a. Write the discrete-time dynamical system for the fraction of organisms with the gene (denote the fraction at time t by f_t and figure out the formula for f_{t+1}).

b. Write the solution with $f_0 = 0.0001$.

c. Will the fraction reach 1.0? Does the discrete-time dynamical system make sense for all values of f_t?

60. The Weber-Fechner law describes how human beings perceive differences. Suppose, for example, that a person first hears a tone with a frequency of 400 hertz (cycles per second). He is then tested with higher tones until he can hear the difference. The ratio between these values describes how well this person can hear differences.

a. Suppose the next tone he can distinguish has a frequency of 404 hertz. What is the ratio?

b. According to the Weber-Fechner law, the next higher tone will be greater than 404 by the same ratio. Find this tone.

c. Write the discrete-time dynamical system for this person. Find the fifth tone he can distinguish.

d. Suppose the experiment is repeated on a musician, and she manages to distinguish 400.5 hertz from 400 hertz. What is the fifth tone she can distinguish?

61–62 ▪ The total mass of a population of bacteria will change if either the number of bacteria changes, the mass per bacterium changes, or both. The following problems derive discrete-time dynamical systems when both change.

61. The number of bacteria doubles each hour, and the mass of each bacterium triples during the same time.

62. The number of bacteria doubles each hour, and the mass of each bacterium increases by 1.0×10^{-9}g. What seems to go wrong with this calculation? Can you explain why?

1.6 Analysis of Discrete-Time Dynamical Systems

We have defined discrete-time dynamical systems that describe what happens during a single time step, and defined the solution as the sequence of values taken on over many time steps. Often enough, finding a formula for the solution is difficult or impossible. Nonetheless, we can often deduce the behavior of the solution with simpler methods. This section introduces two such methods. **Cobwebbing** is a graphical technique that makes it possible to sketch solutions without computing anything. Algebraically, we will learn how to solve for **equilibria**, points where the discrete-time dynamical system leaves the value unchanged.

1.6.1 Cobwebbing: A Graphical Solution Technique

Suppose we have a general discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

with the updating function graphed in Figure 1.6.1. By adding the diagonal (the line $m_{t+1} = m_t$) to the graph, we can find the behavior of solutions graphically. The technique is called **cobwebbing**.

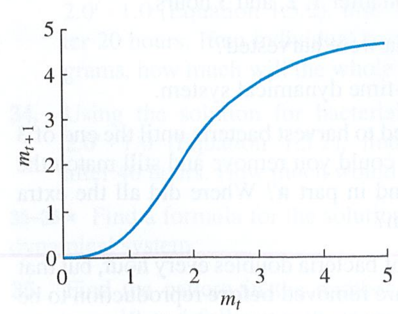


FIGURE 1.6.1
Graph of the updating function

Suppose we are given some initial condition m_0. To find m_1, we evaluate the updating function at m_0, or

$$m_1 = f(m_0).$$

Graphically, m_1 is the point on the graph of the updating function directly above m_0 (Figure 1.6.2a). Similarly, m_2 is the point on the graph of the updating function directly above m_1 and so on.

The missing step is moving m_1 from the vertical axis onto the horizontal axis. The trick is to **reflect** it off the diagonal line that has equation $m_{t+1} = m_t$. Move the point (m_0, m_1) horizontally until it intersects the diagonal. Moving a point horizontally does not change the vertical coordinate. The intersection with the diagonal occurs at the point (m_1, m_1) (Figure 1.6.2b). The point $(m_1, 0)$ lies directly below (Figure 1.6.2c).

What have we done? Starting from the initial value m_0, plotted on the horizontal axis, we used the updating function to find m_1 on the vertical axis and the reflecting trick to project m_1 onto the horizontal axis. We then can find m_2, by moving vertically to the graph of the updating function (Figure 1.6.2d). To find m_3, we move horizontally to the diagonal to reach the point (m_2, m_2), and then vertically to the point (m_2, m_3).

