Problem 4.1

A. If you were playing Deep in the Dungeon, in which room would you put the treasure in order to have the best chance of beating Zark? Explain your choice.

B. Work with a partner to find a way to simulate Deep in the Dungeon so it can be played without a computer. Your simulation should be a two-person game. One person should hide the treasure, and the other should play the role of Zark. You will need to figure out a way for Zark to make a random selection at each fork.

C. Play your simulation of Deep in the Dungeon 20 times with your partner. Take turns hiding the treasure and playing Zark. For each game, record the room that Zark ends in.

D. Based on your results from part C, what is the experimental probability that Zark will end in room A? What is the experimental probability that Zark will end in room B?

FIGURE 5.5.2 Simulating multistage events

This *Classroom Connection* is a very interesting one that you can really spend a lot of time exploring. A nice follow-up activity is to create your own dungeon and then work through the prompts (parts A through D) again. Some people will create very elaborate dungeons that require a lot of thought in order to simulate the game and find the corresponding probabilities for the various rooms. ◆

Focus on Understanding

The *Classroom Connection* in Figure 5.5.2 focused on finding the experimental probability that a player would end up either in room A or room B. We can use the general multiplication rule to find the theoretical probability of ending up in either room A or room B.

1. Determine the theoretical probability of taking each of the paths at each fork. (You have probably already worked this out when you created your simulation.)
2. Use the general multiplication rule to determine the theoretical probability that a player would end up in room A.
3. Use the general multiplication rule to determine the theoretical probability that a player would end up in room B.
4. Using the complement rule, find the probability that a player would NOT end up in room A. Is this probability the same as the probability that a player would end up in room B? Why or why not?
5. Carmen believes that she has a shortcut to find the theoretical probability that a player will end up in room A. She says that since there are six paths total that end up at room A or B, and since three of them lead to room A, then the probability that a player would end up in room A must be $\frac{3}{6}$. Is Carmen correct? Why or why not?

5.6 GEOMETRIC PROBABILITY

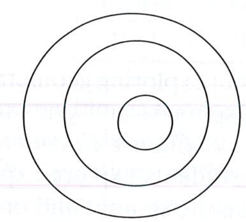
More and more middle school and secondary school mathematics textbooks are paying attention to what are sometimes called **geometric probabilities**. Geometric probabilities are typically represented through some type of geometric figure like a circle (spinners) or a rectangular grid. Different areas in the figure represent the probabilities associated with particular outcomes of an experiment. Geometric probabilities may also be represented by lengths or by volumes, though area models are the most common in most curricula. We defined simple probability earlier as:

$$\text{Probability} = \frac{\textit{number of favorable outcomes}}{\textit{number of possible outcomes}}$$

The idea is the same for geometric probability, but instead of talking about the number of outcomes we use the length, area, or volume associated with a particular outcome. We will focus on area models since those are more typical:

$$\text{Geometric Probability} = \frac{\textit{area associated with a favorable outcome}}{\textit{area for all the outcomes}}$$

EXAMPLE 5.6.1 One of the classic examples of geometric probability is that of finding the probability that a dart will hit a particular ring on a target. Such a target is shown below with various measurements given:



Radius of outside circle = 6 inches
Radius of middle circle = 4 inches
Radius of inner circle = 2 inches

What is the probability of a dart landing in the inner circle? In order to find this probability we need to compare the favorable area (that of the inner circle) to the total area of the circle. In this case, since the radius of the inner circle is 2 inches, we know that its area is $\pi \cdot (2\ \textit{inches})^2$ or 4π squared inches. The area of the whole circle would be $\pi \cdot (6\ \textit{inches})^2$ or 36π squared inches. Hold on a second before you hyperventilate about π showing up in our geometric probability. Take a look at what happens here:

$$\text{Geometric Probability} = \frac{\textit{area associated with a favorable outcome}}{\textit{area of all the outcomes}}$$

So in this case we get:

$$\text{Probability of a dart landing in the inner circle} = \frac{\textit{area of inner circle}}{\textit{area of entire circle}}$$

$$= \frac{4\pi\ \textit{square inches}}{36\pi\ \textit{square inches}} = \frac{1}{9}$$

By simplifying this ratio we not only get rid of the π, but our units of measurement (the square inches) "cancel" each other out. Probability is ***unit-less***. ■

EXAMPLE 5.6.2 See if you can find the probability that a dart would land in the middle ring. This is a little trickier, since the middle ring is really a ring and not a complete circle. Take a minute and see what you get before reading our solution. We'll wait for you.

