be a collection of sets. For each $a \in A$, the function $\pi_a : \prod \mathbf{X} \to X_a$ defined by $\pi_a(x) = x(a)$ is called the **projection map** of $\prod \mathbf{X}$ onto X_a.

Example 6.3

Let $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ and let $x = (x_1, x_2, \ldots, x_n) \in \prod \mathbf{X}$. Then for each j, $1 \leq j \leq n$, the image of x under the projection map π_j is $\pi_j(x) = \pi_j((x_1, x_2, \ldots, x_n)) = x_j$.

Definition

Let $\{(X_a, \mathbf{T}_a) : a \in A\}$ be a collection of topological spaces. If $\mathbf{X} = \{X_a : a \in A\}$, the **product topology** for $\prod \mathbf{X}$ is the weak topology by the collection $\mathbf{F} = \{\pi_a : a \in A\}$ of all the projection maps.

Example 6.4

Let $\mathbf{R}$ be the space of real numbers with the usual topology. Then the product $\mathbf{R} \times \mathbf{R}$ is the plane. For each open interval $(a, b) \subset \mathbf{R}$, we have $\pi_1^{-1}((a, b)) = \{(x, y) \in \mathbf{R} \times \mathbf{R} : a < x < b\}$ and $\pi_2^{-1}((a, b)) = \{(x, y) \in \mathbf{R} \times \mathbf{R} : a < y < b\}$. The product topology is thus the topology generated by the collection of all horizontal and vertical "open strips" in the plane. This is, of course, the usual topology for the plane (see Example 3.22(b)).

Proposition 6.5

Let $\mathbf{X} = \{X_a : a \in A\}$ be a collection of sets, and let $U \subset X_b$ be a subset of some $X_b \in \mathbf{X}$. Then for the projection map π_b, we have $\pi_b^{-1}(U) = \prod\{S_a : a \in A\}$, where $S_b = U$ and $S_a = X_a$ for $a \neq b$.

Proof: This is almost obvious. If $x \in \pi_b^{-1}(U)$, then $\pi_b(x) = x(b) \in U = S_b$, and so $x \in \prod\{S_a : a \in A\}$. On the other hand, if $x \in \prod\{S_a : a \in A\}$, then $\pi_b(x) = x(b) \in S_b = U$, so $x \in \pi_b^{-1}(U)$.

Theorem 6.6

If $\{(X_a, \mathbf{T}_a) : a \in A\}$ is a collection of topological spaces, then the collection of all sets of the form $\prod\{U_a : a \in A\}$, where $U_a \in \mathbf{T}_a$ for every $a \in A$, and $U_a = X_a$ for all but a finite number of the elements $a \in A$, is a base for the product topology on $\prod\{X_a : a \in A\}$.

Proof: We know the collection of sets of the form

$$\pi_{b(1)}^{-1}(U_{b(1)}) \cap \pi_{b(2)}^{-1}(U_{b(2)}) \cap \ldots \cap \pi_{b(k)}^{-1}(U_{b(k)}),$$

where each $U_{b(j)}$ is an open subset of $X_{b(j)}$, is a base for the weak topology by the collection $\{\pi_a : a \in A\}$ of all projection maps. This topology is, of course, the product topology. But from Proposition 6.5 we see that

$$\pi_{b(1)}^{-1}(U_{b(1)}) \cap \pi_{b(2)}^{-1}(U_{b(2)}) \cap \ldots \cap \pi_{b(k)}^{-1}(U_{b(k)})$$

is precisely the set $\prod\{U_a : a \in A\}$, where $U_a = U_{b(j)}$ for $a = b(j)$ and $U_a = X_a$ for all $a \notin \{b(j) : j = 1, 2, \ldots, k\}$.

We shall sometimes write $\prod\{U_a : a \in A\}$, where $U_a = U_{b(j)}$ for $a = b(j)$ and $U_a = X_a$ for all $a \notin \{b(j) : j = 1, 2, \ldots, k\}$, as

$$U_{b(1)} \times U_{b(2)} \times \ldots \times U_{b(k)} \times \prod\{X_a : a \in A, a \notin \{b(i) : i = 1, 2, \ldots, k\}\}.$$

Corollary 6.7

If $Z = X_1 \times X_2 \times \ldots \times X_n$ is the product of a finite collection of topological spaces, the collection of all products $U_1 \times U_2 \times \ldots \times U_n$, where each U_i is an open subset of X_i, is a base for the product topology on Z.

