38. Total mass = mass per bacterium × number of bacteria. Find total mass M as a function of mass per bacterium m if the total number is 10^6.

39–42 ▪ A ski slope has a slope of −0.2. You start at an altitude of 10,000 ft.

39. Write the equation giving altitude a as a function of horizontal distance moved d.

40. Write the equation of the line in meters.

41. What will be your altitude when you have gone 2000 ft horizontally?

42. The ski run ends at an altitude of 8000 ft. How far will you have gone horizontally?

43–46 ▪ The following data give the elevation of the surface of the Great Salt Lake in Utah.

Year, y	Elevation, E (ft)
1965	4,193
1970	4,196
1975	4,199
1980	4,199
1985	4,206
1990	4,203
1995	4,200

43. Graph these data.

44. During which periods is the surface elevation changing linearly?

45. What was the slope between 1965 and 1975? What would the surface elevation have been in 1990 if things had continued as they began? How different is this from the actual depth?

46. What was the slope during between 1985 and 1995? What would the surface elevation have been in 1965 if things had always followed this trend? How different is this from the actual depth?

47–50 ▪ Graph the following relations between measurements of a growing plant, checking that the points lie on a line. Find the equations in both point-slope and slope-intercept form.

Age, a (days)	Mass, M (g)	Volume, V (cm^3)	Glucose production, G (mg)
0.5	2.5	5.1	0.0
1.0	4.0	6.2	3.4
1.5	5.5	7.3	6.8
2.0	7.0	8.4	10.2
2.5	8.5	9.5	13.6
3.0	10.0	10.6	17.0

47. Mass as a function of age. Find the mass on day 1.75.

48. Volume as a function of age. Find the volume on day 2.75.

49. Glucose production as a function of mass. Estimate glucose production when the mass reaches 20.0 g.

50. Volume as a function of mass. Estimate the volume when the mass reaches 30.0 g. How will the density at that time compare with the density when $a = 0.5$?

51–44 ▪ Consider the data in the following table (adapted from *Parasitoids* by H. C. F. Godfray), describing the number of wasps that can develop inside caterpillars of different weights.

Weight of Caterpillar (g)	Number of Wasps
0.5	80
1.0	115
1.5	150
2.0	175

51. Graph these data. Which point does not lie on the line?

52. Find the equation of the line connecting the first two points.

53. How many wasps does the function predict would develop in a caterpillar weighing 0.72 g?

54. How many wasps does the function predict would develop in a caterpillar weighing 0.0 g? Does this make sense? How many would you really expect?

55–58 ▪ The world record times for various races are decreasing at roughly linear rates (adapted from *Guinness Book of Records*, 1990).

55. The men's Olympic record for the 1500 meters was 3:36.8 in 1972 and 3:35.9 in 1988. Find and graph the line connecting these. (Don't forget to convert everything into seconds.)

56. The women's Olympic record for the 1500 meters was 4:01.4 in 1972 and 3:53.9 in 1988. Find and graph the line connecting these.

57. If things continue at this rate, when will women finish the race in exactly no time? What might happen before that date?

58. If things continue at this rate, when will women be running this race faster than men?

Computer Exercises

59. Try Exercise 58 on the computer. Compute the year when the times will reach 0. Give your best guess of the times in the year 1900.

60. Graph the ratio of temperature measured in Fahrenheit to temperature measured in Celsius for $-273 \le °C < 200$. What happens near °C = 0? What happens for large and small values of °C? How would the results differ if the zero for Fahrenheit were changed to match that of Celsius?

1.5 Discrete-Time Dynamical Systems

Suppose we collect data on how much several bacterial cultures grow in one hour, or how much trees grow in one year. How can we predict what will happen in the long run? In this section, we begin addressing these dynamic problems, which form the theme of this chapter and indeed of much of this book. We follow the basic steps of applied mathematics: **quantifying the basic measurement** and describing the **dynamic rule**. We will learn how to summarize the rule with a **discrete-time dynamical system** or an **updating function** that describes change. From the discrete-time dynamical system and a starting point, called an **initial condition**, we will compute a **solution** that gives the values of the measurement as a function of time.

1.5.1 Discrete-Time Dynamical Systems and Updating Functions

A discrete-time dynamical system describes the relation between a quantity measured at the beginning and the end of an experiment or a time interval. If the measurement is represented by the variable m, we will use the notation m_t to denote the measurement at the beginning of the experiment and m_{t+1} to denote the measurement at the end of the experiment (Figure 1.5.1). Think of t as the current time, and $t+1$ as the time one step into the future. The relation between the initial measurement m_t and the final measurement m_{t+1} is given by the **discrete-time dynamical system**

$$m_{t+1} = f(m_t). \qquad (1.5.1)$$

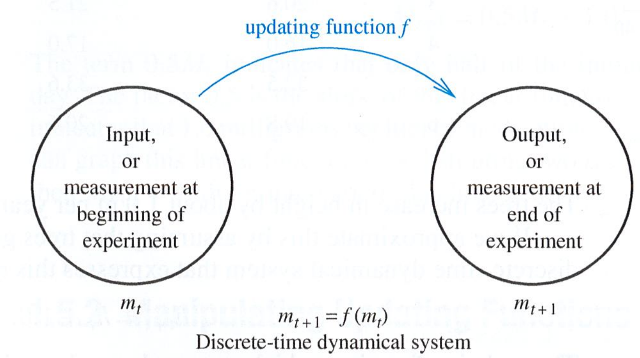


FIGURE 1.5.1
Notation for a discrete-time dynamical system

The **updating function** f accepts the initial value m_t as input and returns the final value m_{t+1} as output.

We will begin by applying this notation to several examples of discrete-time dynamical systems.

Example 1.5.1 A Discrete-Time Dynamical System for a Bacterial Population

Recall the data introduced in Example 1.2.3. Several bacterial cultures with different initial population sizes are grown in controlled conditions for one hour and then carefully recounted.

Colony	Initial Population, b_t	Final Population, b_{t+1}
1	0.47	0.94
2	3.3	6.6
3	0.73	1.46
4	2.8	5.6
5	1.5	3.0
6	0.62	1.24

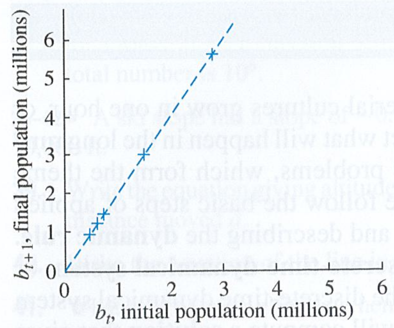


FIGURE 1.5.2
Graph of the updating function for a bacterial population

We have replaced b_i (the initial population) with b_t (the population at time t) and b_f (the final population) with b_{t+1} (the population at time $t+1$).

