Step	Number of new branches added	Length of each new branch (inches)	Height of tree (inches)	Total length of new branches (inches)	Total length of all branches (inches)
1	1	2	2	2	2
2	3	1	3	3	5
3					
4					
5					
6					
7					

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◆

Two types of sequences are particularly important: arithmetic sequences and geometric sequences.

Arithmetic Sequences. A sequence is an **arithmetic sequence** if the next term in the sequence is found by adding a fixed number d to the current term. Hence, $\{x_n\}_{n\geq 1}$ is an arithmetic sequence provided $x_{n+1} = x_n + d$ for $n \geq 1$. Both x_1 (the first term in the sequence) and d are given. Why do you think d is called the **common difference**?

Classroom Discussion 1.1.1: Direct Formula for Arithmetic Sequences

1. Return to the sequences in Classroom Connection 1.1.2 and find a recursive relation for each sequence. Determine which ones are arithmetic. Explain.
2. John asked Michelle, "Is there a way to compute any term in an arithmetic sequence if you know the first term and the common difference?" Michelle answered "You can do that by adding to the first term a suitable multiple of the common difference." Do you agree with Michelle?
3. If the sequence $\{x_n\}_{n\geq 1}$ is an arithmetic sequence with common difference d, explain how to obtain the formula

$$x_n = x_1 + (n - 1)\,d \text{ for each } n \in \mathbb{N}.$$

4. What does this formula give for each arithmetic sequence from Classroom Connection 1.1.2? ◆

Geometric Sequences. A sequence is a **geometric sequence** if the next term in the sequence is found by multiplying the current term by a fixed number r. Thus, $\{x_n\}_{n\geq 1}$ is a geometric sequence provided $x_{n+1} = x_n \cdot r$ for $n \geq 1$. Both x_1 and r are given. Of course, if $x_1 = 0$ or $r = 0$, then $x_n = 0$ for every n. Thus, it is typically assumed that $x_1 \neq 0$ and $r \neq 0$, in which case $x_n \neq 0$ for every n. Why do you think r is called the **common ratio**?

Classroom Discussion 1.1.2: Direct Formula for Geometric Sequences

1. Return to the sequences in Classroom Connection 1.1.2 and determine which ones are geometric. Explain.
2. Alice asked Bob, "Is there a way to compute any term in a geometric sequence if you know the starting term and the common ratio?" Bob answered "You can do that by multiplying the starting term by the common ratio raised to a suitable power." Do you agree with Bob?
3. If $\{y_n\}_{n\geq 1}$ is a geometric sequence with common ratio r, explain how to obtain the formula

$$y_n = y_1 \cdot r^{n-1} \text{ for each } n \in \mathbb{N}.$$

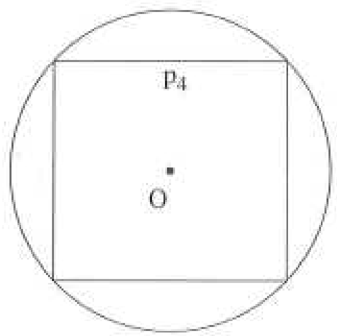
4. What does this formula give for each geometric sequence from Classroom Connection 1.1.2? ◆

Limits of Sequences. When analyzing sequences, we are often interested in understanding what happens to the value of x_n as n gets larger and larger. As you can imagine, the possibilities are numerous. However, sequences whose general term x_n gets closer and closer to some fixed number as n gets larger and larger turn out to be particularly important. In order to better understand this idea, work through the tasks in Classroom Discussion 1.1.3.

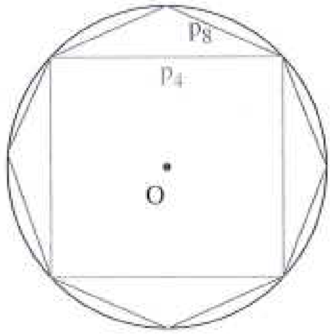
Classroom Discussion 1.1.3: The Circumference of a Circle and Perimeters of Inscribed Polygons

1. Consider a circle of radius r. What is the standard formula in terms of π and r for its circumference L?
2. Take $r = 4$ inches and use the formula you determined from Problem 1 to find an approximate value for L rounded to four decimal places.
3. Now analyze the following construction:

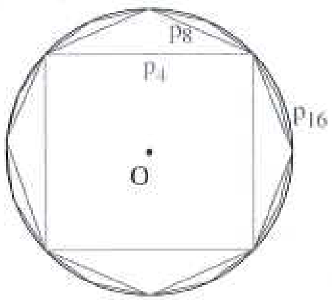
   a. Draw a circle C of radius $r = 4$ and center O, and draw a square inscribed in C. Denote the perimeter of this square by p_4.



b. Inscribe an octagon in the circle C whose vertices contain the square's vertices. Denote the octagon's perimeter by p_8.

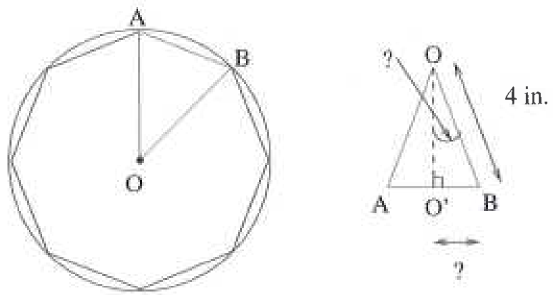


c. Continue this procedure of inscribing regular polygons in the circle C. At each step, the polygon you construct should have twice as many sides and should include all the vertices of the previous polygon. If a polygon has n vertices, its perimeter is denoted by p_n. The following figure shows the result of the first three steps of the procedure.



Describe what you expect to happen to the inscribed polygons as the number of sides increases.

4. Consider the general case of a regular n-gon inscribed in the circle C. Write a formula for its perimeter p_n. Your answer should depend on n. Hint: Look at the following figure.



5. Use your calculator to generate a table with columns n and p_n for $n = 3, 4, 5, \ldots, 102$. Create a scatter plot with the first forty values in the table. On the same graph, draw the line $y = L$, where L is the circumference of the circle you computed in Problem 2.

6. As the number of sides n increases, use the table and the scatter plot to describe what happens to the values of the perimeters of the regular n-gons inscribed in the circle C. How do these values compare to the circle's circumference? Is your conclusion in line with the prediction you made in Problem 3?

7. What is the best approximation for the circumference of the circle you get from the table in Problem 5? What would you suggest doing to get an even better approximation? ◆

Convergent Sequences

Consider what would happen if you "forever" continued constructing inscribed polygons with more and more sides. This nonstop process of letting n become larger and larger is denoted by $n \to \infty$ (n goes to infinity). The values p_n of the perimeters of the regular n-gons inscribed in the circle C would approach the circumference L of the circle C. The larger n is, the better p_n approximates L, the circle's circumference. Think of how the scatter plot looked in Problem 6. You can make p_n as close to L as you want by choosing n sufficiently large. We say that *the limit of the sequence* $\{p_n\}_n$ *as n goes to infinity is* L, and we write $\lim_{n\to\infty} p_n = L$. Alternatively, we say that the sequence $\{p_n\}_n$ **converges** to L.

Important Observation

Return to the scatter plot from Problem 6, that is, the plot of points (n, p_n), $n = 3, 4, 5, \ldots, 40$. Imagine continuing to plot these points for all $n \geq 41$. Next, choose an open interval on the y-axis containing the value $y = L = 8\pi$. We want to know how many p_n's will remain outside this interval. To do so, imagine drawing the horizontal band containing all the points whose y coordinates are in the interval. Given our plot, at most a finite number of the points (n, p_n) will be outside the band. No matter how narrow the band you select, as long as it contains the line $y = 8\pi$, all the points (n, p_n) will be in the band starting from some value $n = N_0$. Thus, any open interval containing L will have all but a finite number of p_n's. This idea is formalized in the following definition.

Definition 1.1.1 A sequence $\{x_n\}_n$ is said to **converge** to a real number L provided any open interval centered at L contains all but finitely many terms of the sequence. In this case, L is called **the limit** of the sequence $\{x_n\}_n$, and we write

$$\lim_{n\to\infty} x_n = L.$$

A sequence that converges to a real number L is called **convergent**; a sequence that does not converge to any real number is called **divergent**.

Remarks

1. A sequence $\{x_n\}_n$ with the property that $x_n < x_{n+1}$ for all n is called **increasing**. For example, the previous sequence $\{p_n\}_n$ is increasing. If a

9. Study this pattern:

row 1: $\frac{1}{2} =$

row 2: $\frac{1}{2} + (\frac{1}{2})^2 =$

row 3: $\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 =$

row 4: $\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + (\frac{1}{2})^4 =$

a. Find the sum for each row.

b. Suppose the pattern continued. Write the expression that would be in row 5, and find its sum.

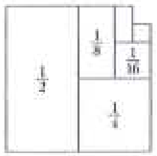
c. What would be the sum of the expression in row 10? In row 20?

d. Describe the pattern of sums in words and with an equation.

e. For which row does the sum first exceed 0.9?

f. As the row number increases, the sum gets closer and closer to what number?

g. Celeste claims that the pattern is related to the pattern of the areas of the ballots cut in Problem 4.1. She drew this picture to explain her thinking.



row 6 $= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6$

What relationship do you think Celeste has observed?

58 **Growing, Growing, Growing**

◆

In the examples discussed so far, you wanted to define a sum of the type $x_1 + x_2 + x_3 + \cdots$ for various choices of the terms $x_1, x_2, x_3, \ldots$. In each case, you have defined a new sequence $\{s_n\}_n$ by computing partial sums of the infinite sum. This is the same as looking at the sum of "more and more" x_i's. Since, in the end, you wanted to add all the x_i's, $i \geq 1$, you should look at the behavior of the partial sums s_n as $n \to \infty$. Thus, the following definition is natural.

Definition 1.2.1 An **infinite series** is an expression of the form

$$x_1 + x_2 + x_3 + \cdots, \tag{2}$$

where the numbers $x_1, x_2, \ldots,$ are the **terms** of the series. The sequence $\{s_n\}_n$ of **partial sums** is

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_n &= x_1 + x_2 + x_3 + \cdots + x_n = \sum_{i=1}^{n} x_i \quad \text{for each } n \geq 1. \end{aligned}$$

If the sequence $\{s_n\}_n$ converges to a real number L, that is, $\lim_{n\to\infty} s_n = L$, then the series (2) is called **convergent**, and L is called the **sum** of the series. In this case, we write $x_1 + x_2 + x_3 + \cdots = L$ or $\sum_{i=1}^{\infty} x_i = L$. Otherwise, we say that the series **diverges**.

Classroom Discussion 1.2.2: The Mysterious Series 1 − 1 + 1 − 1 + 1 − 1 + ⋯

Dan and John are trying to compute the infinite sum

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Dan writes

$$\underbrace{1 - 1}_{=0} + \underbrace{1 - 1}_{=0} + \underbrace{1 - 1}_{=0} + \cdots$$

and concludes that the sum of the series is zero. John writes

$$1 + \underbrace{(-1 + 1)}_{=0} + \underbrace{(-1 + 1)}_{=0} + \underbrace{(-1 + 1)}_{=0} + \cdots$$

and concludes that the sum must be equal to 1. Who do you think is right? Explain. ◆

Geometric Series. In the following discussion, we restrict our analysis to a particular type of series whose terms form a geometric sequence of ratio r. Such a series is called **geometric series of ratio** r. It can be written as $a + ar + ar^2 + \cdots$, where a is the first term of the series. The issue of the convergence of geometric series is well understood. Use the following outline to derive the main results for geometric series.

P4 $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ whenever $\lim_{x\to a} g(x) \neq 0$.

P5 $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ with the additional assumption that $\lim_{x\to a} f(x) \geq 0$ in the case where n is even.

P6 If $f(x) \leq g(x)$ for all x in an open interval containing a, except possibly for $x = a$, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

Note that even if two functions f and g (for which $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist) verify $f(x) < g(x)$ for all x in an open interval containing a, we cannot conclude that $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$. To see this, take $f(x) = x^2$ and $g(x) = 2x^2$ and $a = 0$.

P1–P6 are intuitive and can be proved using Definition 2.2.1, however, we omit their proofs. The preceding statements remain true if $\lim_{x\to a}$ is replaced by $\lim_{x\to a^-}$ or by $\lim_{x\to a^+}$. The following application underscores the usefulness of P1–P6.

Classroom Discussion 2.2.3: Limit of Rational Functions

Let c and a be arbitrary fixed real numbers.

1. Determine $\lim_{x\to a} x$ and $\lim_{x\to a} c$.
2. Use the results in Problem 1 and use P1 and P3 to compute $\lim_{x\to a} c \cdot x^2$, $\lim_{x\to a} c \cdot x^3$, and $\lim_{x\to a} c \cdot x^4$.
3. Let n be a natural number. What is $\lim_{x\to a} c \cdot x^n$?
4. Use the result in Problem 3 and use P1 to compute $\lim_{x\to a} (2x^3 - \frac{4}{3}x^2 + x + \frac{1}{2})$.
5. Now let $f(x)$ be a *polynomial function* of degree n, i.e., $f(x) = b_n x^n + \cdots + b_1 x + b_0$ for all x, where $b_n, \ldots, b_1, b_0$ are fixed real numbers. Using Problem 3 and P1, compute $\lim_{x\to a} f(x)$ and compare with $f(a)$. Fill in the blank:

   If $f(x)$ is a polynomial function, then for any real number a, $\lim_{x\to a} f(x) =$ ______.