Theorem 6.8

Let $Z = \prod\{X_a : a \in A\}$ be the product of a collection of topological spaces. Then each projection map $\pi_a : Z \to X_a$ is a continuous and open function from Z onto X.

Proof: The continuity of each π_a follows from Theorem 3.16, which says that the weak topology by a collection of functions on a set is the smallest topology on the set that makes every member of the collection continuous. That each π_a is onto follows at once from the definition of π_a.

To show π_b is open, let U be an open subset of Z. We shall show that $\pi_b(U)$ is a neighborhood of each of its points. To that end, let $y \in \pi_b(U)$. Then there is an $x \in U$ so that $y = \pi_b(x)$. The set U is open, so there is a basic element

$$B = U_{b(1)} \times U_{b(2)} \times \ldots \times U_{b(k)} \times \prod\{X_a : a \in A, a \notin \{b(i) : i = 1, 2, \ldots, k\}\},$$

with $x \in B \subset U$. But $\pi_b(B) = U_{b(j)}$ for some j if $b \in \{b(i) : i = 1, 2, \ldots, k\}$; otherwise, $\pi_b(B) = X_b$. In either case, $\pi_b(B)$ is open; and

$$y = \pi_b(x) \in \pi_b(B) \subset \pi_b(U),$$

thus making $\pi_b(U)$ open.

Examples 6.9

(a) Let X be the set of real numbers with the usual topology, and let Y be the set of real numbers with the discrete topology. Then the collection of all "horizontal open intervals" $I = (a, b) \times \{c\} = \{(x, c) \in X \times Y : a < x < b\}$ is a base for the product topology on $X \times Y$.

(b) Let X be the set of real numbers with the usual topology, and let Y be the Sorgenfrey line (that is, Y is the set of real numbers with the topology having the set of all half-open intervals $[a, b)$ as a base). Then the collection of all rectangles $(a, b) \times [c, d) = \{(x, y) \in X \times Y : a < x < b, \text{ and } c \leq y < d\}$ is a base for the product topology on $X \times Y$.

Exercises

1. Let $Z = X \times Y$ be the product space described in Example 6.9(a). Find the interior and closure of each of the following subsets of Z.

   (a) $A = \{(x, 0) \in Z : 0 \leq x < 1\}$

   (b) $B = \{(0, y) \in Z : 0 \leq y < 1\}$

   (c) $C = \{(x, y) \in Z : 0 \leq x < 1; 0 \leq y < 1\}$

2. Let $Z = X \times Y$ be the product space described in Example 6.9(b). Find the interior and closure of each of the following subsets of Z.

   (a) $A = \{(x, y) \in Z : 0 < x < 1; 0 < y < 1\}$

   (b) $B = \{(x, y) \in Z : 0 \leq x \leq 1; 0 \leq y \leq 1\}$

3. Let $Z = \prod\{X_a : a \in A\}$ be the product of a collection of topological spaces. Suppose that Z is compact. Prove that each space X_a is compact.

4. Let (X, d) be a pseudometric space. Prove that the pseudometric $d : X \times X \to \mathbf{R}$, from the product space into the space of reals with the usual topology, is a continuous function.

5. Let $Z = \prod\{X_a : a \in A\}$ be the product of a collection of topological spaces. Prove or give a counterexample:

   (a) If each U_a is an open subset of X_a, then the product $\prod\{U_a : a \in A\}$ is an open subset of Z.

   (b) If each F_a is a closed subset of X_a, then the product $\prod\{F_a : a \in A\}$ is a closed subset of Z.

6. Let U be a nonempty open subset of the product space $Z = \prod\{X_a : a \in A\}$. Prove that $\pi_a(U) = X_a$ for all but a finite number of the projection maps π_a.

7. Let $\mathbf{X} = \{X_a : a \in A\}$ be a collection of discrete spaces, each containing more than one point. Prove that the product space $\prod \mathbf{X}$ is discrete if and only if $\mathbf{X}$ is finite.

8. Let X and Y be topological spaces, and suppose $A \subset X$ and $B \subset Y$. Prove that in the product space $X \times Y$, the derived set $(A \times B)' = (A' \times \text{cl } B) \cup (\text{cl } A \times B')$.

9. Let X and Y be topological spaces, and suppose $A \subset X$ and $B \subset Y$. Prove that in the product space $X \times Y$, the interior $\text{int}(A \times B) = (\text{int } A) \times (\text{int } B)$.