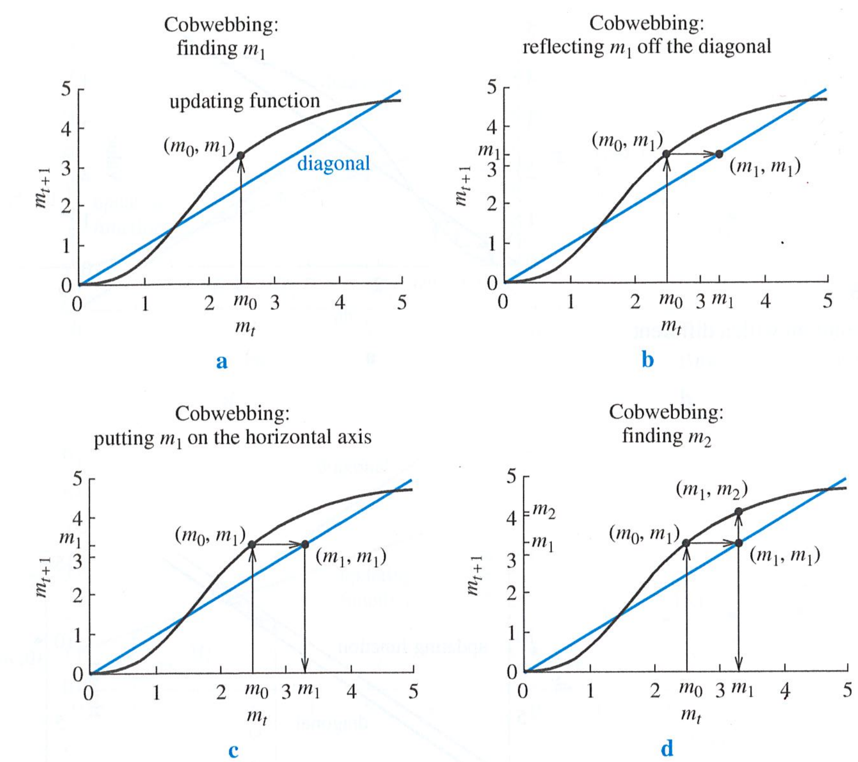


FIGURE 1.6.2
Cobwebbing: The first steps

Because the lines reaching all the way to the horizontal axis are unnecessary, they are generally omitted to make the diagram more readable (Figure 1.6.3).

Having found m_1, m_2, and m_3 on our cobwebbing graph, we can sketch a graph of the solution that shows the measurement as a function of time. In Figure 1.6.2, we began at $m_0 = 2.5$. This is plotted as the point $(0, m_0) = (0, 2.5)$ in the solution (Figure 1.6.4) The value m_1 is approximately 3.2 and is plotted as the point $(1, m_1)$ in the solution. The values of m_2 and m_3 increase more slowly, and are plotted thus on the graph. Without plugging numbers into the formula, we have used the **graph** of the updating function to figure out the behavior of a solution starting from a given initial condition.

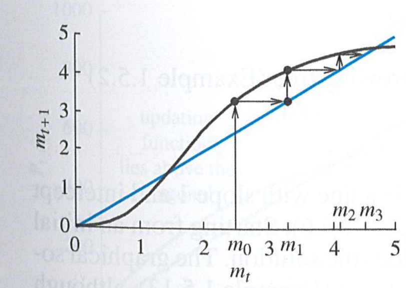


FIGURE 1.6.3
Cobwebbing

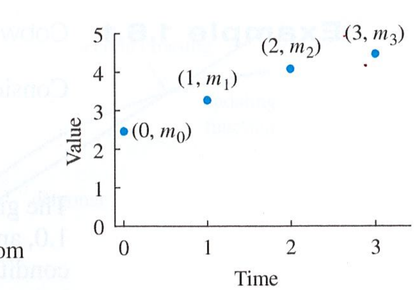


FIGURE 1.6.4
The solution derived from a cobweb diagram

Similarly, we can find how the concentration would behave over time if we started from the different initial condition $m_0 = 1.2$ (Figure 1.6.5). In this case, the diagonal lies below the graph of the updating function, so reflecting off the diagonal moves points to the left. Therefore, the solution decreases.