Okay, to find the area of the middle ring we must first find the area of a circle of radius 4 inches (the complete circle) and then subtract out the area of the inner circle. You should get something like this:

$$\text{Area of middle ring} = 16\pi \text{ square inches} - 4\pi \text{ square inches} = 12\pi \text{ square inches}$$

The probability, then, of a dart landing in the middle ring would be:

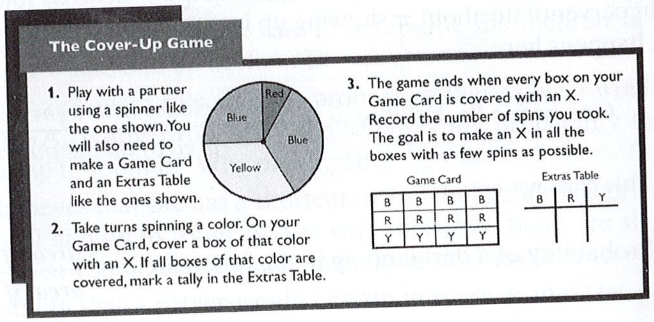
$$\text{Probability of a dart landing in the middle ring} = \frac{\textit{area of middle ring}}{\textit{area of entire circle}} = \frac{12\pi \textit{ square inches}}{36\pi \textit{ square inches}} = \frac{1}{3}$$

You may be intensely bothered by the fact that our circles were not concentric. Does that really matter? Nope. The principle is the same whether our circles are concentric or not! ■

Classroom Connection

Generally, middle school students start out exploring geometric probabilities in a less measurement-intensive situation using spinners and grids to illustrate the various outcomes. Figure 5.6.1 is from *Mathscape: Chance Encounters* (page 14). Spinners and grids are used in this case to have students explore experimental probabilities and the connection between the areas on the spinner and the areas on the grid.

In this activity, middle school students are asked to analyze the results of their spins and design a new game card that minimizes the number of spins required to fill all the boxes. Students use the experimental probabilities from their spins to design a new game card. ◆



The Cover-Up Game

1. Play with a partner using a spinner like the one shown. You will also need to make a Game Card and an Extras Table like the ones shown.
2. Take turns spinning a color. On your Game Card, cover a box of that color with an X. If all boxes of that color are covered, mark a tally in the Extras Table.
3. The game ends when every box on your Game Card is covered with an X. Record the number of spins you took. The goal is to make an X in all the boxes with as few spins as possible.

Game Card

B	B	B	B
R	R	R	R
Y	Y	Y	Y

Extras Table

B	R	Y

FIGURE 5.6.1 The Cover-Up Game

Focus on Understanding

Take a careful look at the spinner in Figure 5.6.1 and the game card shown.

1. Based on the spinner, which color do you think will come up most often? Why? Which color will come up the least? Why?
2. What do you think will happen if the students spin the spinner 12 times? 24 times? Why?
3. How would you design the game card so that in order to completely cover the card you would take fewer spins?
4. How would you change the spinner, if you had to keep the game card the same?

This excerpt is an interesting one because students are really asked to connect the idea of areas on a spinner or circular model to that of areas on a grid. Suppose a student drew a new game card as follows:

Game Card

Blue	Red	Yellow	Red
Yellow	Yellow	Red	Blue
Red	Blue	Blue	Yellow

Is this the same as the game card in Figure 5.6.1? Why or why not? (As you have probably already determined, this new game card is equivalent to the one in Figure 5.6.1.) The "position" of the colors on the card does not matter in this case. It is the area (probability) they represent on both the game card and the spinner that matters.

Classroom Connection

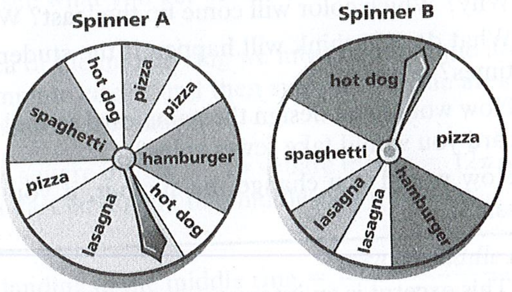
Our next *Classroom Connection* shown in Figure 5.6.2 is from *Connected Mathematics: How Likely Is It?* (page 46). It, too, has middle school students work with spinners to determine the likelihood of certain outcomes. ◆

Focus on Understanding

Use the two spinners pictured in Figure 5.6.2 for the following:

1. Answer the questions posed to the students in Figure 5.6.2. Be sure to explain your reasoning!
2. Are there other ways that you could draw the two spinners but maintain the same probabilities? Draw at least one spinner that is equivalent to Spinner A and at least one spinner that is equivalent to Spinner B.

11. The cooks at Kyla's school made the spinners shown below to help them determine the lunch menu. They let the students take turns spinning to determine the daily menu. In a–c, decide which spinner you would choose, and explain your reasoning.



a. Your favorite lunch is pizza.

b. Your favorite lunch is lasagna.

c. Your favorite lunch is hot dogs.

FIGURE 5.6.2 Lunch spinners

3. How would you go about representing Spinner A and Spinner B using a grid or a game card as in the previous Classroom Connection? Create grids that would approximate Spinner A and Spinner B.