6.2 Continuous Functions and Slices in Product Spaces

The next two theorems are useful in working with product spaces. The first is a direct application of Theorem 3.19, about the continuity of a function into a space having the weak topology by a collection of functions.

Theorem 6.10

Suppose $f : X \to Z = \prod\{X_a : a \in A\}$ is a function from a topological space X into a product of topological spaces Z. Then f is continuous if and only if each composition $\pi_a \circ f : X \to X_a$ is continuous.

The next result shows that the product of a collection includes many subspaces homeomorphic with each member of the product. This is useful for, among other things, showing that each space in a collection inherits certain properties of the product of the collection.

CHAPTER 10

Quotient Spaces

In this chapter, we begin the study of topological spaces in which the points of the space are themselves subsets of a topological space.

10.1 The Strong Topology and the Quotient Topology

Let $f : X \to Y$ be a function from one set into another. In case Y has a topology, it is easy to endow X with a topology $\mathbf{T}$ with respect to which f is continuous: Simply let $\mathbf{T}$ be the discrete topology, in which every subset of X is open. A more interesting and useful topology is, as we have seen, the smallest topology for X that makes f continuous. This is the weak topology by f. We turn now to a kind of dual problem in which X has a topology, and we wish to endow Y with a topology $\mathbf{S}$ so as to have f be continuous. Again, this is simple: Let $\mathbf{S}$ be the trivial topology, in which Y and $\emptyset$ are the only open sets. A more exciting and useful topology here is the largest topology for Y that makes f continuous. This is the strong topology by f.

Proposition 10.1

Let $f : X \to Y$ be a function from a topological space $(X, \mathbf{T})$ into a set Y. Then the collection $\mathbf{S} = \{U \subset Y : f^{-1}(U) \in \mathbf{T}\}$ is a topology for Y.

Proof: This proposition follows easily from the facts that $f^{-1}(\cup\mathbf{C}) = \cup f^{-1}(\mathbf{C})$ and that $f^{-1}(\cap\mathbf{C}) = \cap f^{-1}(\mathbf{C})$ for any collection $\mathbf{C}$ of subsets of Y.

Definition

The topology $\mathbf{S}$ in the previous proposition is the **strong topology by** f. In case $Y = f(X)$, the strong topology by f is called the **quotient topology by** f.

Proposition 10.2

Suppose $f : X \to Y$ is a function from a topological space $(X, \mathbf{T})$ into a topological space $(Y, \mathbf{V})$, and let $\mathbf{S}$ be the strong topology by f. Then $f : (X, \mathbf{T}) \to (Y, \mathbf{V})$ is continuous if and only if $\mathbf{V} \subset \mathbf{S}$.

Proof: First, suppose f is continuous, and let $U \in \mathbf{V}$. Then $f^{-1}(U) \in \mathbf{T}$ because f is continuous, and so $U \in \mathbf{S}$. Thus $\mathbf{V} \subset \mathbf{S}$.

Now suppose $\mathbf{V} \subset \mathbf{S}$ and let $U \in \mathbf{V}$. Then $U \in \mathbf{S}$, and so $f^{-1}(U) \in \mathbf{T}$, which makes f continuous.

This proposition shows that for a function $f : X \to Y$ from a topological space X into a set Y, the strong topology for Y is the largest topology for which f is continuous.

Examples 10.3

(a) Let $X = \mathbf{R}$ be the set of real numbers with the usual topology, and let $Y = \{y \in \mathbf{R} : y \geq 0\}$. Let $f : X \to Y$ be defined by setting $f(x) = x^2$. Then the strong topology by f is the usual topology for Y. To see this, let $U \subset Y$ be such that $f^{-1}(U)$ is open. Then $f^{-1}(U) = \cup\mathbf{I}$, where $\mathbf{I}$ is a collection of open intervals. It is clear that for any interval $(a, b) \in I$, then

$$f((a,b)) = \begin{cases} (a^2, b^2) & \text{if } 0 \leq a < b; \\ (b^2, a^2) & \text{if } a < b \leq 0; \text{ and} \\ (0, \max\{a^2, b^2\}) & \text{if } a < 0 < b. \end{cases}$$

In any case, $f((a,b))$ is a member of the usual topology for Y. Thus

$$U = f(f^{-1}(U)) = f(\cap\mathbf{I}) = \cup f(\mathbf{I}) = \cup\{f((a,b)) : (a,b) \in \mathbf{I}\},$$

which, being a union of open sets, is open. This shows that the strong topology is included in the usual topology, and the usual topology is included in the strong topology because f is continuous with respect to the usual topology for Y.