In each colony, the population doubled in size. We can describe this with the discrete-time dynamical system

$$b_{t+1} = 2.0b_t.$$

The updating function f describes the rule applied to the initial population,

$$f(b_t) = 2.0b_t.$$

The graph of the updating function plots the initial measurement b_t on the horizontal axis and the final measurement b_{t+1} on the vertical axis (Figure 1.5.2).

Example 1.5.2 A Discrete-Time Dynamical System for Tree Growth

Suppose you measure the heights of several trees in one year, and then again the next year (Figure 1.5.3). Denoting the initial height by h_t and the final height by h_{t+1}, we might find the data in the following table (all measured in meters).

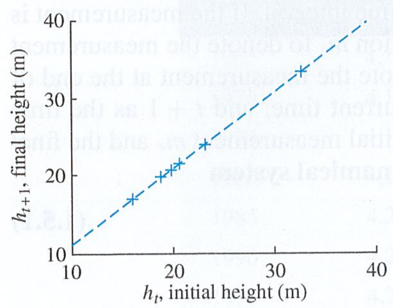


FIGURE 1.5.3
Data describing the growth of six trees

Tree	Initial Height, h_t	Final Height, h_{t+1}	Change in Height
1	23.1	24.1	1.0
2	18.7	19.8	1.1
3	20.6	21.5	0.9
4	16.0	17.0	1.0
5	32.5	33.6	1.1
6	19.8	20.6	0.8

The trees increase in height by about 1.0 m per year.

If we approximate this by assuming that trees grow by exactly 1.0 m per year, the discrete-time dynamical system that expresses this relation is

$$h_{t+1} = h_t + 1.0.$$

The updating function, which we can denote by g, has the formula

$$g(h_t) = h_t + 1.0.$$

For example, for a tree beginning with height 12.2 m, the discrete-time dynamical system predicts a final height of

$$h_{t+1} = g(12.2) = 12.2 + 1.0 = 13.2 \text{ m}.$$

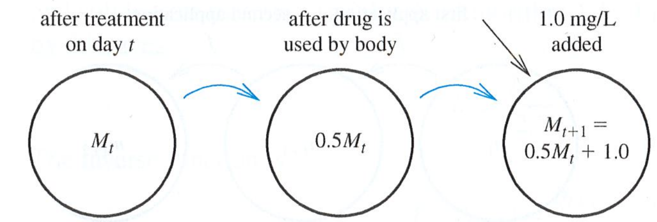
In this example, the data points do not exactly match the discrete-time dynamical system. The updating function captures the major trend in the data while ignoring the noise. Including only the trend corresponds to the use of a **deterministic** dynamical system to describe the behavior. To include the noise, we must use a **probabilistic** dynamical system (Chapter 6). We will specifically address the problem of finding an updating function that captures the major trends in the data when we study the technique of data-fitting called linear regression (Section 8.9).

Example 1.5.3 Discrete-Time Dynamical System for Mites

Recall the lizards infested by mites (Example 1.4.12). The final number of mites x_{t+1} is related to the initial number of mites x_t by the formula

$$x_{t+1} = 2x_t + 30.$$

FIGURE 1.5.4
The dynamics of medication concentration in the blood



This discrete-time dynamical system has the associated updating function

$$h(x_t) = 2x_t + 30.$$

The discrete-time dynamical systems for bacterial populations, tree height, and mite number were all derived from data. Often, dynamical rules can instead be derived directly from the principles governing a system.

Example 1.5.4 A Discrete-Time Dynamical System for Medication Concentration

Suppose we know the following facts about the dynamics of medication. Each day, a patient uses up half of the medication in his bloodstream. However, he is given a new dose sufficient to raise the concentration in the bloodstream by 1.0 milligrams per liter (Figure 1.5.4). Let M_t denote the concentration at time t. The discrete-time dynamical system is

$$M_{t+1} = 0.5M_t + 1.0.$$

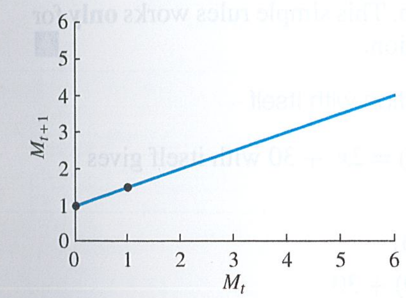


FIGURE 1.5.5
A graph of the updating function for medication concentration

The term $0.5M_t$ indicates that only half of the initial medication **remains** the next day. The factor 0.5 is the slope of this linear function. The second term, the intercept, indicates that 1.0 milligrams per liter of medication is added each day (Figure 1.5.5). We can graph this linear function by substituting two reasonable values for M_t. If $M_t = 0$, then $M_{t+1} = 1$, the y-intercept of this line. If $M_t = 1$, then $M_{t+1} = 1.5$.

1.5.2 Manipulating Updating Functions

All of the operations that can be applied to ordinary functions can be applied to updating functions, but with special interpretations. We will study **composition** of an updating function with itself, find the **inverse** of an updating function, and **convert the units** or **translate the dimensions** of a discrete-time dynamical system.

Composition Consider the discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

with updating function f. What does the composition $f \circ f$ mean? The updating function **updates** the measurement by one time step. Then

$$\begin{aligned}(f \circ f)(m_t) &= f(f(m_t)) && \text{definition of composition}\\ &= f(m_{t+1}) && \text{definition of updating function}\\ &= m_{t+2}. && \text{updating function applied to } m_{t+1}\end{aligned}$$

Therefore,

$$(f \circ f)(m_t) = m_{t+2}.$$

The composition of an updating function with itself corresponds to a two-step updating function (Figure 1.5.6).

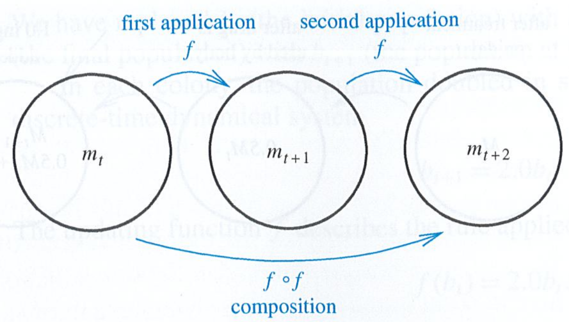


FIGURE 1.5.6
Composition of an updating function with itself

Example 1.5.5 Composition of the Bacterial Population Updating Function with Itself

The bacterial updating function is $f(b_t) = 2b_t$. The function $f \circ f$ takes the population size at time t as input and returns the population size 2 hours later, at time $t = 2$, as output. We can compute $f \circ f$ with the steps

$$\begin{aligned}(f \circ f)(b_t) &= f(f(b_t))\\ &= f(2.0b_t)\\ &= 4.0b_t.\end{aligned}$$

After two hours, the population is four times larger, having doubled twice. In this case, composition of f with itself looks like multiplication. This simple rules works **only** for an updating function expressing a proportional relation.