6. Let $p(x)$ and $q(x)$ be two polynomial functions and consider the rational function $\frac{p(x)}{q(x)}$. Suppose that $q(a) \neq 0$. Use Problem 5 and P4 to compute $\lim_{x\to a} \frac{p(x)}{q(x)}$. Fill in the blank:

   If $p(x)$ and $q(x)$ are polynomial functions, then $\lim_{x\to a} \frac{p(x)}{q(x)} =$ ______ for any real number a satisfying $q(a) \neq 0$. ◆

EXAMPLES

Evaluate the limits using P1–P6 and the results from Classroom Discussion 2.2.3.

1. $\lim_{x\to -2} (4x^2 - x - 3)$
2. $\lim_{x\to 1} \sqrt{3x - 1}$
3. $\lim_{x\to 1} \frac{x+3}{2x-3}$

Solutions

1. The function $f(x) = 4x^2 - x - 3$ is a polynomial function. Applying the results from Classroom Discussion 2.2.3, we obtain

$$\lim_{x\to -2} (4x^2 - x - 3) = f(-2) = 15.$$

2. Using P5, we have that $\lim_{x\to 1} \sqrt{3x - 1} = \sqrt{\lim_{x\to 1} (3x - 1)}$. In addition, since $3x - 1$ is a polynomial function, we also have $\lim_{x\to 1} (3x - 1) = 3 \cdot 1 - 1 = 2$. Thus, $\lim_{x\to 1} \sqrt{3x - 1} = \sqrt{2}$.
3. The functions $p(x) = x + 3$ and $q(x) = 2x - 3$ are polynomial functions, and since $q(1) = -1 \neq 0$, the results from Classroom Discussion 2.2.3 imply that

$$\lim_{x\to 1} \frac{x + 3}{2x - 3} = \frac{p(1)}{q(1)} = -4.$$ ■

Practice Problems

Evaluate the limits using P1–P6 and the results from Classroom Discussion 2.2.3.

1. $\lim_{x\to -1} (5x^4 + x^3 - \frac{1}{2}x + 5)$
2. $\lim_{x\to 0} \sqrt{1 + 6x - 5x^2}$
3. $\lim_{x\to 3} \frac{x^2-4}{x^2+2}$

Classroom Discussion 2.2.4: Limits and Intersecting Streets

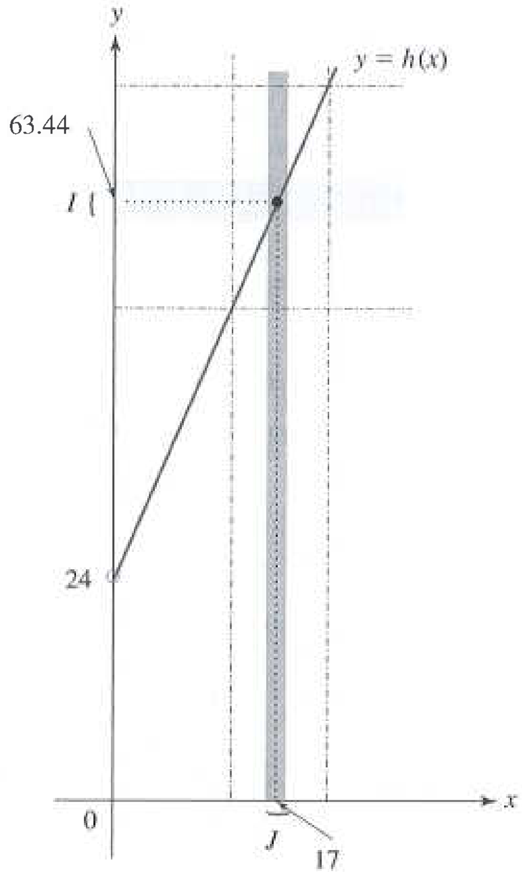
This investigation's goal is to provide a geometric interpretation of the definition of the limit of a function at a point.

1. Return to the function h from Problem 1 in Classroom Discussion 2.2.1. Do you agree with the following statement?

   If we want a person's height $h(x)$ to be within 0.5 inches of 63.44, it suffices to have that person's femur length x within 0.21551724 inches of 17.

2. Does it suffice to choose x within 0.01 of 17 in order to get $h(x)$ within 0.001 of 63.44?
3. Determine $\delta > 0$ so that if $|x - 17| < \delta$ with $x \neq 17$, then $|h(x) - 63.44| < 0.001$.
4. In general, let $\varepsilon > 0$ be a small number. Determine $\delta > 0$ so that if $|x - 17| < \delta$ with $x \neq 17$, then $|h(x) - 63.44| < \varepsilon$.

The computation in Problem 4 shows that we can get $h(x)$ arbitrarily close to 63.44 by choosing x sufficiently close to 17; each time we decide how close to 63.44 we want $h(x)$ to be, we can find an open interval containing 17 such that if $x \neq 17$ is in that interval, then $h(x)$ is as close as we want to 63.44. Geometrically, this is how we can understand this statement.



a. If $h(x)$ is close to 63.44, that means $h(x)$ is contained in an open interval I centered at 63.44. The length of this interval is determined by how close $h(x)$ is to 63.44.

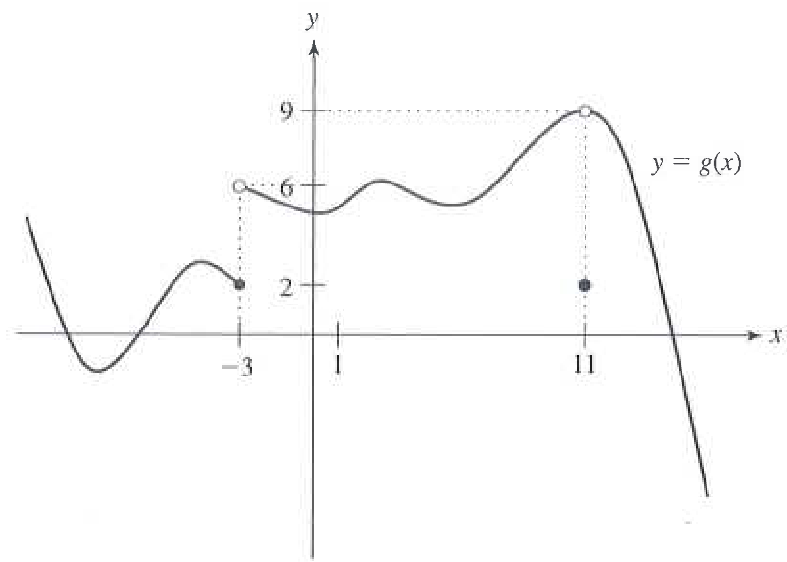
b. The horizontal band, whose intersection with the y-axis is I, cuts a portion of the graph of h.

c. The portion of the graph of h cut by the horizontal band is projected onto the x-axis.

d. Now the goal is to select an open interval J on the x-axis containing 17, $J\setminus\{17\}$ being contained in both the domain of h and the projection obtained in c, with the following property: for all $x \in J$, with $x \neq 17$, the points $(x, h(x))$ are in the portion of the graph of h cut by the horizontal band. Observe that this last condition guarantees that $h(x)$ will be in I for all x in J.

If this construction can be completed for all intervals I of arbitrarily small length, then "$h(x)$ can be made arbitrarily close to 63.44 by taking x in the domain of h sufficiently close but not equal to 17."

5. Use this geometric interpretation to show that for the function g given by the following graph, $\lim_{x \to 11} g(x) = 9$ while $\lim_{x \to -3} g(x) \neq 6$.



Definition 2.2.1 captures the essence of the notion of the limit of a function at a point. However, the phrases "as close as we want" and "sufficiently close" are not mathematically precise. A more rigorous definition follows.

Definition 2.2.3 Let f be a real valued function whose domain is a subset of the real line. We say that a real number L is the limit of f as x approaches a; we write $\lim_{x \to a} f(x) = L$ if for every open interval I centered around L there exists an open interval J containing a, with $J\setminus\{a\}$ contained in the domain of f, such that for every $x \in J\setminus\{a\}$, we have $f(x) \in I$.

EXAMPLE Use Definition 2.2.3 to show that $\lim_{x \to 2} f(x) = 8$ where

$$f(x) = \begin{cases} x^3 & \text{if } x \neq 2 \\ 100 & \text{if } x = 2. \end{cases}$$

CHAPTER 3 Differentiation

3.1 AVERAGE RATES OF CHANGE

Average rate of change of a function

What is a **rate**? How do we use rates in daily life? To address these issues, work in groups and answer the following questions.

Classroom Discussion 3.1.1: Rates of Change in Real Life

1. What is a person's heart rate? How do you compute it? What units of measurement do you use for heart rate? Fill in the blanks:

$$\text{heart rate} = \frac{\text{number of \_\_\_\_\_\_}}{\text{time, measured in \_\_\_\_\_\_}}$$

2. A person's painting rate is the ratio of the area the person paints to the time it takes to paint that area. Two painters must paint a rectangular wooden fence that is 15 yards long and 5 feet tall. The first painter paints at a rate of 25 square feet per minute. The second painter paints at a rate of 20 square feet per minute. How long will it take for the two workers to finish painting both sides of the wooden fence?

3. A person's walking rate is the distance the person walks divided by the time it takes to walk that distance. In other words,

$$\text{walking rate} = \frac{\text{distance}}{\text{time}},$$

What is Adrian's walking rate if he covers 7 kilometers in 1.5 hours? Express your answer in meters per second.

These examples describe how a given entity changes in relationship to another entity. Consider Problem 3 again. To compute a person's walking rate at different times during a walk, look at the change in the distance the person has walked with respect to the change in time: $\frac{\text{change in distance}}{\text{change in time}}$. Suppose that after 1 hour, Adrian has walked 5 kilometers and after 2 hours he has walked 11 kilometers. His walking rate for the second hour is $\frac{11-5}{2-1} = 6$ kilometers per hour.

In order to formalize rates of change, we use the function notation. In each case, you computed the ratio between the change of a given function $f(x)$ and the change of x. This is why it is appropriate to define

the **average rate of change** of $f(x)$ with respect to x as the ratio, $\frac{\text{change in } f(x)}{\text{change in } x}$.

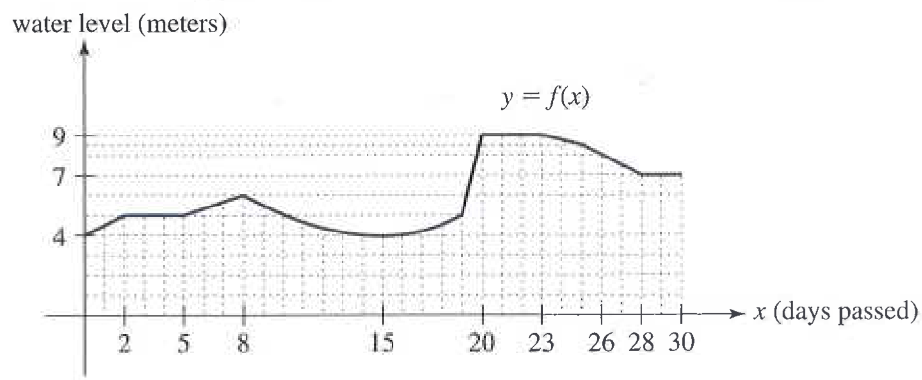
In particular, if $a < b$ and x changes from a to b, then $f(x)$ changes from $f(a)$ to $f(b)$. This change of f is also referred to as the change of f over the interval $[a, b]$. ◆

Definition 3.1.1 The average rate of change of a function $f(x)$ with respect to x over the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

Classroom Discussion 3.1.2: Rates of Change of Linear and Nonlinear Functions

1. **Water Levels**
   The following graph represents a lake's water level over a 30-day period.



Denote the water level by $f(x)$, where x represents the number of days that have passed. In a–c, compute the water level's average rate of change over the given time period. First write the formula for the rate of change using the function notation, and then use the values of f given by the graph.

a. From the end of day 1 to the end of day 2
b. From the beginning of day 3 to the end of day 5
c. From the end of day 8 to the end of day 15, and from the end of day 15 to the end of day 30

How do these average rates of change compare?

2. Walking Rates
Diana walks from home to school in 25 minutes. The distance d in meters she has traveled after a t minute walk is given by $d(t) = 60t$.

a. Compute Diana's average walking rate over the following intervals of time: $[1, 4]$, $[6, 7]$, $[0, 25]$. What do you observe? Interpret the results.
b. Make a conjecture about how the average rates of change of d computed over two arbitrary intervals compare. Prove your conjecture.