(b) Let $f : X \to Z$, where $Z = \mathbf{R}$ is the set of real numbers and where X and f are as in the previous example. Then the strong topology by f is not the usual topology for Z. To see this, let $U = \{-1\}$. Then U is not a member of the usual topology but is a member of the strong topology because $f^{-1}(U) = \emptyset$.

Suppose $f : X \to Y$ is a function from a topological space X into a set Y, and suppose Y has the strong topology by f. Then any $A \subset Y - f(X)$ is open, because its inverse image is empty. Thus the strong topology by f in case $Y - f(X)$ is not empty is of little interest, and we shall see no more of it in this book after this section. Our concern will be exclusively with the case

in which $Y = f(X)$; in this case the strong topology is called the quotient topology.

Proposition 10.4

Suppose $(X, \mathbf{T})$ and $(Y, \mathbf{V})$ are topological spaces, and suppose $f : X \to f(X) = Y$ is an open function. Then the quotient topology by f is included in $\mathbf{V}$.

Proof: Let $\mathbf{S}$ be the quotient topology by f, and let $U \in \mathbf{S}$. Then $f^{-1}(U) \in \mathbf{T}$, and so $U = f(f^{-1}(U)) \in \mathbf{V}$ because f is an open function. Thus $\mathbf{S} \subset \mathbf{V}$.

Proposition 10.5

Suppose $(X, \mathbf{T})$ and $(Y, \mathbf{V})$ are topological spaces, and suppose $f : X \to f(X) = Y$ is a closed map. Then the quotient topology by f is included in $\mathbf{V}$.

Proof: Let $\mathbf{S}$ be the quotient topology by f, and let $U \in \mathbf{S}$. Then $f^{-1}(Y - U) = X - f^{-1}(U)$ is closed because $f^{-1}(U) \in \mathbf{T}$. Thus

$$f(X - f^{-1}(U)) = f(f^{-1}(Y - U)) = Y - U$$

is closed in $(Y, \mathbf{V})$, which means $U \in \mathbf{V}$. This proves $\mathbf{S} \subset \mathbf{V}$.

Exercises

1. Suppose $f : X \to Y$ is a function from a discrete topological space into a set Y. Describe the strong topology by f.
2. Suppose $f : X \to Y$, and suppose X has the trivial topology. Describe the strong topology by f.
3. Suppose $f : X \to Y$ is a function from a topological space X into a set Y. Suppose Y has the strong topology by f. Prove that the subspace $Y - f(X)$ has the discrete topology.
4. Let $X = [0, 1)$ have the topology it inherits from the set of real numbers with the usual topology. Suppose that the subset of the plane $Y = \{(x, y) : x^2 + y^2 = 1\}$ has the quotient topology by the function $f : X \to Y$, where $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Find the closure and the interior of each of the sets $A = f((0, 1/4))$ and $B = f((3/4, 1))$.
5. Let $X = \mathbf{R} \times \mathbf{R}$ be the plane with the usual topology, and let $\pi_1 : X \to \mathbf{R}$ be the projection of X onto $\mathbf{R}$, the set of real numbers (that is, $\pi_1(x, y) = x$). Describe the quotient topology by π_1.
6. Let $f : X \to Y$ and $g : Y \to Z$, where X is a topological space, Y is a set with the strong topology by f, and Z is a set. For Z, let $\mathbf{S_g}$ be the strong

topology by g, and let $\mathbf{S_k}$ be the strong topology by $h = g \circ f$. Prove that $\mathbf{S_g} = \mathbf{S_k}$.

10.2 Quotient Maps

Definition

A continuous function $f : X \to f(X) = Y$ from one topological space onto another is called a **quotient map** if the topology of Y is the quotient topology by f.

Theorem 10.6

Every open or closed continuous function from one topological space onto another is a quotient map.

Proof: Let $f : X \to f(X) = Y$ be continuous and either open or closed. If $\mathbf{S}$ is the quotient topology by f, and if $\mathbf{V}$ is the topology of Y, then $\mathbf{V} \subset \mathbf{S}$ because f is continuous, and $\mathbf{S} \subset \mathbf{V}$ because f is an open or closed function. Thus $\mathbf{V} \subset \mathbf{S}$.

Theorem 10.7

A continuous $f : X \to f(X) = Y$ is a quotient map if and only if $F \subset Y$ is closed whenever $f^{-1}(F)$ is closed.