Example 1.5.6 Composition of the Mite Population Updating Function with Itself

The composition of the mite updating function $h(x_t) = 2x_t + 30$ with itself gives

$$\begin{aligned}(h \circ h)(x_t) &= h(h(x_t))\\ &= h(2x_t + 30)\\ &= 2(2x_t + 30) + 30\\ &= 4x_t + 90.\end{aligned}$$

Suppose we started with $x_t = 10$ mites. After 1 week, we would find $h(10) = 2 \cdot 10 + 30 = 50$ mites. After a second week, we would find $h(50) = 2 \cdot 50 + 30 = 130$ mites. Using the composition of the updating function with itself, we can compute the number of mites after 2 weeks, skipping over the intermediate value of 50 mites after 1 week, finding

$$(h \circ h)(10) = 4 \cdot 10 + 90 = 130.$$

Inverses Consider the general discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

with updating function f. What does the inverse f^{-1} mean? The updating function updates the measurement by one time step, and the inverse function undoes the action of the updating function. Therefore,

$$f^{-1}(m_{t+1}) = m_t.$$

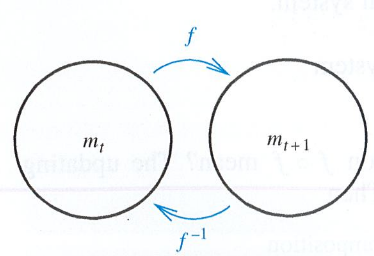


FIGURE 1.5.7
Inverse of an updating function

The inverse of an updating function corresponds to an "updating" function that goes **backwards** in time (Figure 1.5.7).

Example 1.5.7 Inverse of the Bacterial Population Updating Function

The bacterial updating function is $f(b_t) = 2b_t$. We find the inverse by writing the discrete-time dynamical system

$$b_{t+1} = 2.0b_t$$

and solving for the input variable b_t (Algorithm 1.1). In this case, dividing both sides by 2.0 gives

$$b_t = \frac{b_{t+1}}{2.0}.$$

The inverse function is

$$f^{-1}(b_{t+1}) = \frac{b_{t+1}}{2.0}.$$

If **multiplying** by 2.0 describes how the population changes forward in time, **dividing** by 2.0 describes how it changes backward in time.

If $b_t = 2.0$, then $b_{t+1} = 2.0b_t = 2.0 \cdot 2.0 = 4.0$. If we go backwards from $b_{t+1} = 4.0$ using the inverse of the updating function, we find

$$b_t = f^{-1}(4.0) = \frac{4.0}{2.0} = 2.0,$$

exactly where we started.

Example 1.5.8 Inverse of the Mite Updating Function

To find the inverse of the mite updating function $h(x_t) = 2.0x_t + 30$, we use Algorithm 1.1

$$\begin{aligned}2.0x_t + 30 &= x_{t+1} && \text{the original equation}\\ 2.0x_t &= x_{t+1} - 30 && \text{subtract 30 from both sides}\\ x_t &= \frac{x_{t+1} - 30}{2.0}. && \text{divide both sides by 2.0}\end{aligned}$$

Therefore,

$$x_t = h^{-1}(x_{t+1}) = \frac{x_{t+1} - 30}{2.0} = 0.5x_{t+1} - 15.$$

Suppose we started with $x_t = 10$ mites. After 1 week, we would find

$$h(10) = 2 \cdot 10 + 30 = 50.$$

Applying the inverse, we find

$$h^{-1}(50) = \frac{50 - 30}{2.0} = 10.$$

The inverse function takes us back to where we started.

1.5.3 Discrete-Time Dynamical Systems: Units and Dimensions

The updating function $f(b_t) = 2.0b_t$ accepts as input positive numbers with units of bacteria. If we measure this quantity in different units, we must convert the updating function itself into the new units. If we measure a different quantity like total mass or volume, we can translate the updating function into different dimensions.

Example 1.5.9 Describing the Dynamics of Tree Height in Centimeters

Suppose we wish to study tree height (Example 1.5.2) in units of centimeters rather than meters. In meters, the discrete-time dynamical system is

$$g(h_t) = h_t + 1.0 \text{ m}.$$

First, we define a new variable to represent the measurement in the new units. Let H_t be tree height measured in centimeters rather than meters. Then $H_t = 100h_t$, because there are 100 centimeters in a meter. We wish to find a discrete-time dynamical system that gives a formula for H_{t+1} in terms of H_t (Figure 1.5.8).

$$\begin{aligned} H_{t+1} &= 100h_{t+1} && \text{definition of } H_{t+1} \\ &= 100(h_t + 1.0) && \text{discrete-time dynamical system for } h_{t+1} \\ &= 100h_t + 100.0 && \text{multiply through by 100} \\ &= H_t + 100.0. && \text{definition of } H_t \end{aligned}$$

The discrete-time dynamical system in the new units corresponds to adding 100 centimeters to the height, which is equivalent to adding 1 meter. Although the underlying process is the same, the discrete-time dynamical system and the corresponding updating function are different, just as the numerical values of measurements are different in different units.

Example 1.5.10 Describing the Dynamics of Bacterial Mass

Suppose we wish to study the bacterial population in terms of mass rather than number. At the beginning, the mass, denoted by m_t, is

$$m_t = \mu b_t$$

where μ is the mass per bacterium (as in Example 1.3.4). The updated mass m_{t+1} is

$$\begin{aligned} m_{t+1} &= \mu b_{t+1} && \text{the definition of } m_{t+1} \\ &= \mu \cdot 2.0b_t && \text{substituting the original updating function} \\ &= 2.0\mu b_t && \text{rearranging the terms} \\ &= 2.0m_t. && \text{recognize that } m_t = \mu b_t \end{aligned}$$

This new discrete-time dynamical system doubles its input like the original discrete-time dynamical system, but takes mass as its input rather than numbers of bacteria (Figure 1.5.9).