3. Linear Functions versus Nonlinear Functions

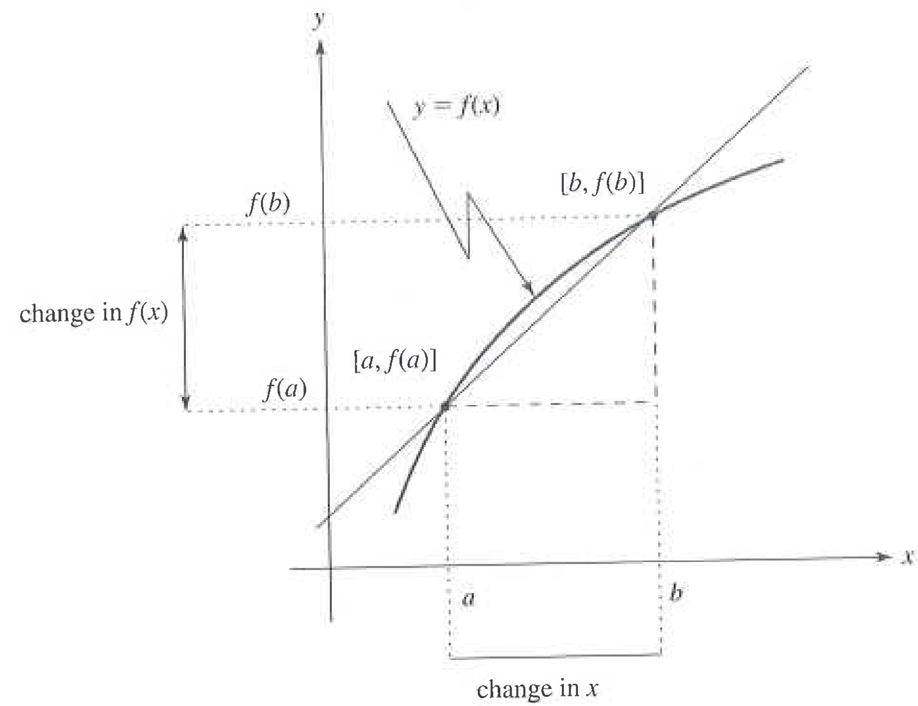
a. Let m and c be arbitrary real numbers, and consider the linear function $g(x) = mx + c$. Compute the average rate of change of g over the interval $[a, b]$. What do you observe?
b. Consider a function g for which the average rate of change over any interval $[a, b]$ is constant and equal to a real number m. Prove that g is linear.

Hint: Fix a point a and let x be arbitrary. What is the average rate of change of g over the interval with endpoints a and x?

Observation. Combine a and b to obtain the following result: A function is linear if and only if the function's average rate of change over any interval is constant.

c. Recall the example in Problem 1. Is f a linear function? What did you observe about the rates of change of f over different intervals?
d. The following graph shows the change in x and the change in $f(x)$ over the interval $[a, b]$. Use this sketch to fill in the blank:

The average rate of change of a function f over an interval $[a, b]$ is equal to the slope of ___________.



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EXERCISES 3.1

1. One criterion for judging a football team is the team's winning percentage:

$$\text{winning percentage} = \frac{\text{number of games won}}{\text{number of games played}}.$$

The winning percentage is the rate of change of the number of games won with respect to the number of games played. What can you conclude if you know that the winning percentage for a football team in a season was 0.4? Can you find the number of total games the team played that season if the team won eight games?

2. Altitudes and the Heart Rate. The human body is optimally equipped for existence at an air pressure close to 760 millimeters of mercury (the air pressure at sea level) with an oxygen concentration of 21%. When the altitude increases, the atmospheric pressure decreases, which in turn leads to a decrease in the number of oxygen molecules per breath. Consequently, the amount of oxygen available in the body's blood and tissue decreases. Such a lack of oxygen can cause potentially life-threatening illness at high altitudes. The following graph (from a report by Catherine M. Quinn) shows the results of studies conducted at the University of Limerick. Subjects breathed into air bags containing concentrations of oxygen that would be found at various altitudes. The relationship between the heart rate

To decide if a graph $y = f(x)$ admits a tangent line at a point $(a, f(a))$, proceed as follows. Consider another point $(b, f(b))$ on the graph and analyze the behavior of the line passing through $(a, f(a))$ and $(b, f(b))$ as $b \to a$. If there exists a line L passing through $(a, f(a))$ such that the slope of the line passing through $(a, f(a))$ and $(b, f(b))$ approaches the slope of L as $b \to a$, then L is by definition called the tangent line to $y = f(x)$ at $(a, f(a))$.

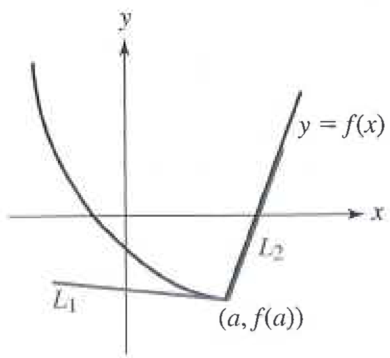
In particular, if we assume that:

i) the slope of the line passing through $(a, f(a))$ and $(b, f(b))$ approaches as $b \to a^-$ the slope of a line L_1 passing through $(a, f(a))$ and

ii) the slope of the line passing through $(a, f(a))$ and $(b, f(b))$ approaches as $b \to a^+$ the slope of a line L_2 passing through $(a, f(a))$,

then a tangent line to $y = f(x)$ at $(a, f(a))$ exists if and only if $L_1 = L_2$.

For example, the function f whose graph is sketched here does not have a tangent line at $(a, f(a))$.



Classroom Discussion 3.2.1: Instantaneous Rate of Change of the Function $f(x) = x^2$

1. Sketch the graph of $f(x) = x^2$ on a sheet of graph paper.
2. Compute the average rate of change of $f(x)$ over the interval $[1, 1.1]$, draw the line passing through the points $(1, f(1))$ and $(1.1, f(1.1))$.
3. Use your calculator to compute the average rates of change of f over the intervals $[1, 1 + h]$ for $h > 0$, taking the values $10^{-1}, 10^{-2}, \ldots, 10^{-6}$. What do you observe? Describe what you expect to happen to the line passing through points $(1, f(1))$ and $(1 + h, f(1 + h))$ as h decreases.
4. Use your calculator to compute the average rates of change of f over the intervals $[1 + h, 1]$ for $h < 0$, taking the values $-10^{-1}, -10^{-2}, \ldots, -10^{-6}$. What do you observe? Describe what you expect to happen to the line passing through points $(1 + h, f(1 + h))$ and $(1, f(1))$ as the absolute value of h decreases.

5. Assuming that your calculator can be as precise as you want, what do you expect to happen to the average rates of change of f over the intervals with endpoints 1 and $1 + h$ as $h \to 0$? What is the connection with the tangent line to the graph of f at the point $(1, 1)$?
6. Using calculus, you can verify the prediction you made in Problem 5. Write the general formula for the average rate of change of f over the interval with endpoints 1 and $1 + h$, then simplify the expression you obtained under the assumption that $h \neq 0$. Next, take the limit as $h \to 0$ of your simplified expression. What is the geometric interpretation of the value you obtained?
7. Why do you think the value you obtained in Problem 6 is called the *instantaneous rate of change* of f at 1, or simply the rate of change of f at 1?
8. Fill in the blank:

   The slope of the tangent line to the graph $y = x^2$ at the point (1,1) is equal to ______.

9. Consider now an arbitrary point $(x, f(x))$ on the graph of f. Compute the average rate of change of f over the interval with endpoints x and $x + h$. Simplify your expression and then take the limit as $h \to 0$.
10. Fill in the blanks:

    The instantaneous rate of change of f at x is $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The value of this limit coincides with the slope of the ______ to the graph $y = f(x)$ at the point ______. ◆

In the preceding example, you constructed a new function, which associates to each x the value of $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. This new function is called the **derivative** of f and is denoted by f'. You have proved that

$$\text{if } f(x) = x^2, \text{ then } f'(x) = 2x. \qquad (1)$$

In addition, you have seen that the slope of the tangent line to the graph of f at a point $(x, f(x))$ is equal to $f'(x)$.

EXAMPLE Determine the equation of the tangent line to the graph $y = x^2$ at the point $(-2, 4)$.

Solution The slope of the tangent line to the graph of the function $f(x) = x^2$ at the point $(-2, 4)$ is $m = f'(-2) = 2(-2) = -4$. Thus, the equation of the tangent line is $y - 4 = -4(x + 2)$, or, equivalently, $y = -4x - 4$. ■

Practice Problem

Determine the equation of the tangent line to the graph $y = x^2$ at the point $(3, 9)$.

7. Explain why the following statement is true:

> For a differentiable function f, the tangent line to the graph of f at a point $(c, f(c))$ is horizontal if and only if $f'(c) = 0$. ◆

You discovered that for a differentiable function f, being nondecreasing/nonincreasing provides information about the sign of the derivative f'. Should we expect the sign of f' to provide information about the behavior of f? Intuitively, this seems to be the case. For example, if $f' < 0$ on an interval (a, b), then the slopes of the tangent lines to the graph of f corresponding to points $x \in (a, b)$ are negative. This suggests that f is decreasing on the interval (a, b). Similarly, if $f' > 0$ on an interval (a, b), then the slopes of the tangent lines to the graph of f corresponding to points $x \in (a, b)$ are positive, and we expect that f is increasing on the interval (a, b). Transform this informal reasoning into a proof by carefully using Definition 3.2.1. The section "More on the Connection between the Sign of f' and the Behavior of f" in Projects and Extensions 3.2 addresses this issue. In many applications, we determine the sign of f' in order to identify the intervals on which f is increasing/decreasing (as you will see in Chapter 4). The following is a summary of the relationship between the behavior of a function f and the sign of its derivative, including the results you proved in Classroom Discussion 3.2.3.

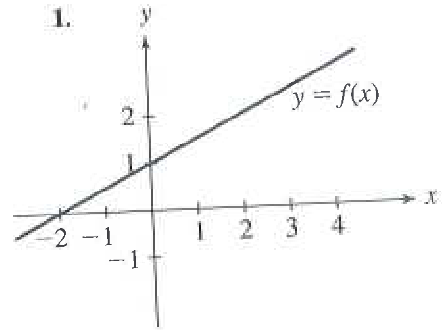
The Relationship between a Function's Behavior and the Sign of Its Derivative. Let f be a differentiable function on an interval (a, b). Then the following hold.

1. f is nonincreasing on (a, b) if and only if $f' \le 0$ on (a, b).
2. f is nondecreasing on (a, b) if and only if $f' \ge 0$ on (a, b).
3. If $f' < 0$ on (a, b), then f is decreasing on (a, b).
4. If $f' > 0$ on (a, b), then f is increasing on (a, b).
5. The tangent line to the graph of f at a point $(c, f(c))$ $(c \in (a, b))$ is horizontal if and only if $f'(c) = 0$.

Classroom Discussion 3.2.4: Sketching Graphs of Derivatives

In each example here, sketch the graph of the derivative based on the graph of the function. (Your sketch cannot be exact in some of these cases, but you can draw a rough approximation for the graph of the derivative.) Describe what happens to the derivative function as x increases. To do so, look at the graph from left to right and answer the following questions:

a. Where is the derivative positive?
b. Where is the derivative negative?
c. Where is the derivative zero?
d. Are there any points $(x, f(x))$ where the graph does not have a tangent line?

1.


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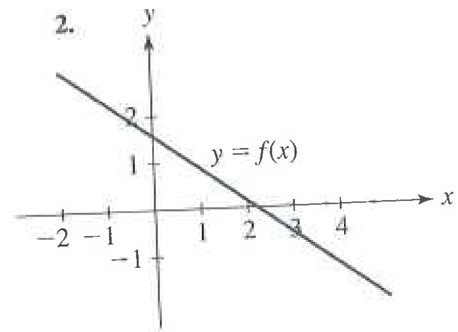

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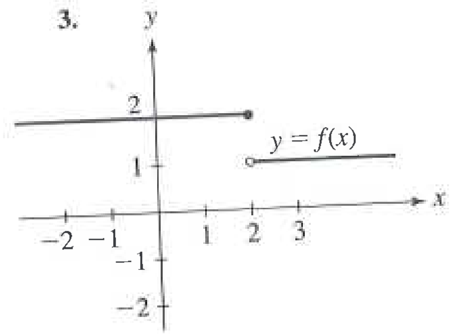

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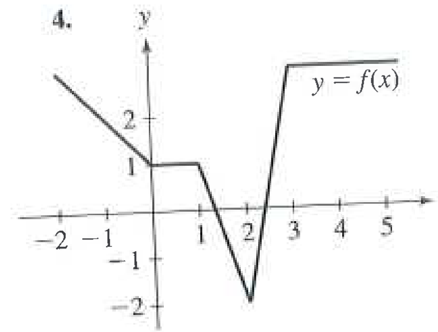

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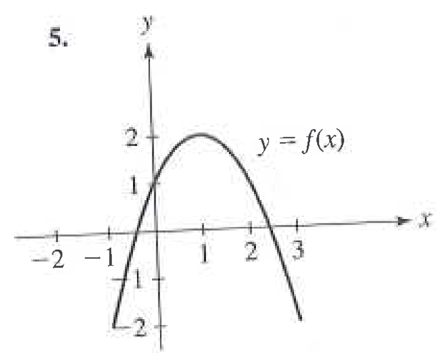

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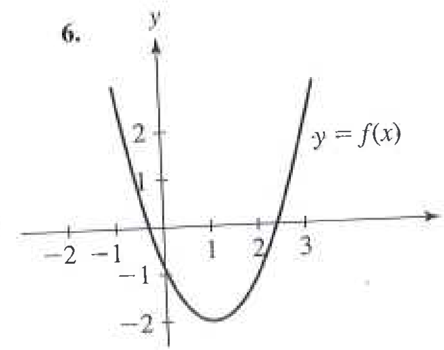