Proof: Since f is a quotient map if and only if U is open whenever $f^{-1}(U)$ is open, the theorem follows from the fact that $f^{-1}(Y - A) = X - f^{-1}(A)$ for every set $A \subset Y$.

Example 10.8

Let X and Y be subspaces of the plane with the usual topology defined as follows: $Y = \{(x, 0) : 0 \le x \le 1\}$, and

$$X = \{(x,1) : 0 < x \le 1\} \cup \{(x, 2) : 0 \le x \le 1/2\}.$$

Define $f : X \to Y$ by $f((x, y)) = (x, 0)$. Observe that f is continuous. To see that it is a quotient map, let $\mathbf{U}$ denote the usual topology for Y, and suppose $S \subset Y$ is a member of the quotient topology for Y. Then $V = f^{-1}(S)$ is open. Now let $p \in V$. Then p is an interior point of V, and if $p \ne (1/2, 2)$, then it is clear that $f(p)$ is a $\mathbf{U}$-interior point of $S = f(V)$. If, on the other hand, $p = (1/2, 2)$, then note that it must also happen that $q = (1/2, 1)$ is a member of V. Thus $f(q) = f(p)$ is a $\mathbf{U}$-interior point of S. Every point of S is a $\mathbf{U}$-interior point of S, and so $S \in \mathbf{U}$. This proves that $\mathbf{U}$ is the quotient topology by f.

Note that $F = \{(x, 1) : 0 < x \le 1\}$ is a closed subset of X and $f(F)$ is not a closed subset of Y. Also, $G = \{(x, 2) : 0 \le x \le 1/2\}$ is an open subset of X, and $f(G)$ is not an open subset of Y. The function f is an example of a quotient map that is neither open nor closed.

Theorem 10.9

Let X and Y be topological spaces, and suppose $f : X \to f(X) = Y$ is a quotient map. If X is locally connected, then Y is locally connected.

Proof: We shall show that Y is locally connected by proving that for each open $U \subset Y$, every component of U is open (Theorem 4.20). Let C be a component of the open set $U \subset Y$. Since the quotient map f is continuous, $f^{-1}(U)$ is open. Let

$$\mathbf{K} = \{K \subset X : K \text{ is a component of } f^{-1}(U), \text{ and } f(K) \cap C \ne \emptyset\}.$$

If $f(x) \in C$, then x is in $f^{-1}(U)$, and hence in some component K of $f^{-1}(U)$ such that $f(K) \cap C \ne \emptyset$. In other words, $x \in \cup\mathbf{K}$, so that we have $f^{-1}(C) \subset \cup\mathbf{K}$. Now for each $K \in \mathbf{K}$, we have $f(K) \subset C$, because $f(K)$ is a connected subset of U and $f(K) \cap C \ne \emptyset$. Thus $K \subset f^{-1}(C)$ for each $K \in \mathbf{K}$. This shows that $\cup\mathbf{K} \subset f^{-1}(C)$, and so $f^{-1}(C) = \cup\mathbf{K}$. Each K is a component of the open set $f^{-1}(U)$ and hence is open, because X is locally connected. But this implies that $f^{-1}(C)$ is open, and so C is open because Y has the quotient topology by f.

Corollary 10.10

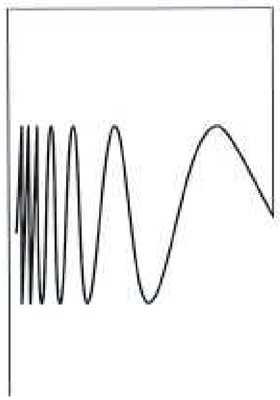
If $f : X \to f(X) = Y$ is a continuous open or closed function and X is locally connected, then Y is locally connected.

Example 10.11

Let $X = \{t \in \mathbf{R} : 0 < t \le 4\}$ with the usual topology, and let Z be the plane with the usual topology. Define the function $f : X \to Z$ as follows:

$$f(t) = \begin{cases} (t, \sin(\pi/t)) & \text{for } 0 < t \le 1; \\ (1, 2t - 2) & \text{for } 1 < t \le 2; \\ (3 - t, 2) & \text{for } 2 < t \le 3; \\ (0, 14 - 4t) & \text{for } 3 < t \le 4. \end{cases}$$

It is straightforward to verify that f is continuous. If $Y = f(X)$, then all sufficiently small neighborhoods (in Y) of any point in the subset $S = \{(0, y) : -1 \le y \le 1\}$ are disconnected. Thus $Y = f(X)$ is not locally connected. This example shows that a continuous image of a locally connected space is not necessarily locally connected, and the requirement in Theorem 10.9 that f be a quotient map is not superfluous.