Classroom Connection

While spinners and grids are pretty common ways to illustrate geometric probabilities, there are also more geometry/measurement "intensive" ways. Take a look at the practice and application exercise in Figure 5.6.3 from *MathThematics, Book 1* (page 569).

Each of these problems in the portion of the exercise set depicted here expects the student to compute various areas and find the probabilities of certain outcomes based on those areas. Though it's more arithmetic intensive perhaps than the spinners and grids of the previous two examples, the idea is the same. ◆

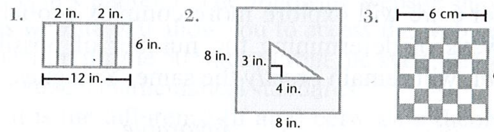
Focus on Understanding

Time to brush up your measurement skills from geometry! Use Figure 5.6.3 for the following:

1. Read through each of the practice and application exercises in the figure. What prior knowledge do you think students (or you) need in order to answer the questions?
2. Which exercises do you think students would find the most difficult to do? Why? Which ones do you think students would find the easiest to do? Why?

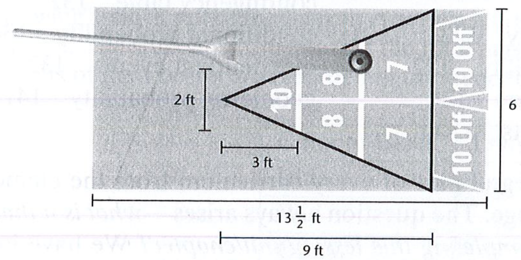
Section 4 Practice & Application Exercises

Suppose an object falls at random on to each target shown below. For each target, what is the probability the object will land in a shaded region?

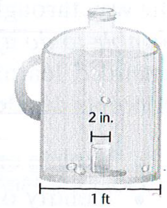


Shuffleboard In a game of shuffleboard players take turns sliding plastic disks onto a scoring area. Players gain or lose the number of points marked on the space their disk is on.

4. If you randomly slide your disk so that it lands somewhere on the court shown, what is the probability that it will land within the triangle that is outlined in black?
5. If you randomly slide your disk so that it lands within the black triangle, what is the probability that you will score 10 points?



6. At a fair there is a jug filled with water with a small glass at the bottom. To win a prize you must drop a quarter into the jug and have it land in the glass. If the quarter falls randomly to the bottom, what is the probability of winning a prize?



7. **Create Your Own** Draw and shade a target. Include at least two different geometric shapes in your design. Find the probability of a dart hitting a shaded part of your target.

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FIGURE 5.6.3 More geometric probabilities

3. Complete the exercises shown in the figure. In the case of exercise 1, the shaded region is the area of the two vertical bars within the rectangle that are marked as having widths of 2 inches. In exercise 2, the shaded region is the area of the square outside of the triangle.

4. Which exercises did you find to be the most difficult? Why? Which ones were the easiest for you? Why?

Chapter 5 Summary

This chapter has introduced you to the fundamental concepts of simulations and simple probability—both theoretical and experimental. While there are many,

EXAMPLE 8.2.1 Change for a Dollar?

x	$P(x)$
1	0.8
5	0.2

When our son gets pennies and nickels in change, he usually doesn't spend them, but lets them pile up in a large container in his room (the population). Suppose that 80% of these coins are pennies and 20% are nickels. If x = *the value of a coin* in this population, the probability distribution for x is given at the right. Let's start by finding the mean of this population.

$$\mu = \sum [x \cdot P(x)]$$
$$= 1(0.8) + 5(0.2)$$
$$= 1.8$$

Suppose for the moment that we didn't know the mean of the population and were planning on estimating μ by using $\bar{x}$ from a random sample. To keep things very simple, we'll start by considering random samples of size 2. We'll also assume that the coins are chosen independently (either the coins are chosen with replacement, or there are enough coins that removing one coin does not significantly affect the probabilities for the next coin). Figure 8.2.2 shows a tree diagram for the process of selecting a random sample of two coins. For each sample, we've computed the value of $\bar{x}$. ■

There are four possible samples of size 2, corresponding to the four different paths through the tree in Figure 8.2.2. Figure 8.2.3 shows the four possible samples along with their means and probabilities. You should understand how the probabilities are computed from the tree diagram.

Based on the information in Figure 8.2.3, you should be able to determine the probability distribution of $\bar{x}$ and answer the questions in the *Focus on Understanding* that follows.