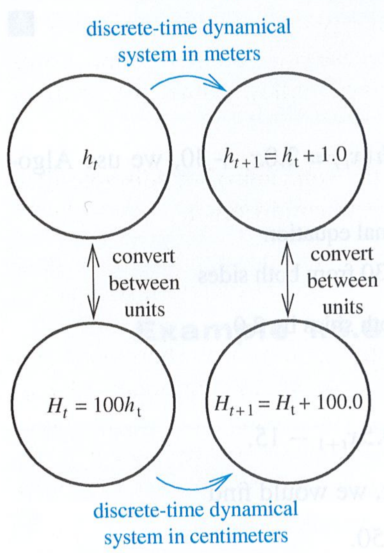


FIGURE 1.5.8
Finding the discrete-time dynamical system for trees in centimeters

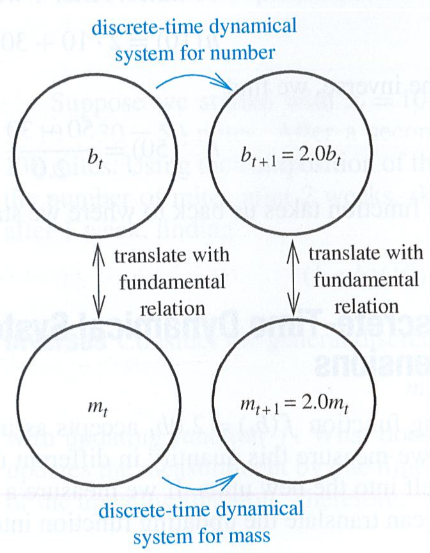


FIGURE 1.5.9
Finding the discrete-time dynamical system for bacteria in terms of mass

1.5.4 Solutions

A discrete-time dynamical system describes some quantity at the end of an experiment as a function of that same quantity at the beginning. What if we were to continue the experiment? A bacterial population growing according to $b_{t+1} = 2.0b_t$ would double again and again. A tree growing according to $h_{t+1} = h_t + 1.0$ would add more and more meters to its height. An infested lizard would become even more infested.

To describe a situation in which a dynamical process is repeated many times, we let m_0 represent the measurement at the beginning, m_1 the measurement after one time step, m_2 the measurement after two time steps, and so forth (Figure 1.5.10). In general, we define

$$m_t = \text{measurement } t \text{ hours after the beginning of the experiment.}$$

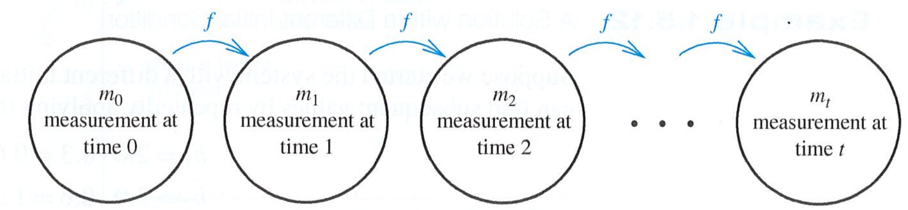


FIGURE 1.5.10
The repeated action of an updating function

Our goal is to find the values of m_t for all values of t. Before we can do so, however, we must know where we **started**. Without knowing where you started, it is impossible to answer a question like "where are you after driving 5 miles south?" The starting value is known as the **initial condition**.

Definition 1.10 The sequence of values of m_t for $t = 0, 1, 2, \ldots$ is the **solution** of the discrete-time dynamical system $m_{t+1} = f(m_t)$ starting from the **initial condition** m_0.

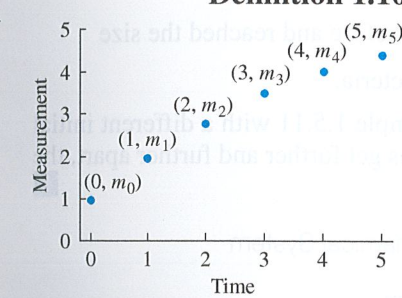


FIGURE 1.5.11
The graph of a solution

There is a mathematical theory devoted to **sequences** like the solutions of discrete-time dynamical system, concerned often with whether the values **converge** to a particular value.

The graph of a solution is a discrete set of points with the time t on the horizontal axis and the measurement m_t on the vertical axis. The initial point has coordinates $(0, m_0)$ to describe the initial condition. The next point, with coordinates $(1, m_1)$, describes the measurement at $t = 1$, and so forth (Figure 1.5.11). It is possible to find a formula for the solution for simple discrete-time dynamical systems, but not in many more complicated cases.

Example 1.5.11 A Solution of the Bacterial Discrete-Time Dynamical System

Suppose we begin with one million bacteria, which corresponds to an initial condition of $b_0 = 1.0$ (with bacterial population measured in millions). If the bacteria obey the discrete-time dynamical system $b_{t+1} = 2.0b_t$,

$$\begin{aligned} b_1 &= 2.0b_0 = 2.0 \cdot 1.0 = 2.0 \\ b_2 &= 2.0b_1 = 2.0 \cdot 2.0 = 4.0 \\ b_3 &= 2.0b_2 = 2.0 \cdot 4.0 = 8.0. \end{aligned}$$

Examining these results, we notice that

$$\begin{aligned} b_1 &= 2.0 \cdot 1.0 \\ b_2 &= 2.0^2 \cdot 1.0 \\ b_3 &= 2.0^3 \cdot 1.0. \end{aligned}$$

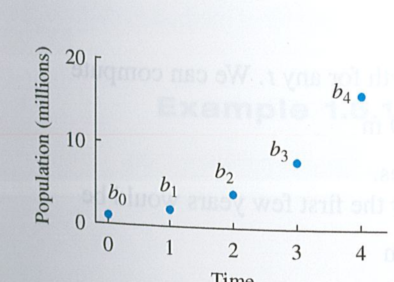


FIGURE 1.5.12
A solution: Bacterial population size as a function of time

After 3 hours, the population has doubled three times, and is $2.0^3 = 8.0$ times the original population. We graph the solution by plotting the time t on the horizontal axis and the number of bacteria after t hours (b_t) on the vertical axis (Figure 1.5.12). The graph consists only of a discrete set of points describing the hourly measurements hence the name "discrete-time dynamical system." Sometimes, we will connect the points in a solution with line segments to make the pattern easier to see.

After t hours, the population has doubled t times and has reached the size

$$b_t = 2.0^t. \quad (1.5.2)$$

This formula describes the solution of the discrete-time dynamical system with initial condition $b_0 = 1.0$. It predicts the population after t hours of reproduction for any value of t. For example, we can compute

$$b_8 = 2.0^8 \cdot 1.0 = 256.0$$

without ever computing b_1, b_2 or other intermediate values.

Example 1.5.12 A Solution with a Different Initial Condition

Suppose we started the system with a different initial condition of $b_0 = 0.3$ million. We can find subsequent values by repeatedly applying the discrete-time dynamical system,

$$b_1 = 2.0 \cdot 0.3 = 0.6$$
$$b_2 = 2.0 \cdot 0.6 = 1.2$$
$$b_3 = 2.0 \cdot 1.2 = 2.4.$$

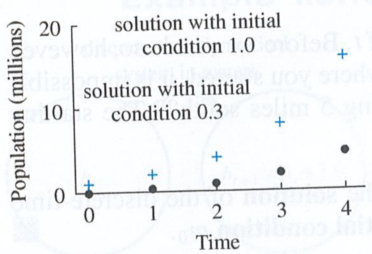


FIGURE 1.5.13
Solutions starting from two different initial conditions

If we look for the pattern in this case,

$$b_1 = 2.0 \cdot 0.3$$
$$b_2 = 2.0^2 \cdot 0.3$$
$$b_3 = 2.0^3 \cdot 0.3.$$

After t hours, the population has doubled t times as before and reached the size

$$b_t = 2.0^t \cdot 0.3 \text{ million bacteria.}$$

The solution is different from the one found in Example 1.5.11 with a different initial condition (Figure 1.5.13). Although the two solutions get further and further apart, the ratio always remains the same (Exercise 55).