$Y = f(X)$

Figure 10.1 Example 10.11

Theorem 10.12

Let $f : X \to f(X) = Y$ be a quotient map, and let $A \subset Y$ be either open or closed. Then $f \mid f^{-1}(A)$ is a quotient map.

Proof: Observe that $f \mid f^{-1}(A)$ is a continuous function from $f^{-1}(A)$ onto A.

Suppose that A is open, and let $U \subset A$ be such that $f^{-1}(U)$ is an open subset of the subspace $f^{-1}(A)$. Then $f^{-1}(U)$ is an open set in X because $f^{-1}(A)$ is open. Thus U is an open subset of Y, which makes $U = U \cap A$ an open subset of the subspace A. This shows that the quotient topology by $f \mid f^{-1}(A)$ is included in the subspace topology for A, and so the two topologies coincide.

The proof for the case in which A is closed is very similar and is omitted.

Exercises

7. Let $f : X \to f(X) = Y$ be a continuous function from a compact space onto a Hausdorff space Y. Prove that f is a quotient map.
8. Prove that the composition of two quotient maps is a quotient map.
9. Let X, Y, and Z be topological spaces, and suppose $f : X \to f(X) = Y$ is a quotient map. Prove that a function $g : Y \to Z$ is continuous if and only if $g \circ f : X \to Z$ is continuous.
10. Prove or give a counterexample: If $f : X \to f(X) = Y$ is a quotient map and $A \subset X$ is closed, then $f \mid A$ is a quotient map.

11. Suppose $f : X \to f(X) = Y$ is one-to-one. Prove f is a homeomorphism if and only if it is a quotient map.
12. Let $f : X \to f(X) = Y$ be a quotient map, and suppose that the set $f^{-1}(y)$ is connected for every $y \in Y$. Prove that if U is an open connected subset of Y, then $f^{-1}(U)$ is connected.

10.3 Quotient Spaces

Definition

Suppose $\mathbf{D}$ is a collection of nonempty subsets of a set X. If the elements of $\mathbf{D}$ are pairwise disjoint, and if $X = \cup\mathbf{D}$, then $\mathbf{D}$ is called a **decomposition** of X. The function $p : X \to \mathbf{D}$ defined by $p(x) = D$, where $D \in \mathbf{D}$ is the unique element of $\mathbf{D}$ such that $x \in D$ is called the **natural map.**

If R is an equivalence relation on a set X, then the collection of all equivalence classes is a decomposition of X, called the **decomposition induced by R**. Conversely, if $\mathbf{D}$ is a decomposition of a set X, then the relation

$$R = \{(x, y) \in X \times X : \text{There is a } D \in \mathbf{D} \text{ with } x \text{ and } y \text{ in } D\}$$

is an equivalence relation, and the collection of all equivalence classes is precisely $\mathbf{D}$. In this case R is said to be the **equivalence relation induced by D**. If R is an equivalence relation, the collection of all equivalence classes is denoted $\mathbf{X/R}$ and is called the **quotient set**.

Definition

Suppose X is a topological space, and suppose R is an equivalence relation on X. The quotient set X/R with the quotient topology by the natural map p is called the **quotient space** determined by X and R. A quotient space is sometimes called a **decomposition space** or an **identification space.**

The next important theorem many times provides a nice way to recognize quotient spaces. First we give yet another definition.

Definition

Let $f : X \to f(X) = Y$ be a function from one topological space onto another, and let

$$R = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}.$$

Clearly, R is an equivalence relation on X. The quotient space X/R is called the **point inverse decomposition by f** and is denoted by X/f.

Theorem 10.13

Suppose $f : X \to f(X) = Y$ is continuous, and let $p : X \to X/f$ denote the natural map onto the point inverse decomposition by f. Then the function $h : X/f \to Y$ given by $h(p(x)) = f(x)$ is a homeomorphism if and only if f is a quotient map.

Proof: First, note that the function h is well-defined, is one-to-one, and is such that $h(X/f) = Y$. We begin by assuming f is a quotient map. Suppose U is an open subset of Y; then $f^{-1}(U) = p^{-1}(h^{-1}(U))$, and $f^{-1}(U)$ is open. Thus $h^{-1}(U)$ is a member of the quotient topology by p, showing that h is continuous. To prove that h is an open map, let V be an open subset of X/f. Then $p^{-1}(V) = f^{-1}(h(V))$, which is open because p is continuous. Thus $h(V)$ is open since Y has the quotient topology by f. We see that h is continuous, open, and one-to-one. In other words, h is homeomorphism.