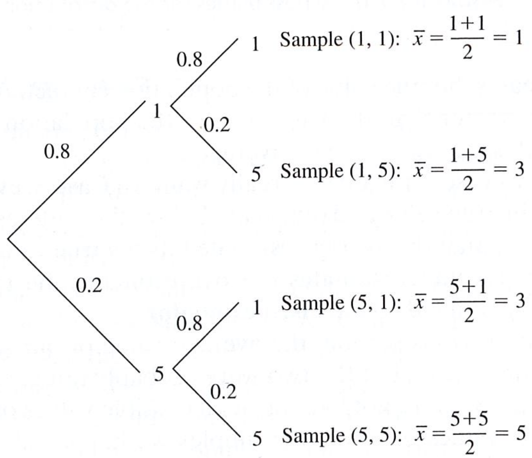


FIGURE 8.2.2 Selecting a random sample of size 2

Sample	$\bar{x}$	Probability
(1, 1)	$\frac{1+1}{2} = 1$	$(0.8)(0.8) = 0.64$
(1, 5)	$\frac{1+5}{2} = 3$	$(0.8)(0.2) = 0.16$
(5, 1)	$\frac{5+1}{2} = 3$	$(0.2)(0.8) = 0.16$
(5, 5)	$\frac{5+5}{2} = 5$	$(0.2)(0.2) = 0.04$

FIGURE 8.2.3 The four possible samples of size 2

Focus on Understanding

$\bar{x}$	$P(\bar{x})$

1. Use the information in Figure 8.2.3 to complete the table at the right for the probability distribution of $\bar{x}$.
2. Recall that the samples came from a population with $\mu = 1.8$. In this case, how likely is it for the sample mean to be the same as the population mean? In other words, what is $P(\bar{x} = \mu)$?
3. For some samples, the sample mean underestimates the population mean. Find $P(\bar{x} < \mu)$.
4. For other samples, the sample mean overestimates the population mean. Find $P(\bar{x} > \mu)$.

In this case, the sample mean is more likely to underestimate the population mean than overestimate it. Because of this, we might expect that $\bar{x}$ would tend to underestimate μ on average. Let's look at the long-term average value of $\bar{x}$ and see. To find the long-term average value of $\bar{x}$, symbolized by $E[\bar{x}]$ or $\mu_{\bar{x}}$, we find the expected value of $\bar{x}$ as a random variable, just as we did in Section 7.2.

$$E[\bar{x}] = \mu_{\bar{x}} = \sum [\bar{x} \cdot P(\bar{x})]$$
$$= 1(0.64) + 3(0.32) + 5(0.04)$$
$$= 1.8 = \mu$$

Notice that the expected value of the sample mean is the same as the mean of the population. Even though some samples underestimate the population mean and others overestimate the population mean, in the long run the overestimates and underestimates balance out. The average value of the sample mean hits the population mean exactly. The sample mean is an unbiased estimator of the population mean. This is true in general, not just in this situation, and is important enough to put in a special box like the one below.

So, if Lauren used a random sample 50 simulations to estimate the average number of points Nicky scores in one-and-one situations, she would have a 58.2% probability of getting within 0.1 of Nicky's true average ($\mu = 0.96$). I suppose that's pretty good, but that still leaves a probability of more than 40% that the estimate would be off by more than 0.1. Suppose we wanted to improve Lauren's chance of getting close to the true mean? Based on our work earlier in this section, we know that the values of $\bar{x}$ tend to get closer to μ as the sample size increases. In the following *Focus on Understanding*, you'll look at what happens if we use a larger sample.

Focus on Understanding

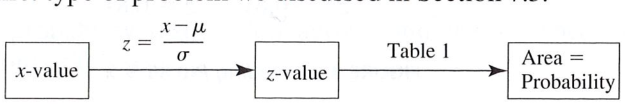
Suppose that four students each simulate Nicky's one-and-one situation 50 times, then combine their data together to get a more accurate estimate.

1. You are going to be finding the probability that the mean from their combined data will be within 0.1 of the true mean, just as we did earlier using Lauren's single sample of size 50, but before you do, decide whether this new probability will be smaller or larger than 0.582 or 58.2%, the probability we got earlier. Explain.
2. Okay, now go ahead and find the probability that the combined data will have a mean within 0.1 of the true mean μ. Is your answer consistent with what you decided in #1?

You've now seen that, by increasing the sample size, we can increase the probability that the sample mean $\bar{x}$ will be close to the population mean μ. In general, larger samples give more accurate estimates. Notice that we judge the accuracy of an estimate by looking at two different numbers:

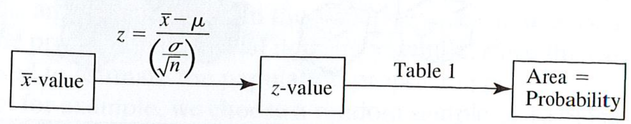
- The maximum distance that we expect between our estimate and the true value (0.1 in the example above). This distance is called the **error bound** or **margin of error**.
- The probability that our estimate is within this distance of the true value (in the example above, 58.2% when $n = 50$, 89.5% for $n = 200$). This probability is called the **confidence level**. In general, as the sample size increases, we grow more confident (as measured by the confidence level) that our estimate will be close to the true value.