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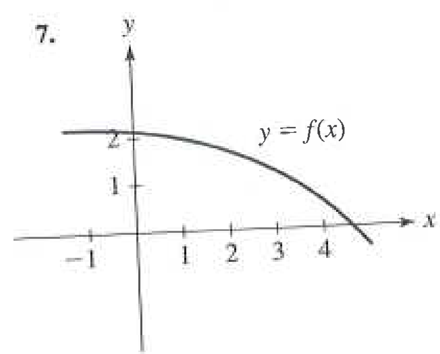

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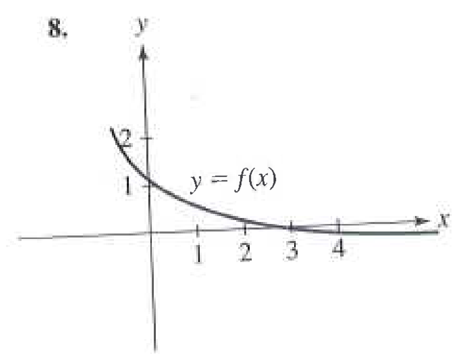

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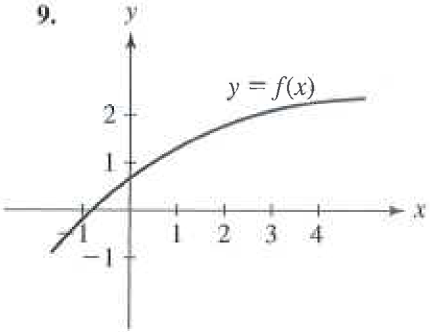

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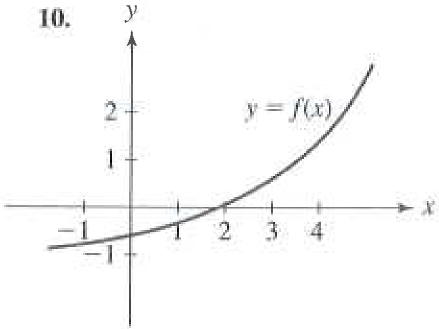


◆

Classroom Discussion 3.2.5: Differentiability and Continuity

From Problem 4 in the previous Classroom Discussion, you can see that not all continuous functions are differentiable. In particular, if the graph of a function contains a corner at some point $(a, f(a))$, then the function does not have a tangent line at that point, thus, the function is not differentiable at $x = a$. Hence, continuity does not imply differentiability. How about the converse? Is it natural to expect that differentiable functions are continuous? From a geometric point of view, the question can be restated as follows: If a graph has a tangent line at every point, does it follow that the graph can be traced without lifting the pen? This discussion's goal is to answer this question.

Suppose f is a function that is differentiable at x. By definition, f is continuous at x provided $\lim_{y\to x} f(y) = f(x)$. In particular, if we denote by h the difference $y - x$, then $y = x + h$ and $y \to x$ if and only if $h \to 0$. Hence, f is continuous at x provided $\lim_{h\to 0} f(x + h) = f(x)$. The following outline is structured around investigating the validity of this equality. First, observe that the algebraic identity

$$f(x + h) - f(x) = \frac{f(x + h) - f(x)}{h} \cdot h$$

holds for any $h \neq 0$.

a. What can you say about $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0} h$?

b. Use a to compute $\lim_{h\to 0} [f(x + h) - f(x)]$.

c. Use b to compute $\lim_{h\to 0} f(x + h)$.

d. Fill in the blanks:

If f is differentiable at x, then $\lim_{h\to 0} f(x + h) =$ ______; that is, f is ________ at x. ◆

EXERCISES 3.2

In Exercises 1–4, compute the derivative of the given function.

1. $f(x) = x^2 + 2x$

2. $g(x) = x^2 - 3x$

3. $A(s) = 2s^2 + s - 2$

4. $f(t) = -t^2 + 3t - 10$

5. Determine the equation of the tangent line to the curve $y = 2x^2 - x + 10$ at the point $(-1, 13)$.

6. Determine the equation of the tangent line to the curve $y = -3x^2 + 5x - 2$ at the point $(1, 0)$.

7. Scientists have found that radioactive carbon-14 (C14) has a half-life of 5,730 years. This means that if the amount of C14 now is α, then the amount 5,730 years from now will be $\frac{1}{2}\alpha$, the amount 11,460 years from now will be $\frac{1}{2^2}\alpha$, and so on. The amount of C14 remains constant in living organisms due to metabolic processes but decreases once the organism dies. This is the idea behind carbon dating.

a. Make a sketch of the amount $A(t)$ of C14 in an organism t years after it has died, if the amount of C14 present while the organism was living is α. Is A increasing, decreasing, or neither? Will $A(t)$ ever be zero? As $t \to \infty$, what value does $A(t)$ approach?

b. Use the graph you have sketched in a to answer the following questions: Is A' positive or negative? Will $A'(t)$ ever be zero? Is A' increasing, decreasing, or neither? As $t \to \infty$, what value does $A'(t)$ approach? Make a sketch of A'.

8. Match each limit a–c with the corresponding derivative from i–iii.

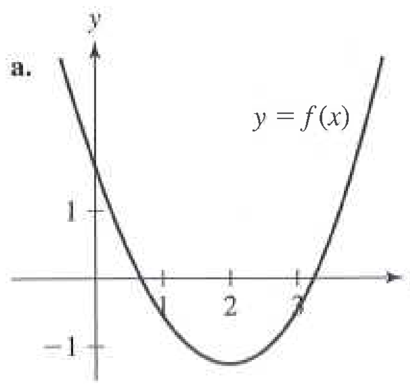
a. $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$ **i.** $(x^3)'$

b. $\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$ **ii.** $(\sqrt{x})'$

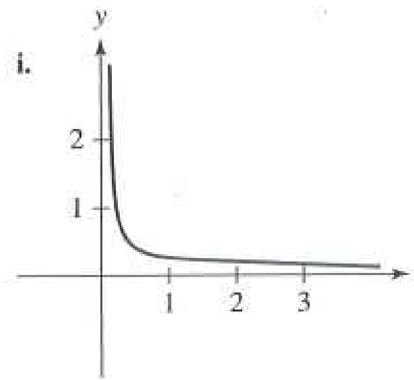
c. $\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ **iii.** $\left(\frac{1}{x}\right)'$

9. Match each function a–c with the graph of its derivative i–iii.

a.



i.



b. Calculus Approach

Using calculus, compute the maximum value of $f(x) = -3x^2 + 6x + 2$ when x takes values on the real line.

3. **The General Case $f(x) = ax^2 + bx + c$**

Let a, b, c be arbitrary fixed real numbers, with $a \neq 0$.

a. Algebraic Approach

i. Show that after completing the square, the expression for $f(x) = ax^2 + bx + c$ becomes

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

ii. What can you say about the values of $\left(x + \frac{b}{2a}\right)^2$ for values of x on the real line?

iii. Assume $a > 0$. Use the conclusion in ii to decide if $f(x)$ has a maximum or minimum value for some value of x. How about if $a < 0$?

iv. For what value of x does f attain the corresponding minimum or maximum? What is the value of that minimum or maximum?

b. Calculus Approach

Use calculus to optimize $f(x) = ax^2 + bx + c$ on the real line. Make sure you distinguish between the cases $a < 0$ and $a > 0$. ◆

Classroom Discussion 4.1.4: Local Maximum/Minimum versus Absolute Maximum/Minimum

The goal in this discussion is to determine the largest and smallest values attained by the function $f(x) = 3x^4 - 4x^3 - 12x^2$ on the interval $[-2, 3]$. Follow the outline here.

1. Compute the critical points of $f(x)$ and find the subintervals of $[-2, 3]$ determined by the critical points.
2. Determine the sign of $f'(x)$ on each subinterval from Problem 1. Explain why the sign of f' cannot change within these subintervals.
3. Decide whether f is increasing or decreasing on each subinterval from Problem 1. Compare your results with the following table.

x	−2		−1		0		2		3
$f'(x)$		−	0	+	0	−	0	+	
f	32	↘	−5	↗	0	↘	−32	↗	27

Observe that the function $f(x)$ is decreasing to the left and increasing to the right of both $x = -1$ and $x = 2$. In this case we say that $f(x)$ has **local minima** at $x = -1$ and $x = 2$. Since $f(-1) = -5 > -32 = f(2)$, and the pattern is decreasing-increasing-decreasing-increasing, the smallest value $f(x)$ that takes on the interval $[-2, 3]$ is -32. This is why -32 is called the **absolute minimum** for $f(x)$.

Similarly, because $f(x)$ is increasing to the left of $x = 0$ and decreasing to the right of $x = 0$, we say that $f(x)$ has a **local maximum** at $x = 0$. The local maximum value is $f(0) = 0$. Is this also the **absolute maximum** for f (i.e., is this the largest value taken by f on the interval $[-2, 3]$)? To answer this question, we look at the behavior of f. The decreasing-increasing-decreasing-increasing pattern suggests that the values of f at the endpoints $x = -2$ and $x = 3$ must be taken into account. A direct computation gives $f(-2) = 32$ and $f(3) = 27$, so the absolute maximum of f is $32 = f(-2)$, not $0 = f(0)$.

4. Do critical points always yield local minima or local maxima? To answer this question, analyze the critical points of the function $f(x) = x^3$ defined on the real line. ◆

Summary of Main Steps in Optimization Problems

1. Select the variables and write the expression for the function f to be optimized.
2. Write the constraint and use it to express the function f in terms of one variable.
3. Determine the domain of f.
4. Find all critical points of f.
5. Determine the intervals where f' is positive and the intervals where f' is negative. Determine the intervals where f is increasing and the intervals where f is decreasing.
6. Determine maxima and/or minima for f.
7. Interpret and check your solution.

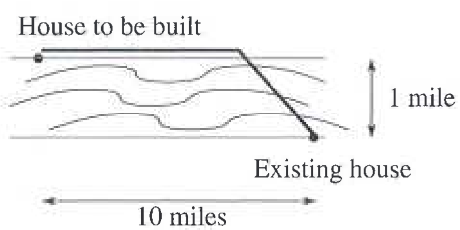
EXERCISES 4.1

In Exercises 1–4, for each given function, determine the critical points, the subintervals in the domain determined by the critical points, the sign of the derivative on each subinterval, and whether the function is increasing or decreasing on each subinterval.

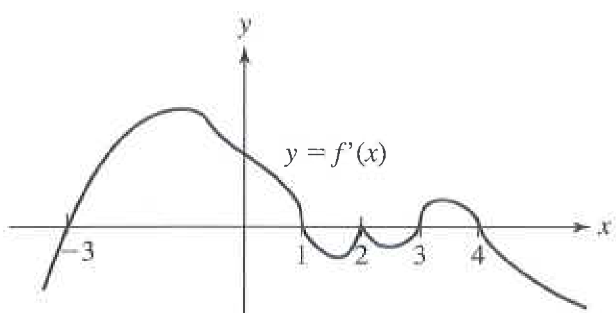
1. $f(x) = 3x - 2$, Domain $= [-4, 12]$
2. $f(x) = x^2 - 3x + 2$, Domain $= [-1, 8]$
3. $f(x) = \frac{1}{3}x^3 - 9x$, Domain $= (-\infty, \infty)$
4. $f(x) = -x^3 + 12x$, Domain $= (0, \infty)$
5. For each function from 1–4, determine local maxima, local minima, absolute maxima, and absolute minima, provided they exist.
6. **Classroom Connection 4.1.1: Fenced In**

The following exploration is from page 26 in the eighth-grade textbook *MathScape, Family Portraits*.

12. A gardener is trying to decide what is the best time to harvest and sell his cantaloupe crop. He estimates that there are approximately 200 pounds of ripe cantaloupe in his garden. Each week an additional 60 pounds of cantaloupe ripens, and 10 pounds go to waste. The cantaloupe's market price is \$1 per pound, but it drops \$0.10 per pound each week that passes. When should the gardener sell his cantaloupe crop in order to maximize revenue? What is the maximum revenue he can make?
13. **Cable Lines:** A house is built along a 1-mile wide river. On the river's other side, 10 miles downstream, there is another building from which cable line will be run to the new house. The underwater cable costs twice as much per foot as the underground cable. How long should the cable line along the river be in order to minimize the cost?



14. Tod runs a chocolate store. He invests \$4 for each pound of chocolate he makes. He sells 500 pounds of chocolate each month for the price of \$12 per pound. Tod discovers that for every 10 cents he takes off the price, he sells 10 more pounds of chocolate each month. What should he charge for 1 pound of chocolate in order to maximize his profit? How many pounds of chocolate will Tod sell at that price?
15. **Inventory Costs:** A retail appliance store sells 500 refrigerators each year. It costs \$50 to store one refrigerator for a year. At each reordering, there is a fixed \$20 fee for the truck rental and an additional \$5 handling fee for each refrigerator ordered. How many times per year should the store place an order to minimize the storage, truck rental, and handling costs, if the number of refrigerators per order is constant? When modeling this problem, assume that at any time during the year, the average number of refrigerators in stock is half the number x of refrigerators ordered each time.
16. The following figure shows the graph of the derivative of a function f. Find all the points where f has a local maximum or a local minimum.