Example 1.5.13 Two Solutions of the Tree Height Discrete-Time Dynamical System

Tree height obeys the discrete-time dynamical system

$$h_{t+1} = h_t + 1.0$$

(Example 1.5.2). Suppose the tree begins with a height of $h_0 = 10.0$ m. Then

$$h_1 = h_0 + 1.0 = 11.0 \text{ m}$$
$$h_2 = h_1 + 1.0 = 12.0 \text{ m}$$
$$h_3 = h_2 + 1.0 = 13.0 \text{ m.}$$

Each year, the height of the tree increases by 1.0 m. After 3 years, the height is 3.0 m greater than the original height. After t years the tree has added 1.0 m to its height t times, meaning that the height will have increased by a total of t m (Figure 1.5.14). Therefore the solution is

$$h_t = 10.0 + t.$$

This formula predicts the height after t years of growth for any t. We can compute

$$h_8 = 10.0 + 8.0 = 18.0 \text{ m}$$

without computing h_1, h_2 or other intermediate values.

If the tree began at a height of 2.0 m, the size for the first few years would be

$$h_1 = h_0 + 1.0 = 3.0 \text{ m}$$
$$h_2 = h_1 + 1.0 = 4.0 \text{ m}$$
$$h_3 = h_2 + 1.0 = 5.0 \text{ m.}$$

Again, the tree adds t meters of height in t years, so the height is

$$h_t = 2.0 + t.$$

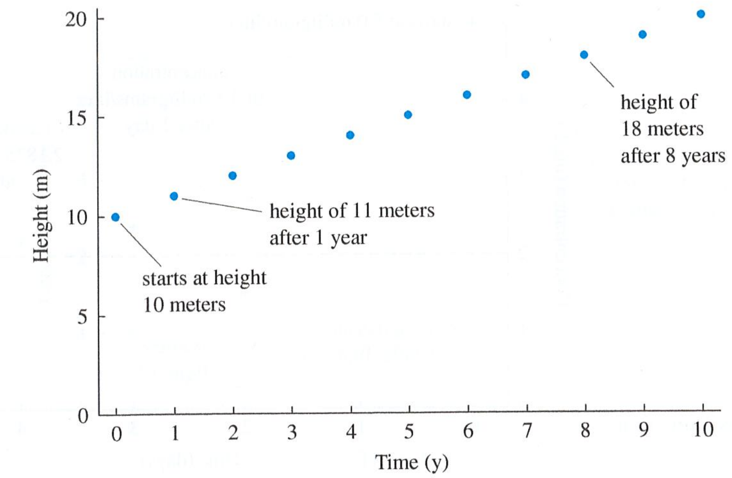


FIGURE 1.5.14
A solution: tree height as a function of time

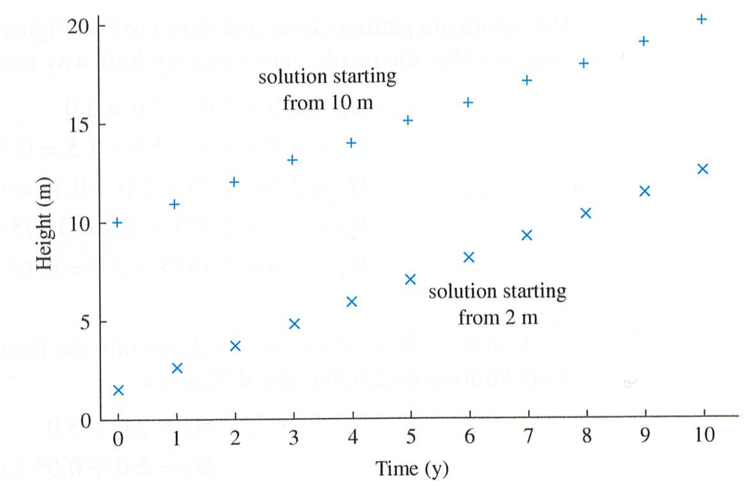


FIGURE 1.5.15
Two solutions for tree height as functions of time

The solution with this smaller initial condition is always exactly 8.0 m less than the solution with the $h_0 = 10$ (Figure 1.5.15).

Is it always possible to guess the formula for a solution in this way? We will next see some cases where computing the solution step by step is straightforward, but finding a solution is tricky. Remarkably, there are simple discrete-time dynamical systems for which it is **impossible** to write a formula for a solution. For example, **chaotic** dynamical systems have solutions that are so unpredictable that no formula can describe them (Subsection 3.2.3).

Example 1.5.14 Finding a Solution of the Medication Discrete-Time Dynamical System

Consider the discrete-time dynamical system for medication (Example 1.5.4), given by

$$M_{t+1} = 0.5 M_t + 1.0.$$

Suppose we begin from an initial condition of $M_0 = 5.0$ milligrams per liter. Then

$$M_1 = 0.5 \cdot 5.0 + 1.0 = 3.5$$
$$M_2 = 0.5 \cdot 3.5 + 1.0 = 2.75$$
$$M_3 = 0.5 \cdot 2.75 + 1.0 = 2.375$$
$$M_4 = 0.5 \cdot 2.375 + 1.0 = 2.1875.$$

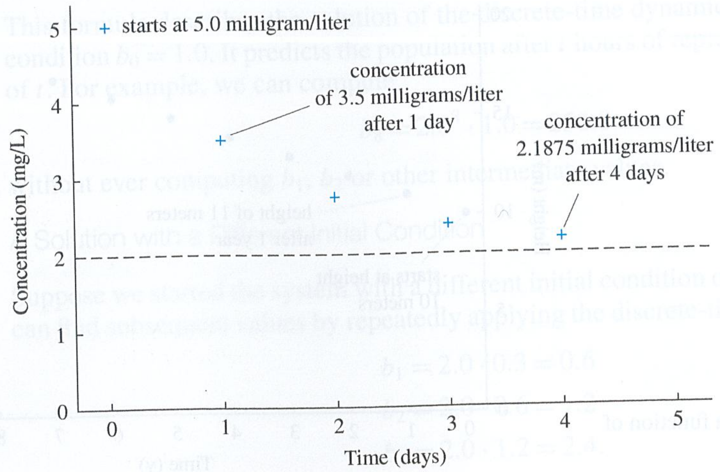


FIGURE 1.5.16
Medication concentration as a function of time

The values are getting closer and closer to 2.0 (Figure 1.5.16). More careful examination indicates that the results move exactly **half way** toward 2.0 each step. In particular,

$$
\begin{aligned}
M_0 - 2.0 &= 5.0 - 2.0 = 3.0 \\
M_1 - 2.0 &= 3.5 - 2.0 = 1.5 = 0.5 \cdot 3.0 \\
M_2 - 2.0 &= 2.75 - 2.0 = 0.75 = 0.5 \cdot 1.5 \\
M_3 - 2.0 &= 2.375 - 2.0 = 0.375 = 0.5 \cdot 0.75 \\
M_4 - 2.0 &= 2.1875 - 2.0 = 0.1875 = 0.5 \cdot 0.375.
\end{aligned}
$$