Both the error bound and the confidence level will be discussed more fully in Chapter 9. You will see that the sample size, the error bound, and the confidence level are related to each other; if we know any two of these, we can determine the third. So far, we have been given the error bound and the sample size and been asked to determine the confidence level. In such a situation, we follow a procedure similar to the first type of problem we discussed in Section 7.5:



The difference is that we are dealing with the distribution of $\bar{x}$, not x, so when we compute the z-score, we need to use the mean and standard deviation of the distribution of $\bar{x}$ $\left(\mu_{\bar{x}} = \mu \text{ and } \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\right)$. Therefore, $z = \frac{\bar{x}-\mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$.

This type of problem could be diagrammed as follows:



In Chapter 9, we will be reversing this procedure, starting with the confidence level (the probability) and working from right to left in the diagram above. Before we get there, however, we need to know some things about the distribution of a sample proportion (an experimental probability). That is the subject of the next section.

8.3 THE DISTRIBUTION OF SAMPLE PROPORTIONS

Just as we use the mean of a random sample to estimate a population mean, we can use a proportion from a random sample to estimate a population proportion. In fact, that is exactly what we do whenever we use an experimental probability to approximate the true (theoretical) probability, as in the *Classroom Connection* that follows.

Classroom Connection Read through the excerpt from *Connected Mathematics: How Likely Is It?* (page 32), as shown in Figure 8.3.1.

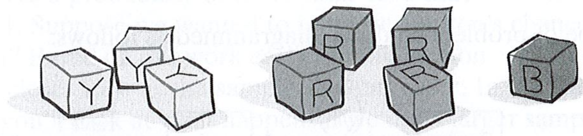
Classroom Exploration 8.3

1. Simulate drawing a random sample of size 10 (with replacement) from the bucket in *Drawing More Blocks* (Figure 8.3.1). Describe your simulation method, and give the experimental probability of drawing a yellow block, based on this single sample of size 10.
2. Suppose that you didn't know the contents of the bucket, but only the result of your sample of size 10. Based on your sample, what would be your best guess for the proportion of yellow blocks in the bucket?
3. Record your sample proportion on the board, along with those of the other students in your class.
4. Repeat #1 through #3, but with a sample of size 50.
5. Make dot plots for both of your lists of sample proportions, using the same horizontal axis. Comment on the center and spread of the two distributions.
6. Based on your dot plots, explain why samples of size 50 might be better for estimating the proportion of yellow blocks in the bucket than samples of size 10.

Drawing More Blocks deals with the relationship between theoretical probability and experimental probability. It illustrates the idea that experimental

4.2 Drawing More Blocks

Your teacher put eight blocks in a bucket. All the blocks are the same size. Three are yellow, four are red, and one is blue.



Problem 4.2

A. When you draw a block from the bucket, are the chances equally likely that it will be yellow, red, or blue? Explain your answer.

B. What is the total number of blocks? How many blocks of each color are there?

C. What is the *theoretical probability* of drawing a blue block? A yellow block? A red block? Explain how you found each answer.

Now, as a class or in groups, take turns drawing a block from the bucket. After each draw, return the block to the bucket. Keep a record of the blocks that are drawn. If you work in a group, take turns drawing blocks until you have 40 trials.

D. Based on your data, what is the *experimental probability* of drawing a blue block? A yellow block? A red block?

E. Compare the theoretical probabilities you found in part C to the experimental probabilities you found in part D. Are the probabilities for each color close? Are they the same? If not, why not?

Problem 4.2 Follow-Up

Suppose you and your classmates each took three turns drawing a block from the bucket, replacing the block each time, and then used the large amount of data you collected to find new experimental probabilities for drawing each color. You found the theoretical probability of drawing each color in part C. Do you think these new experimental probabilities would be closer to the theoretical probabilities than the experimental probabilities you found in part D were? Explain your reasoning.

FIGURE 8.3.1 Theoretical vs. experimental probability

probabilities can be used to estimate theoretical probabilities and that this estimate improves if we collect more data by including more repetitions of the experiment. These are the same ideas we addressed in the previous section about the distribution of sample means, except this time we're dealing with proportions, rather than means.

Before we go further, we need to introduce some symbols and terminology. Suppose we have a population, and some members of the population have a characteristic which makes them special, different from the rest of the population. For example, in a population of blocks, some are yellow. The proportion of yellow blocks in the population would be symbolized by the letter p. This proportion is exactly the same as the theoretical probability of choosing a yellow block from the population. In this case:

$$p = \frac{\textit{number of yellow blocks in the population}}{\textit{number of blocks in the population}} = \frac{3}{8} = 0.375$$

When we use a random sample from the population to find the experimental probability of choosing a yellow block, we are using the proportion of yellow blocks in the sample to estimate p, the population proportion of yellow blocks. Statisticians often use a "hat" symbol to indicate an estimate. In the previous section, we might have used the symbol $\hat{\mu}$ (*mu-hat*) for the sample mean, instead of $\overline{x}$, since the sample mean is used as an estimate for μ. In the same way, we will use the symbol $\hat{p}$ (*p-hat*) to stand for the proportion of special items in a sample, since the sample proportion is generally used to estimate the population proportion p.