4.2 CURVE SKETCHING

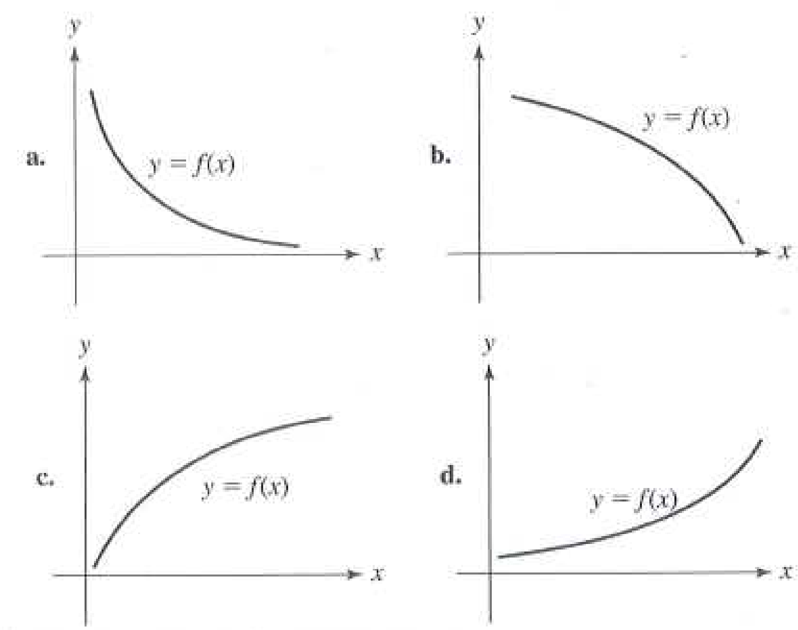
Second derivative • Concave up and concave down • Inflection points • Sketching graphs

You saw in Section 3.2 that the sign of a function's derivative yields substantial information about the shape of the graph of the function. Take this principle one step further and look at the derivative of $f'(x)$ to get information about $f'(x)$. The derivative of $f'(x)$, also called the **second derivative** of $f(x)$, is denoted by $f''(x)$. Thus, $f''(x) = (f'(x))'$, whenever this makes sense. For example, if we consider the functions $f(x) = 2x^2 - 4x + 7$ and $g(x) = x^5 - 3x^4 + 100$, then $f'(x) = 4x - 4$, $f''(x) = 4$, $g'(x) = 5x^4 - 12x^3$, and $g''(x) = 20x^3 - 36x^2$. The goal is to determine what information the second derivative f'' provides about the shape of the graph of f.

Classroom Discussion 4.2.1: Concavity and the Second Derivative

Let f be a function that is twice differentiable on an interval (a, b); that is, f is differentiable on (a, b), and, in turn, f' is also differentiable on (a, b).

1. Recall the relationship between a function's behavior and the function's derivative. Use this relationship to fill in the blanks:
   f' is nondecreasing on (a, b) if and only if f''_____ on (a, b).
   f' is nonincreasing on (a, b) if and only if f''_____ on (a, b).
2. The goal is to understand how a function's behavior relates to the sign of the function's second derivative. Look at the following graphs. Decide in each case whether the derivative is nondecreasing or nonincreasing.



3. Which are the graphs for which $f'' \geq 0$?
4. Which are the graphs for which $f'' \leq 0$? ◆

In Problem 8, you had to join the plots corresponding to integer values of x to get a "smooth curve." Is there a particular curve passing through all the points you plotted for integer values of x that would be best for our model? If yes, what function has this "smooth curve" as its graph? To answer these questions, return to Problem 7a in Classroom Connection 4.3.1. The expression you found for the number of layers in terms of x was $y = 2^x$. The function $f(x) = 2^x$ is a great candidate. However, the definition of 2^x when x is an irrational number has not yet been provided. The next discussion's goal is to clarify the issue of how one can raise a given positive number to an irrational power. First, recall how powers with rational exponents were defined.

Rational Exponents. Fix a real number $b > 0$ and take m, n to be whole numbers. Then, rational powers of b are defined as follows.

1. $b^m = b \cdot b \cdots b$ is the product of m copies of b.
2. $b^{-m} = \frac{1}{b^m}$ is the reciprocal of b^m.
3. $b^{\frac{1}{n}} = \sqrt[n]{b}$ is the positive number whose nth power is b. The case $n = 2$ has been discussed in the project "*Computing $\sqrt{x}$ for x a Real Positive Number*" in Section 1.1. The nth root can be defined similarly.
4. $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ is the number whose nth power is b^m.
5. $b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$ is the reciprocal of $b^{\frac{m}{n}}$.

Thus, for any rational number x, b^x is meaningful. In addition, you can check that for any x, y rational numbers, we have $b^{-x} = \frac{1}{b^x}$, $b^{x+y} = b^x b^y$, and $(b^x)^y = b^{xy}$. The goal is to extend this definition to the case when the exponent x is an irrational number.

Classroom Discussion 4.3.1: Irrational Exponents

Fix an irrational number x. For example, take $x = \pi$. Starting with its decimal representation

$$\pi = 3.14159265358979323846264338327950288419...,$$

you can construct sequences $\{y_n\}_n$ and $\{z_n\}_n$ of rational numbers:

$$y_0 = 3,\ y_1 = 3.1,\ y_2 = 3.14,\ y_3 = 3.141,\ y_4 = 3.1415,\ y_5 = 3.14159,$$

$$y_6 = 3.141592, \ldots \text{ and}$$

$$z_0 = 4,\ z_1 = 3.2,\ z_2 = 3.15,\ z_3 = 3.142,\ z_4 = 3.1416,\ z_5 = 3.1416,$$

$$z_6 = 3.141593, \ldots$$

1. For each n, compare the values of y_n, z_n and π.
2. What can you say about $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} z_n$? Explain.
3. According to the definition of rational exponents, the number 2^{y_n} is meaningful for each $n \geq 0$. What happens to the values of 2^{y_n} as $n \to \infty$?
4. Is the sequence $\{2^{y_n}\}_n$ convergent or divergent?
5. Similarly, for each $n \geq 0$, 2^{z_n} is well defined since z_n is rational. What happens to the values of $\{2^{z_n}\}_n$ as $n \to \infty$?
6. Is the sequence $\{2^{z_n}\}_n$ convergent or divergent?
7. Intuitively, it is expected that for $a > 0$, $\lim_{n\to\infty} \sqrt[n]{a} = 1$. Use this fact to compute $\lim_{n\to\infty} (2^{z_n} - 2^{y_n})$.
8. What does the preceding construction suggest as a definition for 2^π?

   Take now the general case. Let x be an arbitrary irrational number. Then, there exist two sequences $\{y_n\}_n$ and $\{z_n\}_n$ with the following properties:

   a. y_n and z_n are rational numbers for each n,
   b. $y_1 \leq y_2 \leq y_3 \leq \ldots$ and $z_1 \geq z_2 \geq z_3 \geq \ldots,$
   c. $y_n \leq x \leq z_n$ for each n,
   d. $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x$

   as seen in the project "*Real Numbers as Limits of Sequences of Rational Numbers*" in Section 1.1.
9. How can you define 2^x?
10. How can you define b^x for $b > 0$, an arbitrary real number, and for x, an arbitrary irrational number?

    For each $b > 0$, the function $f(x) = b^x$ is now well defined on the set of real numbers. This function is the **exponential function with base b**. Observe that, due to the properties of powers with rational exponents, this construction also yields $b^{-x} = \frac{1}{b^x}$, $b^{x+y} = b^x b^y$, and $(b^x)^y = b^{xy}$ for all real numbers x, y. ◆

Classroom Discussion 4.3.2: Exponential versus Linear

Recall that a linear function is a function of the form $g(x) = mx + n$, where m and n are real numbers. Fix $b > 0$ and consider the exponential function $f(x) = b^x$.

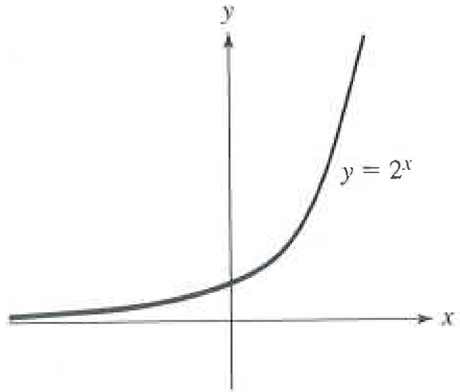
1. Compute $g(3) - g(2)$ and $g(70) - g(69)$. What do you observe?
2. What is the value of $g(x + 1) - g(x)$ for an arbitrary x?
3. Compute $\frac{f(5)}{f(4)}$ and $\frac{f(-5)}{f(-6)}$. What do you observe?
4. What is the value of $\frac{f(x+1)}{f(x)}$ for an arbitrary x?
5. Explain the difference between exponential and linear functions. ◆

Classroom Discussion 4.3.3: Graphs of Exponential Functions

1. Use a graphing calculator to trace the graphs of the functions $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 4^x$, and $l(x) = 9^x$. Describe the differences and similarities in the curves.
2. Use a graphing calculator to trace the graphs of the functions $F(x) = (\frac{1}{2})^x$, $G(x) = (\frac{1}{3})^x$, $H(x) = (\frac{1}{4})^x$, and $L(x) = (\frac{1}{9})^x$. Describe the differences and similarities in the curves.
3. Make a prediction about the shape of the graph of the exponential function $f(x) = b^x$ for $b > 0$, depending on whether $b < 1$ or $b > 1$. ◆

Classroom Discussion 4.3.4: Derivatives of Exponential Functions

1. Let $f(x) = 2^x$ be defined for all real numbers x. Here is the graph of $f(x)$:



a. For what values of x is $f'(x)$ positive, negative, or zero?

b. What can you say about the values of $f'(x)$ as $x \to \infty$ and as $x \to -\infty$?

c. On another set of coordinates, sketch roughly the graph of $f'(x)$; use the geometric interpretation of the derivative at a point x as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

d. Compare the graph of f with the graph of f'. What do you observe?

e. The two graphs look very similar. Is $f'(x)$ an exponential function? To answer this question, you must compute $f'(x)$ using the definition of the derivative. To do so, first write the average rate of change for the function $f(x) = 2^x$ over the interval with endpoints x and $x + h$.

f. Check that for each $h \neq 0$,

$$\frac{2^{x+h} - 2^x}{h} = 2^x \frac{2^h - 1}{h}. \qquad (1)$$

g. Use your calculator to investigate $\lim_{h\to 0} \frac{2^h-1}{h}$. To do so, generate a table of values for $\frac{2^h-1}{h}$ with $h \neq 0$ and varying from -0.01 to 0.01 with step size $\Delta h = 0.001$. What do you observe? Does $\lim_{h\to 0} \frac{2^h-1}{h}$ exist? If yes, give an estimate for its value.

h. Return to (1). What can you conclude about $f'(x)$?

2. Use the outline in a–h to determine the derivative of the function $g(x) = (\frac{1}{5})^x$.

3. Fix $b > 0$.

a. Prove the following statement:

$$\text{if } F(x) = b^x, \text{ then } F'(x) = b^x k, \text{ where } k = \lim_{h\to 0} \frac{b^h - 1}{h}.$$

b. Use your calculator to estimate the values of k when $b = 0.2$, $b = 0.6$, $b = 1$, $b = 1.5$, $b = 2$, and $b = 5$. What do you observe? How do the values of k change as b increases? Compare k with 0 to fill in the blanks.

base b	constant k
$0 < b < 1$	k ___ 0
$b = 1$	k ___ 0
$b > 1$	k ___ 0

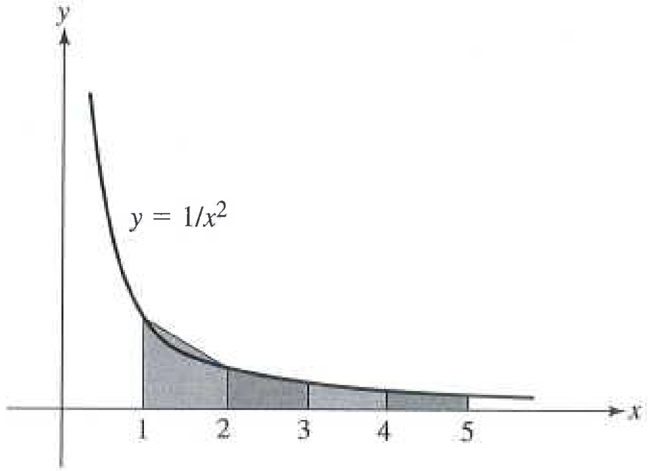
c. Is the preceding table consistent with the geometric interpretation of the derivative? Think about how the shape of the graph of the exponential function b^x varies with the values of $b > 0$.