Can we convert these observations into the formula for a solution? If we write the concentration as 2.0 plus the difference,

$$
\begin{aligned}
M_0 &= 2.0 + 3.0 \\
M_1 &= 2.0 + 0.5 \cdot 3.0 \\
M_2 &= 2.0 + 0.5^2 \cdot 3.0 \\
M_3 &= 2.0 + 0.5^3 \cdot 3.0
\end{aligned}
$$

we might see that

$$M_t = 2.0 + 0.5^t \cdot 3.0.$$

Finding patterns in this way and translating them into formulas can be tricky. It is much more important to be able to **describe** the behavior of solutions with a graph or in words. In this case, our calculations quickly revealed that the solution moved closer and closer to 2.0. In Section 1.6, we will develop a powerful graphical method to deduce this pattern with a minimum of calculation.

Example 1.5.15 A Second Solution of the Medication Discrete-Time Dynamical System

If we begin with an initial concentration of $M_0 = 1.0$ milligrams per liter, then

$$
\begin{aligned}
M_1 &= 0.5 \cdot 1.0 + 1.0 = 1.5 \\
M_2 &= 0.5 \cdot 1.5 + 1.0 = 1.75 \\
M_3 &= 0.5 \cdot 1.75 + 1.0 = 1.875 \\
M_4 &= 0.5 \cdot 1.875 + 1.0 = 1.9375.
\end{aligned}
$$

Unlike bacterial populations (Example 1.5.12) and tree size (Example 1.5.13), the graphs of solutions starting from different initial conditions look completely different (Figure 1.5.17).

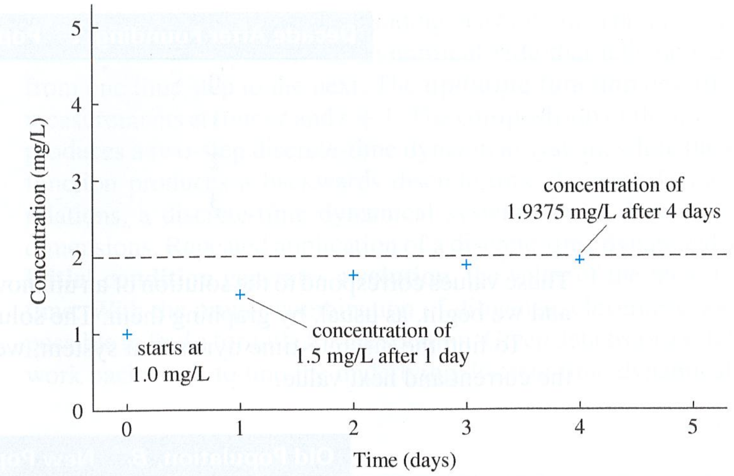


FIGURE 1.5.17
Another solution of medication concentration as a function of time

However, the values still get closer and closer to 2.0, and the difference from 2.0 is reduced by a factor of 0.5 each day,

$$
\begin{aligned}
M_0 - 2.0 &= 1.0 - 2.0 = -1.0 \\
M_1 - 2.0 &= 1.5 - 2.0 = -0.5 \\
M_2 - 2.0 &= 1.75 - 2.0 = -0.25 \\
M_3 - 2.0 &= 1.875 - 2.0 = -0.125 \\
M_4 - 2.0 &= 1.9375 - 2.0 = -0.0625.
\end{aligned}
$$

We can find the formula using the same idea as before. If we write

$$
\begin{aligned}
M_0 &= 2.0 - 1.0 \\
M_1 &= 2.0 - 0.5 \cdot 1.0 \\
M_2 &= 2.0 - 0.5^2 \cdot 1.0 \\
M_3 &= 2.0 - 0.5^3 \cdot 1.0
\end{aligned}
$$

we can see that

$$M_t = 2.0 - 0.5^t \cdot 1.0.$$

In Section 2.2, we will use the fundamental idea of the **limit** to study more carefully what it means for the sequence of values that define a solution to get closer and closer to 2.0.

Example 1.5.16 A Solution of the Mite Discrete-Time Dynamical System

Recall the discrete-time dynamical system

$$x_{t+1} = 2x_t + 30$$

for mites. If we started our lizard off with $x_0 = 10$ mites, we compute

$$
\begin{aligned}
x_1 &= 2.0x_0 + 30 = 50 \\
x_2 &= 2.0x_1 + 30 = 130 \\
x_3 &= 2.0x_2 + 30 = 290.
\end{aligned}
$$

The pattern is not too obvious in this case. There is a pattern, which is a good challenge to find (Exercise 35).

Example 1.5.17 Finding an Updating Function from a Solution

Suppose we have measured a sequence of values over time and wish to find the discrete-time dynamical system that generated them. Consider the following population of birds in a newly founded refuge.

Decade After Founding, t	Population Size, B_t
0	300
1	400
2	700
3	1600

These values correspond to the solution of an unknown discrete-time dynamical system, and we begin, as usual, by graphing them. The solution rises quickly.

To find the discrete-time dynamical system, we can rewrite these data in terms of the current and next value.

Old Population, B_t	New Population, B_{t+1}
300	400
400	700
700	1600

We can seek the equation describing this relationship with Algorithm 1.3.