Suppose, for example, we choose a random sample of 10 blocks (with replacement), and we got 3 yellow blocks, 5 red, and 2 blue. From this sample, our best estimate of the proportion of yellow blocks in the population would be the proportion of yellow blocks in the sample:

$$p \approx \hat{p} = \frac{\textit{number of yellow blocks in the sample}}{\textit{number of blocks in the sample}} = \frac{3}{10} = 0.3$$

Just as $\overline{x}$ is a random variable, so is $\hat{p}$; its value varies from sample to sample. In this section, we will look at the probability distribution of $\hat{p}$ under random sampling. Our first goal will be to show that $\hat{p}$ has properties that make it extremely useful for estimating p:

- $\hat{p}$ is an unbiased estimator of p.
- The value of $\hat{p}$ tends to get closer to p as the sample size n increases.
- When n is large enough, $\hat{p}$ has a familiar type of distribution. (You can probably guess what it will be.)

Is $\hat{p}$ an Unbiased Estimator?

Let's go back to the situation in *Change for a Dollar*. We'll return to *Drawing More Blocks* later.

EXAMPLE 8.3.1 Change for a Dollar Revisited

Recall that we are dealing with a population that is 80% pennies and 20% nickels. Before, we were interested in the mean value of the coins. This time, we want to focus on the probability of selecting a nickel. The theoretical probability of choosing a nickel is the same as the proportion of nickels in the population: $p = 0.20$ or 20%.

But suppose we didn't know the proportion of nickels in the population. We might try to estimate the probability of choosing a nickel by doing an experiment. We could choose coins repeatedly from the population and use the experimental probability as an estimate of the theoretical probability. Or, to put it another way, we could take a random sample and use the sample proportion as an estimate of the population proportion. For example, if we chose a random sample of 100 coins and got 22 nickels in the sample, we would estimate the population proportion of nickels as:

$$p \approx \hat{p} = \frac{22}{100} = 0.22$$

Sample	$\hat{p}$	Probability
1	$\frac{0}{1} = 0$ or 0%	0.8
5	$\frac{1}{1} = 1$ or 100%	0.2

FIGURE 8.3.2 The two possible samples of size 1

We're going to start with an even simpler case than in the last section—with a sample of size 1. The two possible samples of size 1 are listed in Figure 8.3.2 (1 if we choose a penny, 5 if we choose a nickel), along with its proportion of nickels and the probability of that sample. ■

Focus on Understanding

$\hat{p}$	$P(\hat{p})$
0	0.8
1	0.2

The probability distribution of the random variable $\hat{p}$ is given at the right. Recall that the population proportion p is 20% or 0.20.

1. What is the probability that $\hat{p}$ is exactly the same as p? That is, what is $P(\hat{p} = p)$?
2. What is the probability that $\hat{p}$ underestimates p? That is, what is $P(\hat{p} < p)$?
3. What is the probability that $\hat{p}$ overestimates p? That is, what is $P(\hat{p} > p)$?
4. To show that $\hat{p}$ is an unbiased estimator of p, we must show that the average or expected value of the random variable $\hat{p}$ is exactly the same as p. Find $E(\hat{p})$. Is it the same as p?

In the last section we found that, although the sample mean $\bar{x}$ sometimes underestimates the population mean μ and sometimes overestimates it, the average value of $\bar{x}$ hits μ exactly; it is an unbiased estimator. You saw the same kind of result in the previous *Focus on Understanding*. Although the sample proportion $\hat{p}$ sometimes underestimates the population proportion p and sometimes overestimates it, the average value of $\hat{p}$ is exactly p. The sample proportion $\hat{p}$ is an unbiased estimator of the population proportion p. We'll verify this again for a larger sample size later, but it does hold true in general, and is important enough to display in the following box.

The Expected Value of $\hat{p}$

If $\hat{p}$ is to be computed from a random sample and used to estimate a population proportion p, then $\hat{p}$ is an unbiased estimator of p:

$$E[\hat{p}] = p \quad \text{or} \quad \mu_{\hat{p}} = p$$

Note: Once again, the previous result depends on the fact that the samples are *random*. If sampling is done in a biased way, the sample proportion will be biased as well.

Does $\hat{p}$ Get Closer to p as n Increases?