It turns out that as b takes all the real positive values, k also takes all real values. Consequently, there is a value of b for which $k = 1$. This is precisely the case when b is the irrational number e. An approximate value for e is 2.71828. The number e arises naturally when modeling investments (see "Compound Interest" in Projects and Extensions 4.3). Using the formula you obtained in Problem 3a, you see that

$$(e^x)' = e^x.$$

d. Compute $(e^{-x})'$ using the formula for the derivative of reciprocals.

e. Suppose that f is a differentiable function. Use the chain rule to compute $(e^{f(x)})'$. ◆



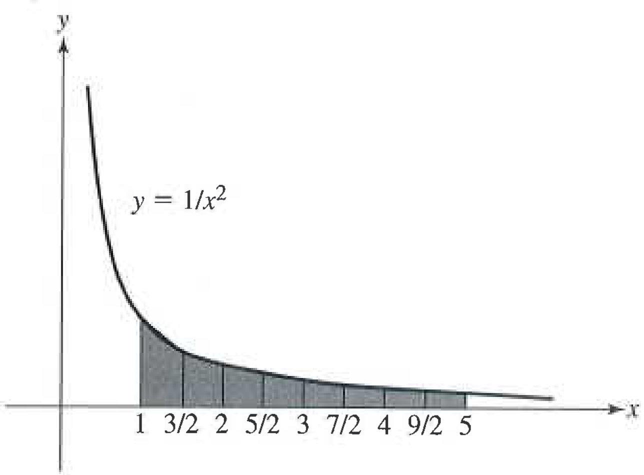
Since the graph of the function is concave upward, T_4 is an upper bound for $\mathcal{A}$; that is, $\mathcal{A} \leq T_4 \approx 0.94361$.

Similarly, the area T_8 may be obtained by applying directly the formula (1) as follows:

$$T_8 = \frac{5-1}{16}\left(1 + 2 \cdot \frac{4}{9} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{4}{25}\right.$$

$$\left. + 2 \cdot \frac{1}{9} + 2 \cdot \frac{4}{49} + 2 \cdot \frac{1}{16} + 2 \cdot \frac{4}{81} + \frac{1}{25}\right) \approx 0.83953.$$

For the same reason, $T_8 \approx 0.83953$ is also an upper bound for $\mathcal{A}$.



2.

n	T_n
10	0.82568
20	0.80656
30	0.80293
40	0.80165
50	0.80106
60	0.80073
70	0.80054
80	0.80041
90	0.80033
100	0.80026

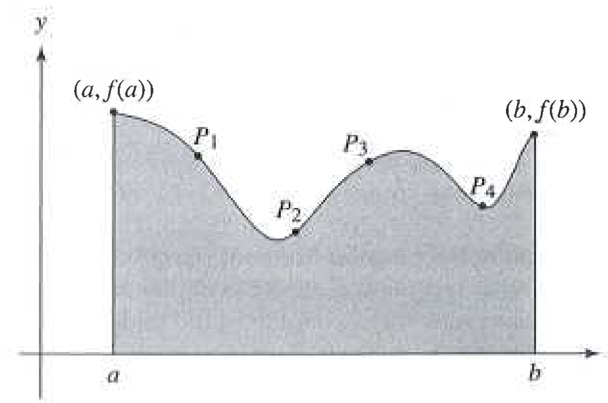
n	T_n
110	0.80023
120	0.80018
130	0.80016
140	0.80013
150	0.80012
160	0.80010
170	0.80009
180	0.80008
190	0.80007
200	0.80006

3. By analyzing the table in the preceding problem, it seems reasonable to conjecture that the approximation $\mathcal{A} \approx 0.800$ generates an error of magnitude less than 10^{-3}; that is, $0.799 < \mathcal{A} < 0.801$.

In Section 6.3, you will learn how to compute the exact value of the area $\mathcal{A}$, and thus you will have the opportunity to check whether your conjecture is true or false. ■

Classroom Discussion 6.2.2. The Rectangular Methods

This Classroom Discussion's goal is to approximate the region in the plane that is under the graph of a continuous nonnegative function with appropriate *rectangular* tiles instead of trapezoidal tiles in order to find approximate values for its area. Pick $n - 1$ arbitrary points $P_1, P_2, \ldots, P_{n-1}$ on the curved path between the endpoints $(a, f(a))$ and $(b, f(b))$ (in the figure $n = 5$). Then, think about how you can use the points that are lying on the curved path to form rectangular tiles to approximate the irregular shape.



Let $x_1, x_2, \ldots, x_{n-1}$ be the x-coordinates of points $P_1, P_2, \ldots, P_{n-1}$. For convenience, set $a = x_0$ and $b = x_n$.

1. **The Left Rectangular Method:** Guided by the following problems, discuss this first variant of the rectangular method in small groups.

   a. For each $1 \leq i \leq n$, the ith rectangle is defined by three vertices $(x_{i-1}, 0)$, $(x_i, 0)$, and $(x_{i-1}, f(x_{i-1}))$. Draw the first, the ith and the nth rectangles.

   b. Find the area of each rectangular tile.

   c. Write down the Riemann sums for f on the interval $[a, b]$ corresponding to your rectangular tiles.

   d. Assume $P_1, P_2, \ldots, P_{n-1}$ are selected such that the coordinates $x_0, x_1, \ldots, x_n$ are equally spaced. Show that in this case, the Riemann sums are given by the formula

$$L_n = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1}). \qquad (2)$$

   e. Analyze what happens as $n \longrightarrow \infty$. Then, write down a formula for the area $\mathcal{A}$ of the irregular shape.

   f. For which functions are the areas L_n necessarily lower bounds for the areas of the resulting irregular shapes? (Hint: Think about the shape of their graphs.)

   g. For which functions are the areas L_n necessarily upper bounds for the areas of the resulting irregular shapes?

   h. Write a calculator program for computing the areas L_n. Set up your program to provide the value of L_n once you enter the given information on f, a, b, and n.

2. **The Right Rectangular Method**

   a. Describe the right rectangular method by analogy with the left one, following similar steps. (Denote by R_n the Riemann sum corresponding to the right rectangular method with n rectangular tiles.)

   b. Is there a link between the trapezoidal method and the left and right rectangular methods? Explain.

   c. As n becomes very large, which approximate values for the area do you think become more accurate, T_n, L_n, or R_n?

3. **The Midpoint Rectangular Method:** Describe this method by analogy with the left and right rectangular methods, following similar steps. (Denote by M_n the Riemann sum corresponding to the midpoint rectangular method with n rectangular tiles.)

4. **Generalization**

   a. How can you describe *simultaneously* the left, the right, and the midpoint methods? Your description should include all three methods as particular cases.

   b. Based on this generalization, describe what needs to be done in order for the Riemann sums to be lower/upper bounds for the area of the irregular shape. For instance, how should the heights of the rectangular tiles be chosen? ◆

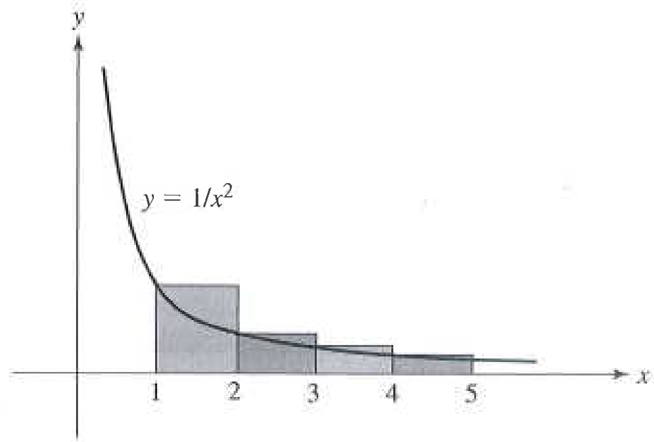
EXAMPLE Let $\mathcal{A}$ be the area of the plane region bounded by the graph of $f(x) = 1/x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 5$; let L_n and R_n be the areas of the polygonal regions with n rectangular tiles as described in the left and right rectangular methods, respectively.

1. Compute by hand L_4 and L_8. How do these values compare to the (unknown) value of the area $\mathcal{A}$?
2. Compute by hand R_4 and R_8. How do these values compare to the area $\mathcal{A}$?
3. Using your calculator program from Classroom Discussion 6.2.2, compute L_n and R_n for $n = 10, 20, \ldots, 200$. Tabulate your results.
4. Among the integers n in your table, determine, if possible, the smallest one that allows you to find an approximate value for $\mathcal{A}$ with an error of magnitude less than 10^{-1}, 10^{-2}, and 10^{-3}, respectively.

Solution

1. The area L_4 is obtained by applying the formula (2) or by directly computing the area of each of the rectangular tiles involved.

$$L_4 = \frac{5-1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right) \approx 1.42361.$$



In Problem 2a, show that

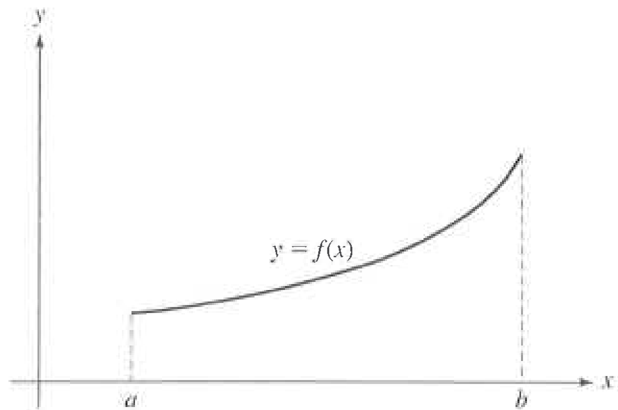
$$R_n = \frac{1}{n^4}\left(1^3 + 2^3 + \cdots + n^3\right).$$

In Problem 2b, consider the identity $(k + 1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$ and use similar ideas to derive a simple expression for R_n.

3. For which other functions do you think a reasoning along the same lines would work? Explain.

III. Riemann Sums for Increasing and Decreasing Functions

Consider the plane region bounded by the graph $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, where f is an increasing, continuous, and nonnegative function defined on the interval $[a, b]$.



1. Sketch the rectangular tiles approximating the given region, as described in the left rectangular method, with $n = 5$.
2. Sketch the rectangular tiles approximating the given region, as described in the right rectangular method, with $n = 5$.
3. Shade the gap that is between the region formed of all the rectangles obtained in 1 and the one formed of all those obtained in 2. What does the area of this gap represent?
4. Stack up all of the rectangles forming the shaded region. What are the height and width of the resulting rectangle?
5. What would be the height and width of the resulting rectangle if $n = 6, 7, \ldots$?
6. What happens to the area of the rectangle in 4 as $n \to \infty$?
7. Use your findings to convince your classmates that the limit of the Riemann sums is indeed the area of the region below the graph $y = f(x)$ and above the x-axis.
8. Does this reasoning still work if the function $f(x)$ is decreasing instead of increasing?

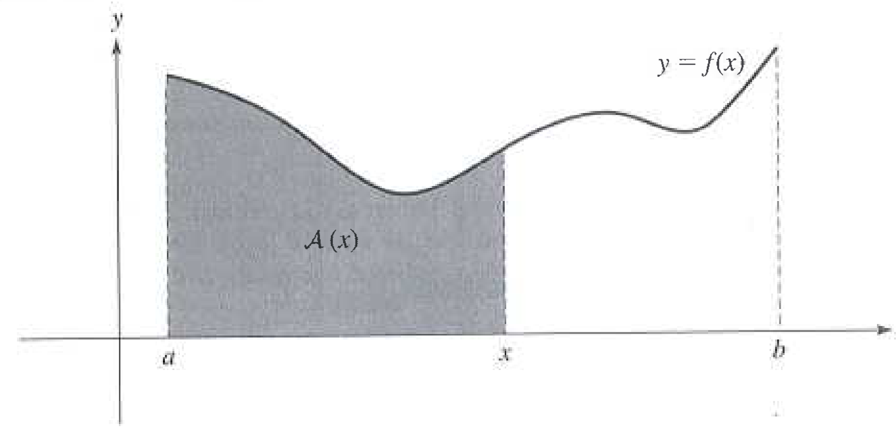
6.3 EXACT VALUE OF THE AREA OF AN IRREGULAR SHAPE

Fundamental Theorem of Calculus • Areas of regions bounded by graphs

Consider the graph $y = f(x)$ where $f(x)$ is a continuous function defined on the interval $[a, b]$. Our task in this section is to find the exact value of the area $\mathcal{A}$ of the region of the plane bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

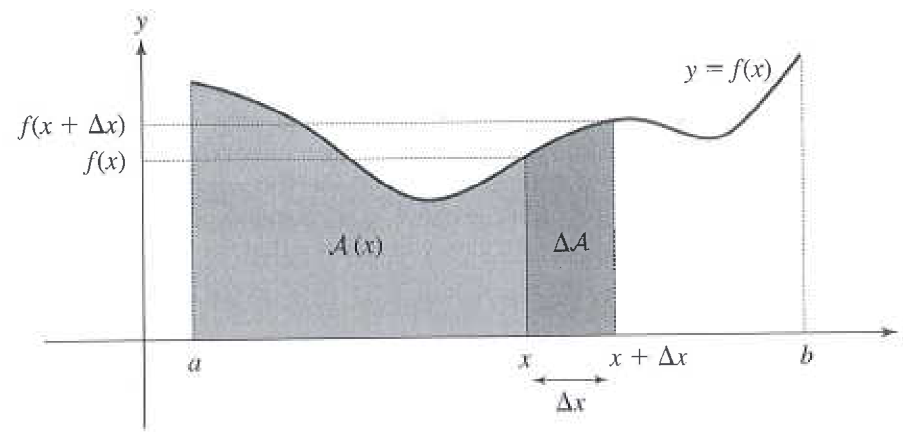
Classroom Discussion 6.3.1: The Fundamental Theorem of Calculus

Throughout this Classroom Discussion, the function f is assumed to be nonnegative. Fix $a \le x \le b$ and denote by $\mathcal{A}(x)$ the area of the region in the plane bounded by the graph of f, the x-axis, and the vertical lines passing through points $(a, 0)$ and $(x, 0)$. The function $\mathcal{A}(x)$ is sometimes called the *area-so-far* function.