Now suppose h is a homeomorphism. Then $f = h \circ p$ is a continuous function from X onto Y. Let $U \subset Y$ be such that $f^{-1}(U)$ is open. Then $f^{-1}(U) = p^{-1}(h^{-1}(U))$, and so $h^{-1}(U)$ is an open subset of the quotient space X/f. Finally, $U = h(h^{-1}(U))$ is open because h is a homeomorphism. This proves Y has the quotient topology by f.

Examples 10.14

(a) Let $X = \mathbf{R} \times \mathbf{R}$ be the plane with the usual topology. For each real number r, let $L_r = \{(r, y) \in X : y \in \mathbf{R}\}$. Each L_r is thus the vertical line through the point $(r, 0)$. The collection $\mathbf{D} = \{L_r : r \in \mathbf{R}\}$ is a decomposition of X. If R denotes the equivalence relation induced by $\mathbf{D}$, the quotient space X/R is the space of reals with the usual topology. This is a consequence of Theorem 10.13: The decomposition $\mathbf{D}$ is the point inverse decomposition by the projection π_1, which is a quotient map because it is continuous and open.

(b) Let $X = [0, 1]$ have the usual topology. Define the relation R by

$$R = \{(x, x) : x \in X\} \cup \{(0, 1)\}.$$

The quotient space X/R is (homeomorphic with) the circle with the usual topology it inherits as a subspace of the plane. Again we appeal to Theorem 10.13. The function $f : X \to C$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is continuous. It is closed because it is a continuous function from a compact space onto a Hausdorff space (Theorem 5.15). Now f is a quotient map, and the point inverse decomposition by f is the quotient space X/R.

(c) Let X be the square $[0, 1] \times [0, 1]$ with the topology it inherits from the plane with the usual topology. Define a decomposition, or equivalence relation R, on X by listing the equivalence classes $p((x, y))$, where p is the natural map of X onto X/R:

$$p((x, y)) = \{(x, y)\} \text{ for } 0 \le x \le 1 \text{ and } 0 < y < 1;$$
$$p((x, 0)) = p((x, 1)) = \{(x, 0), (x, 1)\} \text{ for } 0 \le x \le 1.$$

The quotient space X/R is (homeomorphic with) the subspace Y of $\mathbf{R}^3$ defined by

$$Y = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 = 1, 0 \le x_3 \le 1\}.$$

To see this, observe that the quotient space X/R is the point inverse decomposition by f, where $f : X \to Y$ is given by

$$f((x, y)) = ((\cos(2\pi y), \sin(2\pi y)), x), \text{ for all } (x, y) \in X.$$

Let us see what properties of a topological space X are passed on to a quotient space X/R. Since the natural map is continuous, it is clear that X/R is compact or connected whenever X is compact or connected. The quotient space X/R is also locally connected whenever X is locally connected because the natural map is a quotient map. There is not much beyond this that can be said without making further assumptions about the equivalence relation.

Example 10.15

Let X be the closed interval [0,1] with the usual topology. Let $\mathbf{D} = \{D_1, D_2\}$, where $D_1 = [0, 1/2]$ and $D_2 = (1/2, 1]$. Then the quotient topology for $\mathbf{D}$ consists of the three sets: $\mathbf{D}$, $\emptyset$, and $\{D_2\}$. The quotient space is clearly not Hausdorff, although X is.

Theorem 10.16

Suppose X is a Hausdorff space and R is an equivalence relation on X such that every member $\mathbf{D}$, the decomposition induced by R, is compact. Suppose further that the natural map is a closed function. Then the quotient space X/R is a Hausdorff space.

Proof: Let $p : X \to X/R$ denote the natural map. Suppose w and z are distinct elements of X/R. Then $p^{-1}(w)$ and $p^{-1}(z)$ are disjoint compact subsets of X. Since X is Hausdorff, there are disjoint open sets U_w and U_z so that $p^{-1}(w) \subset U_w$ and $p^{-1}(z) \subset U_z$ (Theorem 5.14). Now since p is a closed function, there is a neighborhood V_w of w such that $p^{-1}(V_w) \subset U_w$, and there is a neighborhood V_z of z such that $p^{-1}(V_z) \subset U_z$ (Theorem 3.11). Then V_w and V_z are disjoint neighborhoods of w and z, respectively. This proves that X/R is Hausdorff.