1. The graph in Figure 1.5.18b looks like a line.

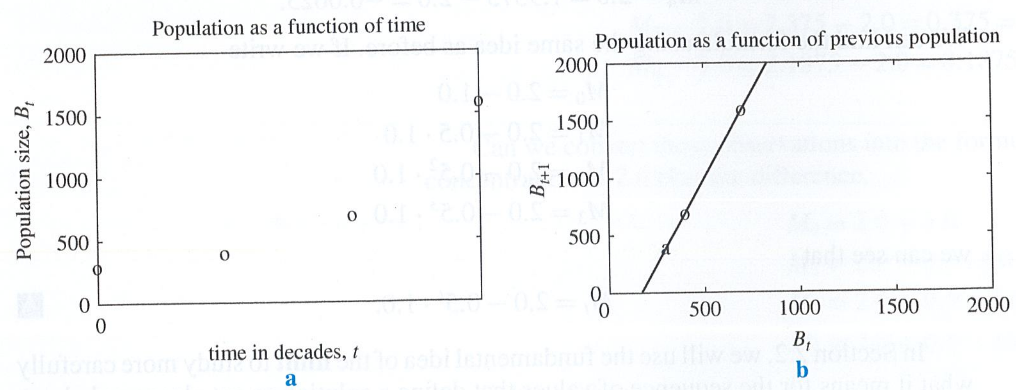


FIGURE 1.5.18
Finding the equation of a discrete-time dynamical system from data

2. Pick the first two data points.
3. The slope m is

$$m = \frac{\Delta B_{t+1}}{\Delta B_t} = \frac{700-400}{400-300} = \frac{300}{100} = 3.0.$$

4. Using the point (300, 400) as the base point in the point-slope form for a line, the equation is

$$B_{t+1} = 3.0(B_t - 300) + 400.$$

5. We can multiply this out to find the slope-intercept form

$$B_{t+1} = 3.0B_t - 500.$$

This population seems to have the potential to triple every decade, but 500 individuals are removed, perhaps by poachers.

Summary Starting from data or an understanding of a biological process, we can derive a **discrete-time dynamical system**, the **dynamical rule** that tells how a measurement changes from one time step to the next. The **updating function** describes the relation between measurements at times t and $t+1$. The **composition** of the updating function with itself produces a two-step discrete-time dynamical system, while the **inverse** of the updating function produces a backwards discrete-time dynamical system. Like all biological relations, a discrete-time dynamical system can be described in different units and dimensions. Repeated application of a discrete-time dynamical system starting from an **initial condition** generates a **solution**, the value of the measurement as a function of time. With the proper combination of diligence, cleverness, and luck, it is sometimes possible to find a formula for the solution. Given data from a solution, we can sometimes work backwards to find the underlying discrete-time dynamical system.

1.5 Exercises

Mathematical Techniques

1–4 ▪ Write the updating function associated with each of the following discrete-time dynamical systems and evaluate it at the given arguments. Which are linear?

1. $p_{t+1} = p_t - 2$, evaluate at $p_t = 5$, $p_t = 10$, and $p_t = 15$.
2. $\psi_{t+1} = \frac{\psi_t}{2}$, evaluate at $\psi_t = 4$, $\psi_t = 8$, and $\psi_t = 12$.
3. $x_{t+1} = x_t^2 + 2$, evaluate at $x_t = 0$, $x_t = 2$, and $x_t = 4$.
4. $Q_{t+1} = \frac{1}{Q_t + 1}$, evaluate at $Q_t = 0$, $Q_t = 1$, and $Q_t = 2$.

5–8 ▪ Compose the updating function associated with each discrete-time dynamical system with itself. Find the two-step discrete-time dynamical system. Check that the result of applying the original discrete-time dynamical system twice to the given initial condition matches the result of applying the new discrete-time dynamical system to the given initial condition once.

5. Volume follows $v_{t+1} = 1.5v_t$, with $v_0 = 1220\mu m^3$.
6. Length obeys $l_{t+1} = l_t - 1.7$, with $l_0 = 13.1$ cm.
7. Population size follows $n_{t+1} = 0.5n_t$, with $n_0 = 1200$.
8. Medication concentration obeys $M_{t+1} = 0.75M_t + 2.0$ with $M_0 = 16.0$.

9–12 ▪ Find the backwards discrete-time dynamical system associated with each discrete-time dynamical system. Use it to find the value at the previous time.

9. $v_{t+1} = 1.5v_t$. Find v_0 if $v_1 = 1220\mu m^3$ (as in Exercise 5).
10. $l_{t+1} = l_t - 1.7$. Find l_0 if $l_1 = 13.1$cm (as in Exercise 6).
11. $n_{t+1} = 0.5n_t$. Find n_0 if $n_1 = 1200$ (as in Exercise 7).
12. $M_{t+1} = 0.75M_t + 2.0$. Find M_0 if $M_1 = 16.0$ (as in Exercise 8).

13–14 ▪ Find the composition of each of the following mathematically elegant updating functions with itself, and find the inverse function.

13. The updating function $f(x) = \frac{x}{1+x}$. Put things over a common denominator to simplify the composition.
14. The updating function $h(x) = \frac{x}{x-1}$. Put things over a common denominator to simplify the composition.

15–18 ▪ Find and graph the solutions of the following discrete-time dynamical systems for five steps with the given initial condition. Compare the graph of the solution with the graph of the updating function.

15. $v_{t+1} = 1.5v_t$, with $v_0 = 1220\mu m^3$.
16. $l_{t+1} = l_t - 1.7$, with $l_0 = 13.1$ cm.
17. $n_{t+1} = 0.5n_t$, with $n_0 = 1200$.
18. $M_{t+1} = 0.75M_t + 2.0$ with $M_0 = 16.0$.

19–22 ▪ Using a formula for the solution, you can project far into the future without computing all the intermediate values. Find the following, and say whether the results are reasonable.

19. Find a formula for v_t for the discrete-time dynamical system in Exercise 15, and use it to find the volume at $t = 20$.
20. Find a formula for v_t for the discrete-time dynamical system in Exercise 16, and use it to find the length at $t = 20$.
21. Find a formula for v_t for the discrete-time dynamical system in Exercise 17, and use it to find the number at $t = 20$.
22. Find a formula for v_t for the discrete-time dynamical system in Exercise 18, and use it to find the concentration at $t = 20$ (use the method in Example 1.5.14 after finding the value it seems to be approaching).

23–26 ▪ Experiment with the following mathematically elegant updating functions and try to find the solution.

23. Consider the updating function

$$f(x) = \frac{x}{1+x}$$

from Exercise 13. Starting from an initial condition of $x_0 = 1$, compute x_1, x_2, x_3, and x_4, and try to spot the pattern.

24. Use the updating function in Exercise 23 but start from the initial condition $x_0 = 2$.

25. Consider the updating function

$$g(x) = 4 - x.$$

Start from initial condition of $x_0 = 1$, and try to spot the pattern. Experiment with a couple of other initial conditions. How would you describe your results in words?

26. Consider the updating function

$$h(x) = \frac{x}{x-1}$$

from Exercise 14. Start from initial condition of $x_0 = 3$, and try to spot the pattern. Experiment with a couple of other initial conditions. How would you describe your results in words?

Applications

27–30 ▪ Consider the following actions. Which of them commute (produce the same answer when done in either order)?

27. A population doubles in size; 10 individuals are removed from a population. Try starting with 100 individuals, and then try to figure out what happens in general.

28. A population doubles in size; population size is divided by 4. Try starting with 100 individuals, and then try to figure out what happens in general.