In order to familiarize yourself with the function $\mathcal{A}(x)$, answer the following two questions.

1. What are the values of $\mathcal{A}(a)$, $\mathcal{A}(\frac{a+b}{2})$, and $\mathcal{A}(b)$?
2. Is the function $\mathcal{A}(x)$ increasing, decreasing, or neither? Explain.



Let x increase by a small amount Δx; then the area $\mathcal{A}(x)$ increases by an amount $\Delta\mathcal{A}$ corresponding to the additional region.

3. If Δx is very small, which polygonal region would you suggest to approximate this additional region?
4. Express the corresponding approximation for $\Delta\mathcal{A}$ in terms of Δx, $f(x)$, and $f(x + \Delta x)$.
5. Use your findings from Problem 4 to approximate the average rate of change $\dfrac{\Delta\mathcal{A}}{\Delta x}$.
6. Evaluate $\lim\limits_{\Delta x \to 0} f(x + \Delta x)$. Recall that $f(x)$ is continuous over the interval $[a, b]$.
7. Show that $\lim\limits_{\Delta x \to 0} \dfrac{\Delta\mathcal{A}}{\Delta x} = f(x)$.
8. What can you say about the derivative $\mathcal{A}'(x)$?
9. Express $\mathcal{A}(x)$ as a definite integral.
10. Evaluate the area $\mathcal{A}$ of the plane region below the graph of f and above the x-axis. ◆

Theorem 6.3.1 (The Fundamental Theorem of Calculus). *Let $f(x)$ be a nonnegative and continuous function defined on the interval $[a, b]$. Denote by $\mathcal{A}(x)$ the area of the plane region bounded by the graph of f, the x-axis, and the vertical lines through points $(a, f(a))$ and $(x, f(x))$. Then,*

$$\mathcal{A}'(x) = f(x), \text{ for all } x \text{ in } [a, b]. \tag{3}$$

Therefore,

$$\mathcal{A}(x) = \int_a^x f(t)\,dt, \text{ for all } x \text{ in } [a, b]. \tag{4}$$

In particular, the area of the plane region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

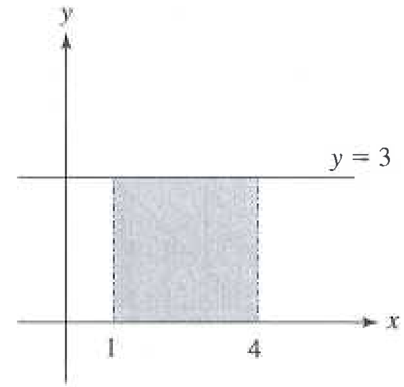
$$\mathcal{A}(b) = \int_a^b f(t)\,dt. \tag{5}$$

Recall that the Riemann sums for a nonnegative and continuous function f defined on an interval $[a, b]$ (when using a large number of tiles) provide approximate values for the area $\mathcal{A}$ of the region that is below the graph of f and above the x-axis. Their limit, as the number of tiles goes to ∞, coincides with the area $\mathcal{A}$. Using the Fundamental Theorem of Calculus, you now see that the Riemann sums for f, when using a large number of tiles, provide approximate values for the definite integral $\int_a^b f(x)\,dx$. Their limit, as the number of tiles goes to ∞, coincides with $\int_a^b f(x)\,dx$.

Historical Note: Sir Isaac Newton (1643–1727; from England)

While still under 25 years old, Newton made revolutionary advances in mathematics, physics, optics, and astronomy. He laid the foundations for differential and integral calculus several years before its independent discovery by Leibniz. Newton's *De Methodis Serierum et Fluxionum* ("On the Methods of Series and Fluxions"), the first book on calculus, was written in 1671, but it did not appear in print until 1736. His later work *Principia* is considered to be the greatest scientific book ever written. In 1705, Newton was knighted by Queen Anne; he was the first scientist ever to receive such an honor for his work.

EXAMPLE Compute the area of the plane region bounded by the graph of $f(x) = 3$, the x-axis, and the vertical lines $x = 1$ and $x = 4$, first using the Fundamental Theorem of Calculus, and then using the area formulas obtained in Section 6.1.



Solution Using the Fundamental Theorem of Calculus, the area of the shaded region is

$$\int_1^4 3\,dx = 3x\Big|_1^4 = (12 - 3) = 9.$$

On the other hand, the shaded region is a square whose sides each have length 3. Using the formula for the area of a square, its area is $3^2 = 9$. ■

Practice Problems

1. Compute the area of the plane region bounded by the graph of $f(x) = 2x$, the x-axis, and the vertical line $x = 2$, first using the Fundamental Theorem of Calculus, and then using the area formulas obtained in Section 6.1.

For Questions 30–31, use 3.14 or the π key on a calculator.

30 Try This as a Class Use your figure from Question 24.

a. Use $A = \pi r^2$ to write an expression that represents the exact area of the *Circle*.

b. Find the approximate area of the *Circle*.

c. If you were planning to make a circle kite with a 4 cm radius, would you use your answer from part (a) or from part (b) to order the material? Why?

d. How does the area of the circle from part (b) compare with the estimated area of the figure in Question 26(a)?

31 ✓ CHECKPOINT A centipede kite has 10 in. diameter circles.

a. Find the exact area of one circle.

b. Find the approximate area of one circle.

c. About how many square inches of silk were used to make all 11 circles of the kite?

✓ QUESTION 31

...checks that you can find the area of a circle.

HOMEWORK EXERCISES See Exs. 16–23 on p. 407.

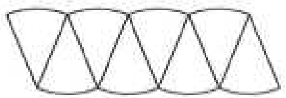
Name Date

MODULE 6 **LABSHEET 2A**

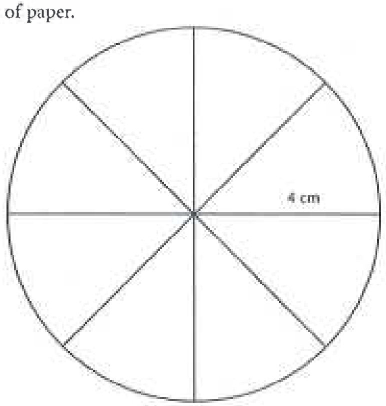
Circle (Use with Question 24 on page 402.)

Directions

- Cut out the circle.
- Cut apart the eight sectors and arrange them to form the figure shown below.



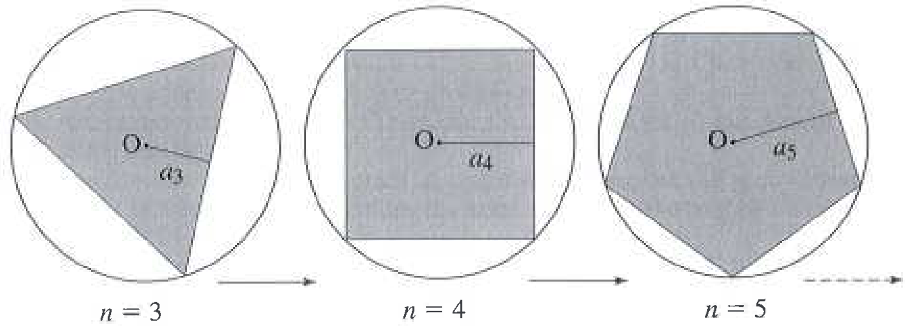
- Tape the figure to a sheet of paper.



Classroom Discussion 6.4.1: Approximating the Circle by Inscribed Regular *n*-gons

To find the area of a disc, we use here the *method of exhaustion*, which was invented by Eudoxus (similar ideas may be found in the project "Archimedes's Computation of π" in Section 1.1). To see how the method works, use the following outline.

1. For each $n \geq 3$, let a_n and p_n denote the apothem and the perimeter of an inscribed regular n-gon, respectively. How can you express the area $\mathcal{A}_n$ inside the n-gon in terms of a_n and p_n?



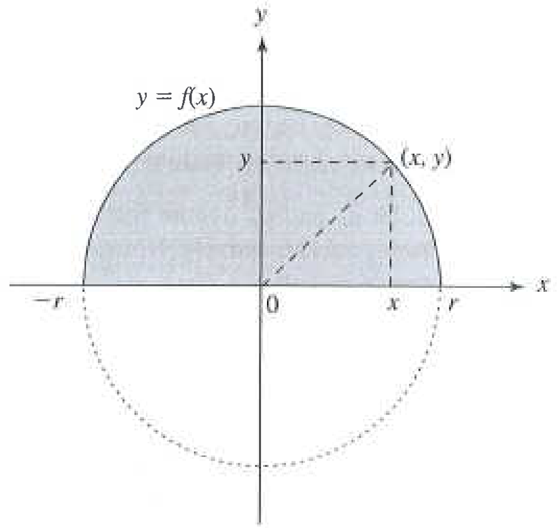
2. What happens to the n-gons as n increases? What happens to the values of a_n, p_n, and $\mathcal{A}_n$ as n increases? What can you say about their limits as $n \longrightarrow \infty$?
3. Using your findings in Problems 1 and 2, find the area $\mathcal{A}$ of the disc.
4. Why do you think this method is called the *method of exhaustion*? ◆

Historical Note: Eudoxus of Cnidus (408–355 BC; from Asia Minor [now Turkey])

Eudoxus, a contemporary of Plato, had a rich and varied academic background in mathematics, music, medicine, astronomy, theology, and meteorology. Early in his career, he developed a theory of *proportion*, which appears in Euclid's *Elements* and facilitated his early work on finding areas. Eudoxus introduced the *method of exhaustion*, which led to important developments in calculus by Archimedes and others; Eudoxus himself was the first to prove that a cone's volume is one-third the volume of a cylinder having the same base and equal height.

Classroom Discussion 6.4.2: Area of a Disc Using the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus allows us to find the area of a region bounded by the graph of a function and the x-axis. The disc is not such a region, but since it has a reflectional symmetry about the x-axis, its area is twice that of the upper semidisc. The upper semidisc is a region for which the Fundamental Theorem of Calculus applies.



1. Use the Pythagorean Theorem to find a relation between x and y for the point (x, y) to lie on the circle centered at the origin with radius r.
2. What additional constraint must be placed on y for the point (x, y) to lie on the upper semicircle?
3. Solve for y in your equation from Problem 1 to obtain the equation of the upper semicircle.
4. Use the Fundamental Theorem of Calculus to express the disc's area as a definite integral.
5. (T) Compute the area A by using your calculator to evaluate the definite integral.

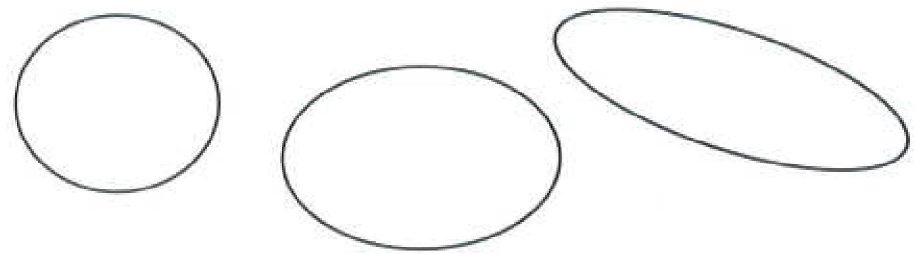
◆

EXERCISES 6.4

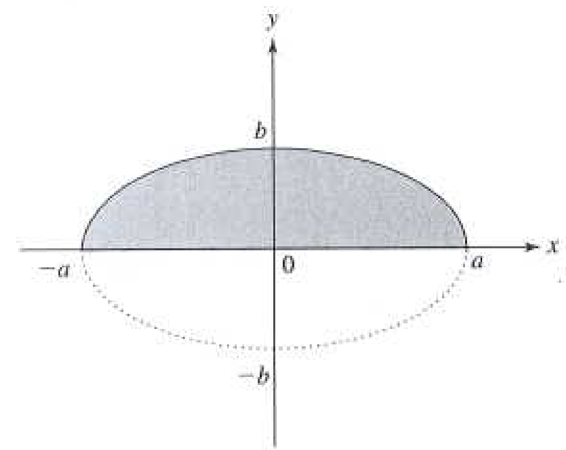
1. Write a paragraph about the Fundamental Theorem of Calculus. In particular, write about your understanding of this theorem, and why you think it is (or is not) useful.
2. In Classroom Connection 6.4.1, we used the formula $A_1 = 3r^2$ to estimate the area A of a disc of radius r. In the old Babylonian civilization, the formula $A_2 = \frac{C^2}{12}$ was used for the area A inside a circle of circumference C.
   a. Express A_2 in terms of r using the classical formula for the circumference of a circle.
   b. Express the errors $A_1 - A$ and $A_2 - A$ in terms of r using the formula obtained for A in Section 6.4. Interpret the results.
   c. Evaluate the relative error in each estimate. The *relative error*, when estimating an exact value v with an approximate value v_0, is the quantity $\frac{v_0 - v}{v} = \frac{v_0}{v} - 1$.
   d. Evaluate the percentage error in each estimate. The *percentage error* in an estimate is 100% times the relative error in the estimate.

The goal of the following problems is to use what you have already done in the case of a disc to find the area of the region inside an ellipse.