$\hat{p}$	$P(\hat{p})$
0	0.8
1	0.2

To determine whether $\hat{p}$ gets closer to p as n increases, we need to look at the variance of the distribution of $\hat{p}$. The distribution of $\hat{p}$ for samples of size 1 is reproduced at the right. Using the computing formula for variance:

$$\sigma_{\hat{p}}^2 = \sum[\hat{p}^2 \cdot P(\hat{p})] - (\mu_{\hat{p}})^2$$
$$= [0^2(0.8) + 1^2(0.2)] - (0.2)^2$$
$$= 0.16$$

Notice that $pq = (0.2)(0.8) = 0.16$ also. This is not a coincidence; there's something going on here. When $n = 1$:

$$\hat{p} = \frac{\textit{number of nickels in sample}}{\textit{number of coins in sample}} = \frac{\textit{number of nickels in sample}}{1}$$
$$= \textit{number of nickels in sample}$$

The number of nickels in the sample is a binomial random variable. For a binomial random variable, the variance is given by $\sigma^2 = npq$:

For $n = 1: Var[\hat{p}] = \sigma_{\hat{p}}^2 = npq = 1 \cdot pq = pq$

We need to see what happens to $Var[\hat{p}]$ as n increases, so let's go to $n = 2$. That will be the subject of the following *Focus on Understanding*.

Focus on Understanding

1. Draw a tree diagram for the process of selecting a random sample of size 2 from a population that is 20% nickels and 80% pennies.
2. Make a table listing the four possible samples of size 2, the value of $\hat{p}$ (the proportion of nickels) for each sample, and the probability of each sample.
3. Give the probability distribution for $\hat{p}$.
4. Use the probability distribution from #3 to find $E[\hat{p}]$. Does $E[\hat{p}] = p$, as expected?
5. Find $Var[\hat{p}]$.

In the previous *Focus on Understanding* you should have found that $Var[\hat{p}] = 0.08$. Recall that when we used samples of size 1, we got $Var[\hat{p}] = pq = 0.16$. Notice that when we take samples of size $\underline{2}$, the variance is *half* as much. That is:

$$\text{For } n = 2{:}Var[\hat{p}] = \frac{pq}{2} = \frac{(0.2)(0.8)}{2} = 0.08$$

What do you suppose would happen if we took samples of size $\underline{3}$? We won't go through that one, but it does work out exactly as you would expect.

The Variance and Standard Deviation of the Distribution of $\hat{p}$

If $\hat{p}$ is to be computed from a random sample of size n and used to estimate a population proportion p, then:

$$Var[\hat{p}] = \frac{pq}{n} \quad \text{or} \quad \sigma^2_{\hat{p}} = \frac{pq}{n}$$

It follows that:

$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}}$$

So, just as we saw with sample means, increasing the sample size n will increase the denominators in the previous formulas, making the variance and standard deviation of $\hat{p}$ decrease. Since variance and standard deviation are measures of spread, the values of $\hat{p}$ will become less spread out as the sample size increases, getting closer and closer to p, the mean of the distribution. We have shown that $\hat{p}$ tends to get closer to p as the sample size increases. Once again, larger samples give more accurate estimates.

Does $\hat{p}$ Have a Familiar Distribution?

You probably know what's coming here, so we'll get right to it. Figure 8.3.3 shows a histogram of the distribution of $\hat{p}$ for samples of size 100. Does it look like a normal distribution? You bet! Just as with the distribution of $\bar{x}$, as the sample size increases, the distribution of $\hat{p}$ gets closer and closer to a normal distribution. However, the rule of thumb for how large a sample is necessary is different for proportions than for means ($n \geq 30$). For proportions, it is better to use the same criteria we used to decide whether a binomial distribution is approximately normal, that is: $n \cdot \min(pq) > 5$.

Summary of Facts about the Distribution of $\hat{p}$

If $\hat{p}$ is computed from a random sample of size n used to estimate a population proportion p, the distribution of $\hat{p}$ has these characteristics:

1. $E[\hat{p}] = \mu_{\hat{p}} = p$

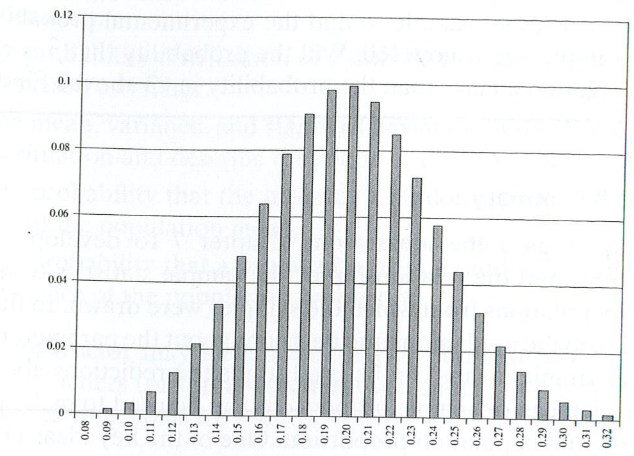


FIGURE 8.3.3 The distribution of sample proportions

The mean of the distribution of $\hat{p}$ is equal to the population proportion p. $\hat{p}$ is an unbiased estimator of p.