Exercises

13. Let X be the subspace of the plane with the usual topology given by $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$. Define the equivalence relation R by listing the elements $p((x_1, x_2))$ in the decomposition induced by R as follows:

$$p((x_1, x_2)) = \{(x_1, x_2)\} \text{ for } x_1^2 + x_2^2 < 1; \text{ and}$$
$$p((x_1, x_2)) = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\} \text{ for } x_1^2 + x_2^2 = 1.$$

Describe the quotient X/R. (It is a familiar space.)

14. Define the equivalence relation R on the space of real numbers with the usual topology by listing the elements $p(x)$ of the decomposition induced by R:

$$p(x) = \{x\} \text{ for } |x| > 1; \text{ and}$$
$$p(x) = [-1, 1] \text{ for } |x| \leq 1.$$

Describe the quotient space X/R.

15. Define the equivalence relation R on Euclidean 3-space $\mathbf{R}^3$ as follows:

$$R = \{((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbf{R}^3 \times \mathbf{R}^3 : x_1 = y_1\}.$$

Describe the quotient space X/R.

16. Let R be the equivalence relation defined on $\mathbf{R}$, the space of real numbers with the usual topology, by

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : (x - y)/3 \text{ is an integer}\}.$$

Describe the quotient space X/R.

17. Let Z be a compact metric space. Prove there is an equivalence relation R on the Cantor space C so that the quotient space C/R is (homeomorphic with) Z.

18. Prove there is an equivalence relation R on the interval of real numbers $[0,1]$ with the usual topology so that the quotient space X/R is (homeomorphic with) the product space $[0, 1] \times [0, 1]$.

10.4 The Metric Identification

With every pseudometric space is associated a metric space that reflects much of the topology of the original pseudometric space.

Proposition 10.17

Let (X, d) be a pseudometric space, and let $R \subset X \times X$ be given by $R = \{(x, y) : d(x, y) = 0\}$. Then R is an equivalence relation on X; and if p is the natural map X onto the quotient set X/R, then the function $d^* : X/R \times X/R \to \mathbf{R}$ defined by $d^*(p(x), p(y)) = d(x, y)$ is a metric on the quotient set.

Proof: The verification that R is an equivalence relation is straightfoward and is omitted. To see that d^* is well-defined, suppose $p(x) = p(u)$, and $p(y) = p(v)$. We need to show that $d^*(p(x), p(y)) = d(x, y) = d(u, v) = d^*(p(u), p(v))$. Now,

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y), \text{ and}$$
$$d(u, v) \leq d(u, x) + d(x, y) + d(y, v);$$

and so $d(x, y) \leq d(u, v) \leq d(x, y)$, since $d(x, u) = 0$ and $d(y, v) = 0$.

It is clear that d^* is a pseudometric. To see that it is a metric, suppose $p(x) \neq p(y)$. This means that $(x, y) \notin R$; thus $d^*(p(x), p(y)) = d(x, y) > 0$.

Definition

Let (X, d) be a pseudometric space, and let R and d^* be as in Proposition 10.17. The metric space $(X/R, d^*)$ is called the **metric identification** of (X, d).

The following proposition follows at once from the definition of the metric d^* for the metric identification.

Proposition 10.18

If (X, d) is a pseudometric space, then the natural map $p : X \to X/R$ is an isometry from X onto X/R.

Theorem 10.19

Let $(X/R, d^*)$ be the metric identification of the pseudometric space (X, d). Then the topology for the quotient space X/R is the topology generated by the metric d^*.

Proof: We prove that the collection of all d^* cells is a base for the quotient topology. This follows almost directly from the fact that for any d-cell $C(x; r)$, we have $p(C(x; r)) = C(p(x); r)$ because the natural map p is an isometry. First, let U be a member of the quotient topology, and let $p(x) \in U$. Then $x \in p^{-1}(U)$, which is open and so there is a d-cell $C(x; r) \subset p^{-1}(U)$, and so $p(C(x; r)) = C(p(x); r) \subset U$. This shows that every member of the quotient topology is a union of d^*-cells. Each d^*-cell $C(p(x); r)$ is a member of the quotient topology because $p^{-1}(C(p(x); r)) = C(x; r)$ is an open set in X. This proves that the collection of all d^*-cells is a base for the quotient topology.