3. What is an ellipse? Research its definition and its Cartesian equation.



4. Can the methods in Classroom Connection 6.4.2 and in Classroom Discussion 6.4.1 be adapted to ellipses? Why or why not?
5. Adapt the method used in Classroom Connection 6.4.1 to the case of a region inside an ellipse.
6. Use the Fundamental Theorem of Calculus to compute the area of the region inside an ellipse. To do so, follow the same steps as in the case of a circle.



PROJECTS AND EXTENSIONS 6.4

I. History of Finding Areas

The problem of finding areas of plane regions has a rich and fascinating history, which started with the early Greek philosophers and continued for thousands of years afterward. Research the history of this problem and write a detailed report describing, in chronological order, the main advances that were made on the problem. Be sure to include biographical information about the mathematicians responsible for these advances.

II. Leibniz vs. Newton

The controversy over who discovered calculus, Leibniz or Newton, caused bitter disputes among their followers for many years. Eventually, both mathematicians

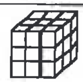
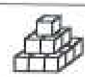
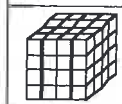
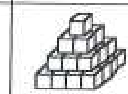
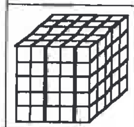
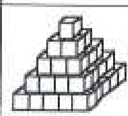
Name Date

MODULE 6 **LABSHEET 3B**

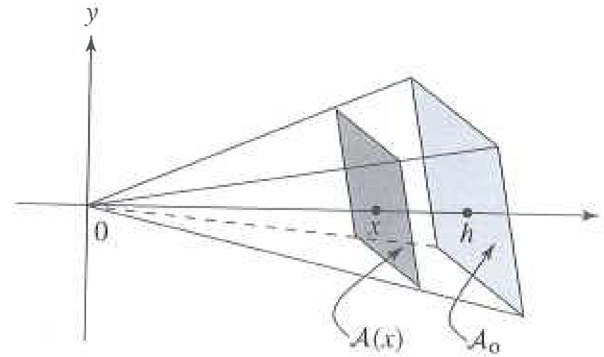
Table of Volumes (Use with Questions 12–14 on page 423.)

Directions

- Find the height, the area of the base, and the volume of each block prism in the table.
- The block pyramids have the same heights and bases as the corresponding prisms. Complete the block pyramid portion of the table.
- For each row, find the ratio of the volume of the block pyramid to the volume of the prism. In the last column, write each answer in decimal form and round to the nearest thousandth.

Block Prisms				Block Pyramids				Volume Ratio
	Height h	Area of base B	Volume of prism $V = B \cdot h$		Height h	Area of base B	Volume of block pyramid	Volume of block pyramid ÷ Volume of prism
	1	1	1		1	1	1	1 ÷ 1 = 1.000
	2	4	8		2	4	5	5 ÷ 8 = 0.625
								
								
								
	6							
	7							
	8							
	9							
	10							

3. Volumes of Pyramids Using Calculus



a. Explain why, when computing a pyramid's volume, the pyramid can be assumed to be right without loss of generality.

b. Using similarity, express the area $\mathcal{A}(x)$ of the cross-section corresponding to x in terms of x, the height h, and the base's area $\mathcal{A}_0$.

c. Use Theorem 7.3.1 to determine the volume of the given pyramid. Compare this value with the volume of a prism that has the same base area and height as the pyramid. ◆

Classroom Discussion 7.3.5: Volumes of Spheres

The goal is to find the volume of a sphere of radius $r > 0$. We present three different approaches: one for middle-school students based on comparing the volumes of spheres and cylinders, one for high school students based on approximating the sphere with tiny "pyramids," and one for college students based on calculus.

1. Classroom Connection 7.3.4: Volumes of Spheres and Cylinders

The following exploration is taken from pages 47–48 in the seventh-grade textbook *Connected Mathematics, Filling and Wrapping*. Discuss it in small groups. ◆

Although spheres may differ in size, they are all the same shape. We can describe a sphere by giving its radius.

In this investigation, you will explore ways to determine the volume of cones and spheres.

Comparing Spheres and Cylinders

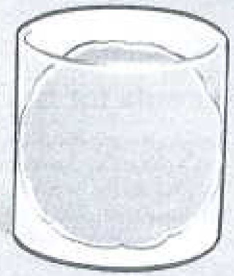
In this problem, you will make a sphere and a cylinder with the same radius and height and then compare their volumes. (The "height" of a sphere is just its diameter.) You can use the relationship you observe to help you develop a method for finding the volume of a sphere.

Did you know?

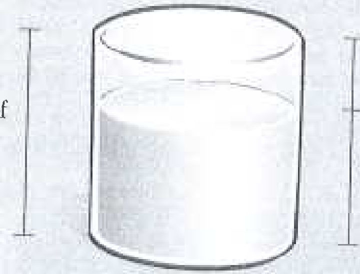
The Earth is nearly a sphere. You may have heard that, until Christopher Columbus's voyage in 1492, most people believed the Earth was flat. Actually, as early as the fourth century B.C., scientists in Greece and Egypt had figured out that the Earth was round. They observed the shadow of the Earth as it passed across the Moon during a lunar eclipse. It was clear that the shadow was round. Combining this observation with evidence gathered from observing constellations, these scientists concluded that the Earth was indeed spherical. In fact, in the third century B.C., Eratosthenes, a scientist from Alexandria, Egypt, was actually able to estimate the circumference of the Earth.

Problem 5.1

- Using modeling dough, make a sphere with a diameter between 2 inches and 3.5 inches.
- Using a strip of transparent plastic, make a cylinder with an open top and bottom that fits snugly around your sphere. Trim the height of the cylinder to match the height of the sphere. Tape the cylinder together so that it remains rigid.



- Now, flatten the sphere so that it fits snugly in the bottom of the cylinder. Mark the height of the flattened sphere on the cylinder.



A. Measure and record the height of the cylinder, the height of the empty space, and the height of the flattened sphere.

B. What is the relationship between the volume of the sphere and the volume of the cylinder?

Remove the modeling dough from the cylinder, and save the cylinder for the next problem.

Problem 5.1 Follow-Up

Compare your results with the results of a group that made a larger or smaller sphere. Did the other group find the same relationship between the volume of the sphere and the volume of the cylinder?

2. Classroom Connection 7.3.5: Volume of a Sphere Using Tiny "Pyramids"

The following exploration is taken from pages 546–547 in the textbook *Discovering Geometry*, third edition, by Michael Serra. In this exploration, the sphere's volume is known, and it is used to find its surface area. However, a similar reasoning works to find the sphere's volume if its surface area is known. Replace Steps 3 and 4 in the exploration with Step 3 here, and discuss it in small groups.

Step 3: The sphere's surface area is $S = 4\pi r^2$. Use the equation in Step 2 to find the sphere's volume V. ◆

Investigation

The Formula for the Surface Area of a Sphere

In this investigation you'll visualize a sphere's surface covered by tiny shapes that are nearly flat. So the surface area, S, of the sphere is the sum of the areas of all the "nearly polygons." If you imagine radii connecting each of the vertices of the "nearly polygons" to the center of the sphere, you are mentally dividing the volume of the sphere into many "nearly pyramids." Each of the "nearly polygons" is a base for a pyramid, and the radius, r, of the sphere is the height of the pyramid. So the volume, V, of the sphere is the sum of the volumes of all the pyramids. Now get ready for some algebra.

Step 1 Divide the surface of the sphere into 1000 "nearly polygons" with areas $B_1, B_2, B_3, \ldots, B_{1000}$. Then you can write the surface area, S, of the sphere as the sum of the 1000 B's:

$$S = B_1 + B_2 + B_3 + \ldots + B_{1000}$$

Step 2 The volume of the pyramid with base B_1 is $\frac{1}{3}(B_1)(r)$, so the total volume of the sphere, V, is the sum of the volumes of the 1000 pyramids:

$$V = \frac{1}{3}(B_1)(r) + \frac{1}{3}(B_2)(r) + \ldots + \frac{1}{3}(B_{1000})(r)$$

What common expression can you factor from each of the terms on the right side? Rewrite the last equation showing your factoring.

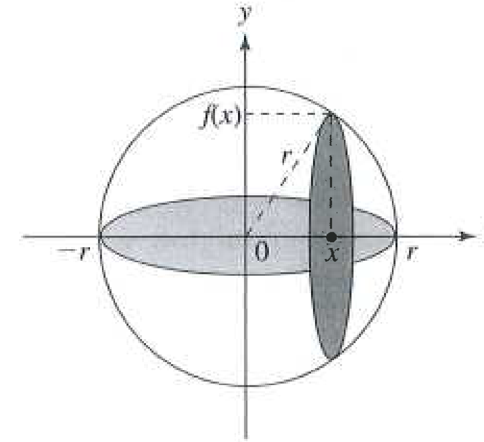
Step 3 But the volume of the sphere is $V = \frac{4}{3}\pi r^3$. Rewrite your equation from Step 2 by substituting $\frac{4}{3}\pi r^3$ for V and substituting for S the sum of the areas of all the "nearly polygons."

Step 4 Solve the equation from Step 3 for the surface area, S. You now have a formula for finding the surface area of a sphere in terms of its radius. State this as your next conjecture and add it to your conjecture list.

Sphere Surface Area Conjecture C-91

The surface area, S, of a sphere with radius r is given by the formula __?__.

3. Volumes of Spheres Using Calculus

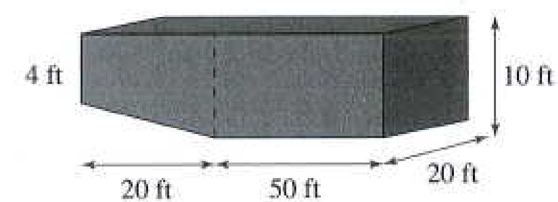


a. For each $-r \leq x \leq r$, describe the cross-section corresponding to x, then express its area $A(x)$ in terms of x and r.

b. Use Theorem 7.3.1 to determine the volume of the given sphere. ◆

EXERCISES 7.3

1. A swimming pool is in the shape of a right prism as in the following figure. How many cubic feet of water can this swimming pool hold?



2. **Classroom Connection 7.3.6: Revisiting Cylinders**

The following exploration is taken from pages 502–503, page 7-56, and page 7-57 in the sixth-grade textbook *Math Thematics, Book 1*. Answer the questions therein. ◆

GOAL

LEARN HOW TO...
- recognize a cylinder
- find the volume of a cylinder

AS YOU...
- explore the size and shape of a kiva

KEY TERM
- cylinder

Exploration 2

Volume of a Cylinder

SET UP *Work in a group of three. You will need:*
- *Labsheets 5B and 5C* • *scissors* • *tape* • *rice* • *ruler*

In the summer of 1891, Gustaf Nordenskiöld of Sweden and his team began to uncover the ruins at Mesa Verde. Part of their task was to remove the layers of dust and rubbish that had piled up over the centuries. After digging to a depth of $\frac{1}{2}$ m at one location, they began to see a kiva take shape.

12 How do you think Nordenskiöld could have estimated the amount of dust and rubbish in the kiva without removing it?

A kiva is shaped like a circular **cylinder**. A circular cylinder is a space figure that has two circular bases that are parallel and congruent.

The bases are parallel and congruent.

13 **Use Labsheets 5B and 5C.** Cut out the nets for the open-topped *Prism A, Prism B and Cylinder.* Fold and tape each net.

14 How is the cylinder like a prism? How is it different?

15 Which has a larger volume, prism A or prism B? Explain.

16 Which do you think holds more, the cylinder or prism A? the cylinder or prism B? Explain your thinking in each case.

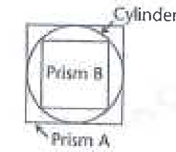
17 a. Fill prism B with rice and then pour the rice into the cylinder. Does the rice completely fill the cylinder, or is there too much or not enough rice?

b. Fill the cylinder with rice and then pour the rice into prism A. Does the rice completely fill prism A?

c. What can you conclude about the volume of the cylinder?

18 a. Place the cylinder inside the larger prism. Then place the smaller prism inside the cylinder.

b. For each of the prisms and the cylinder, find the area of a base and the height. Make a table to record your results.



top view

19 **Discussion** Add on to the table you completed in Question 18.

a. Find the volumes of prism A and prism B. Explain your method.

b. Use the same method you used in part (a) to find the volume of the cylinder.

c. Use your models and your results with rice to decide whether the volume you found for the cylinder is reasonable.

▶ **You can find the volume V of a cylinder with height h and a base with area B in the same way you find the volume of a prism.**

$V = Bh$, or $V = \pi r^2 h$.

area of circular base

EXAMPLE

Find the volume of the cylinder shown to the nearest cubic centimeter. Use 3.14 for π.

SAMPLE RESPONSE

$V = \pi r^2 h$

$\approx 3.14 \cdot 4^2 \cdot 5.3 = 266.272$

The volume is about 266 cm^3.

Volume is measured in cubic units.

20 **✓ CHECKPOINT** Find the volume of the cylinder to the nearest cubic meter. Use 3.14 for π.

✓ QUESTION 20

...checks that you can find the volume of a cylinder.

21 Gustaf Nordenskiöld reported that one of the kivas he uncovered had walls 2 m high with a diameter of 4.3 m. If this kiva was completely full of dust and rubbish, about how much material did Nordenskiöld have to remove?

HOMEWORK EXERCISES ▶ See Exs. 14–22 on p. 506.