2. $Var[\hat{p}] = \sigma^2_{\hat{p}} = \frac{pq}{n}$ and $\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}}$

Therefore, the values of $\hat{p}$ tend to get closer to p as the sample size increases.

3. The distribution of $\hat{p}$ is normal (or close enough) if: $n \cdot \min(pq) > 5$.

Focus on Understanding

Look back at *Drawing More Blocks*, the *Classroom Connection* at the beginning of this section.

1. What is the theoretical probability p for drawing a yellow block from the bucket?

2. In *Drawing More Blocks*, students will select a random sample of 40 blocks (with replacement) and compute $\hat{p}$, the proportion of yellow blocks in the sample. Is the distribution of $\hat{p}$ close to normal? (Show how you check this.) Find the mean and standard deviation of the distribution of $\hat{p}$.

3. We know that it is too much to expect for $\hat{p}$ to be exactly the same as p. Suppose we allowed ourselves a margin of error of 0.10. How low can $\hat{p}$ be and still be within 0.10 of p? How high can $\hat{p}$ be and still be within the margin of error? For a sample of size 40, find the probability that $\hat{p}$ is within 0.10 of the population proportion p.

4. In the follow-up question at the bottom of *Drawing More Blocks*, the students will use a larger sample to find the experimental probability $\hat{p}$. Suppose that the sample size is now 160. Will the probability that $\hat{p}$ is within 0.10 of p now be larger or smaller than the probability in #3 above? Find this probability.

Chapter 8 Summary

This chapter used the ideas from Chapter 7 to develop the idea of sampling distributions and the relationship of the sample statistics to the parameters of the original populations from which the samples were drawn. In the real world, samples are used to make predictions and decisions about the parameters of a population. For example, sample means can be used to make predictions about the corresponding population mean or sample proportions can be used to make predictions about the corresponding population proportion. One of the key ideas in this chapter was that as the sample size increases, the shape of the sampling distribution becomes more "normal." You also saw that, as the sample size increases, the standard error of the sampling distribution decreases.

The key terms and ideas from this chapter are listed below:

Assessment is an integral part of every curriculum from the elementary school all the way through college. The question always arises—*what is it that students should be able to do after completing this lesson/unit/chapter*? We have included here our intended learning goals for Chapter 8. Students who have a good grasp of the concepts developed in this chapter should be successful in responding to these items:

- Explain, describe, or give an example of what is meant by each of the terms in the vocabulary list.
- List or count all the possible samples of a given size that can be taken from a given population.
- Determine the probability of a particular random sample being chosen or the probability that a random sample contains a particular object, item, or person.
- Create dot plots (line plots) using the means or proportions from random samples of a given size.
- Use sample means and sample proportions as estimates for population means and population proportions.

- Determine the probability distribution of $\bar{x}$, $\hat{p}$, or other sample statistics in a given situation.
- Use the probability distribution of a statistic to determine whether the statistic is an unbiased estimator.
- Find the mean, variance, and standard deviation of the distribution of $\bar{x}$ or $\hat{p}$ in a given situation and describe the shape of the distribution.
- Find the probability that the mean of a random sample will be within a given distance of the population mean.
- Find the probability that a proportion from a random sample will be within a given distance of the population proportion.

Your course instructor may have additional or different assessable outcomes for your class. As teachers (or future teachers) you should think about the assessment outcomes and learning goals for each chapter as you work through them.

EXERCISES FOR CHAPTER 8

1. Using the *Data Analysis and Probability Standards for Grades 6–8* from the NCTM's *Principles and Standards for School Mathematics* found at www.nctm.org, identify the middle school objectives that are found in Chapter 8.
2. Using the state standards for mathematics for the content area of data analysis and probability for your state, identify the middle school objectives that are found in Chapter 8. The following website may be useful: www.doe.state.in.us. This website will allow you to access the web pages of the state departments of education for the 50 states. From the state web pages you should be able to find the state's mathematical standards.
3. Suppose we have a list of 30 people, numbered from 1 to 30. We'd like to select a sample of 5 different people from the list. To select the sample, we plan to start by rolling an ordinary number cube. The first person selected will be the person whose number is rolled. Starting with that person, we'll select every sixth person from the list. For example, if we rolled a 4, we'd select person #4, 10, 16, 22, and 28.
   - **a.** Using this plan, how many different samples are possible? List the people who would be in each of the possible samples.
   - **b.** How many different samples contain person #17? What is the probability that a sample containing person #17 is selected?
   - **c.** Does every person on the list have the same probability of being selected? Explain why or why not.
   - **d.** Is a sample selected this way a random sample? Explain why or why not.
   - **e.** Describe a method for choosing a random sample of 5 different people from the list. Explain why your sample would be random.
   - **f.** How many different random samples of size 5 can be chosen from a list of 30 people?
   - **g.** How many of the random samples contain person #17? What is the probability that person #17 is chosen?
   - **h.** Does every person on the list have the same probability of being chosen? Explain why or why not.