29. An organism grows by 2.0 cm; an organism shrinks by 1.0 cm.

30. A person loses half his money. A person gains $10.

31–34 ▪ Use the formula for the solution to find the following, and say whether the results are reasonable.

31. Using the solution for tree height $h_t = 10.0 + t$ m (Example 1.5.13), find the tree height after 20 years.

32. Using the solution for tree height $h_t = 10.0 + t$ m (Example 1.5.13), find the tree height after 100 years.

33. Using the solution for bacterial population number $b_t = 2.0^t \cdot 1.0$ (Equation 1.5.2), find the bacterial population after 20 hours. If an individual bacterium weighs about 10^{-12} grams, how much will the whole population weigh?

34. Using the solution for bacterial population number $b_t = 2.0^t \cdot 1.0$ (Equation 1.5.2), find the bacterial population after 40 hours. How much would this population weigh?

35–36 ▪ Find a formula for the solution of the given discrete-time dynamical system.

35. Find the pattern in the number of mites on a lizard with $x_0 = 10$ and following the discrete-time dynamical system $x_{t+1} = 2x_t + 30$. (*Hint:* Add 30 to the number of mites.)

36. Find the pattern in the number of mites on a lizard with $x_0 = 10$ and following the discrete-time dynamical system $x_{t+1} = 2x_t + 20$.

37–40 ▪ The following tables display data from four experiments:

a. Cell volume after 10 minutes in a watery bath

b. Fish mass after 1 week in a chilly tank

c. Gnat population size after 3 days without food

d. Yield of several varieties of soybean before and after fertilization

For each, graph the new value as a function of the initial value, write the discrete-time dynamical system, and fill in the missing value in the table.

37.

Cell Volume *Parasitoids* (μm^3)	
Initial, v_t	Final, v_t
1220	1830
1860	2790
1080	1620
1640	2460
1540	2310
1420	??

38.

Fish Mass (g)	
Initial, m_t	Final, m_{t+1}
13.1	10.4
18.2	15.5
17.3	14.6
16.0	13.3
20.5	17.8
2.5	??

39.

Gnat Number	
Initial, n_t	Final, n_{t+1}
1.2×10^3	6.0×10^2
2.4×10^3	1.2×10^3
1.6×10^3	8.0×10^2
2.0×10^3	1.0×10^3
1.4×10^3	7.0×10^2
8.0×10^2	??

40.

Soybean Yield per Acre	
Initial, Y_t	Final, Y_{t+1}
100	210
50	110
200	410
75	160
95	200
250	??

41–44 ▪ Recall the data used for Exercises 49–52 in Section 1.2.

Age, a (days)	Length, L (cm)	Tail Length, T (cm)	Mass, M (g)
0.5	1.5	1.0	1.5
1.0	3.0	0.9	3.0
1.5	4.5	0.8	6.0
2.0	6.0	0.7	12.0
2.5	7.5	0.6	24.0
3.0	9.0	0.5	48.0

These data define several discrete-time dynamical systems. For example, between the first measurement (on day 0.5) and the second (on day 1.0), the length increases by 1.5 cm. Between the second measurement (on day 1.0) and the third (on day 1.5), the length again increases by 1.5 cm.

41. Graph the length at the second measurement as a function of length at the first, the length at the third measurement as a function of length at the second, and so forth. Find the discrete-time dynamical system that reproduces the results.

42. Find and graph the discrete-time dynamical system for tail length.

43. Find and graph the discrete-time dynamical system for mass.

44. Find and graph the discrete-time dynamical system for age.

45–48 ▪ Suppose students are permitted to take a test again and again until they get a perfect score of 100. We wish to write a discrete-time dynamical system describing these dynamics.

45. In words, what is the argument of the updating function? What is the value?

46. What are the domain and range of the updating function? What value do you expect if the argument is 100?

47. Sketch a possible graph of the updating function.

48. Based on your graph, how would a student do on her second try if she scored 20 on her first try?

49–50 ▪ Consider the discrete-time dynamical system $b_{t+1} = 2.0b_t$ for a bacterial population (Example 1.5.1).

49. Write a discrete-time dynamical system for the total volume of bacteria (suppose each bacterium takes up $10^4 \mu m^3$).

50. Write a discrete-time dynamical system for the total area taken up by the bacteria (suppose the thickness is 20 μm).

51–52 ▪ Recall the equation $h_{t+1} = h_t + 1.0$ for tree height.

51. Write a discrete-time dynamical system for the total volume of the cylindrical trees in Section 1.3, Exercise 27.

52. Write a discrete-time dynamical system for the total volume of a spherical tree (this is kind of tricky).

53–54 ▪ Consider the following data describing the level of medication in the blood of two patients over the course of several days.

Day	Medication Level in Patient 1	Medication Level in Patient 2
0	20.0	0.0
1	16.0	2.0
2	13.0	3.2
3	10.75	3.92

53. Graph three points on the updating function for the first patient. Find the discrete-time dynamical system for the first patient.

54. Graph three points on the updating function for the second patient and find the discrete-time dynamical system.

55–56 ▪ For the following discrete-time dynamical systems, compute solutions with the given initial condition. Then find the difference between the solutions as a function of time, and the ratio of the solutions as a function of time. In which cases is the difference constant, and in which cases is the ratio constant? Can you explain why?

55. Two bacterial populations follow the discrete-time dynamical system $b_{t+1} = 2.0b_t$, but the first starts with initial condition $b_0 = 1.0 \times 10^6$ and the second starts with initial condition $b_0 = 3.0 \times 10^5$.

56. Two trees follow the discrete-time dynamical system $h_{t+1} = h_t + 1.0$, but the first starts with initial condition $h_0 = 10.0$ m and the second starts with initial condition $h_0 = 2.0$ m.

57–60 ▪ Follow the steps to derive discrete-time dynamical systems describing the following contrasting situations.

57. A population of bacteria doubles every hour, but 1.0×10^6 individuals are removed after reproduction to be converted into valuable biological by-products. The population begins with $b_0 = 3.0 \times 10^6$ bacteria.

a. Find the population after 1, 2, and 3 hours.

b. How many bacteria were harvested?

c. Write the discrete-time dynamical system.

d. Suppose you waited to harvest bacteria until the end of 3 hours. How many could you remove and still match the population b_3 found in part **a**? Where did all the extra bacteria come from?

58. Suppose a population of bacteria doubles every hour, but that 1.0×10^6 individuals are removed **before** reproduction to be converted into valuable biological by-products. Suppose the population begins with $b_0 = 3.0 \times 10^6$ bacteria.

a. Find the population after 1, 2, and 3 hours.

b. Write the discrete-time dynamical system.

c. How does the population compare with that in the previous problem? Why is it doing worse?

59. Suppose the fraction of individuals with some superior gene increases by 10% each generation.