The steps for cobwebbing are summarized in the following algorithm.

Algorithm 1.4 Using Cobwebbing to Find a Solution

1. Graph the updating function and the diagonal.
2. Starting from the initial condition go "up or down to the updating function and over to the diagonal."

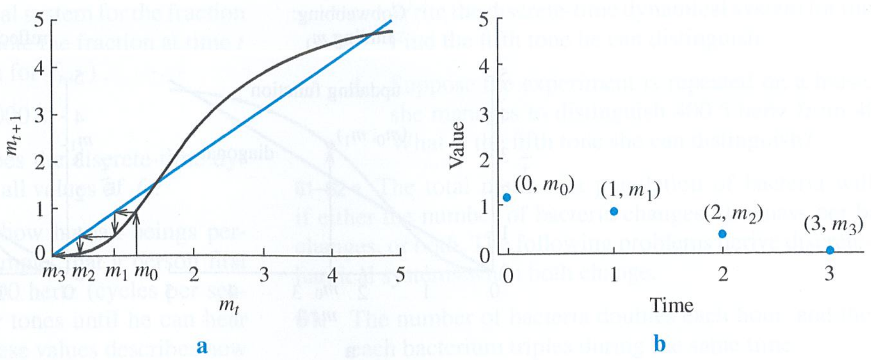


FIGURE 1.6.5

Cobweb and solution with a different initial condition

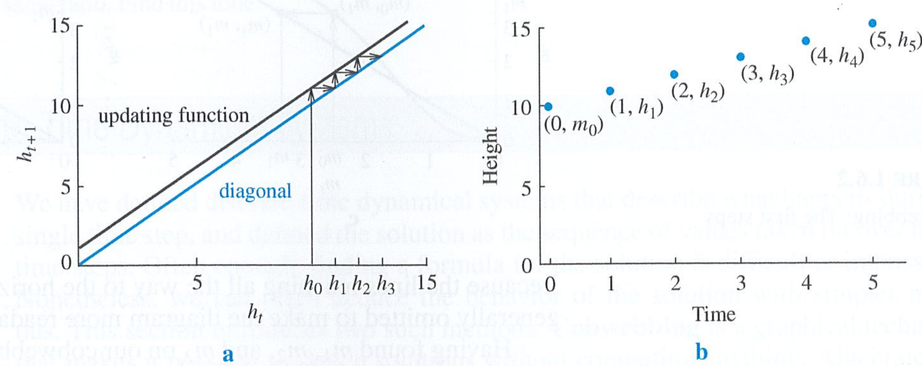


FIGURE 1.6.6

Cobweb and solution of tree growth model

3. Repeat for as many steps as needed to find the pattern.
4. Sketch the solution at times 0, 1, 2, and so forth as the consecutive horizontal coordinates of intersections with the diagonal.

Example 1.6.1 Cobwebbing and Solution of the Tree Growth Model

Consider the discrete-time dynamical system for a growing tree (Example 1.5.2)

$$h_{t+1} = h_t + 1.0.$$

The graph of the updating function $g(h_t) = h_t + 1.0$ is a line with slope 1 and intercept 1.0, and is thus parallel to the diagonal $h_{t+1} = h_t$ (Figure 1.6.6). Starting from an initial condition of 10.0, the cobweb moves up steadily, as does the solution. The graphical solution is consistent with the exact solution $h_t = 10.0 + t$ m (Example 1.5.13), although it does not provide exact **quantitative** predictions.

Example 1.6.2 Cobwebbing and Solution of the Medication Model

Consider the discrete-time dynamical system for medication (Example 1.5.4)

$$M_{t+1} = 0.5M_t + 1.0.$$

The updating function is a line with slope 0.5 and intercept 1, and is thus less steep than the diagonal $M_{t+1} = M_t$. If we begin at $M_0 = 5$, the cobweb and solution decrease more and more slowly over time (Figure 1.6.7). If we begin instead at $M_0 = 1$, the cobweb and solution increase over time (Figure 1.6.8).

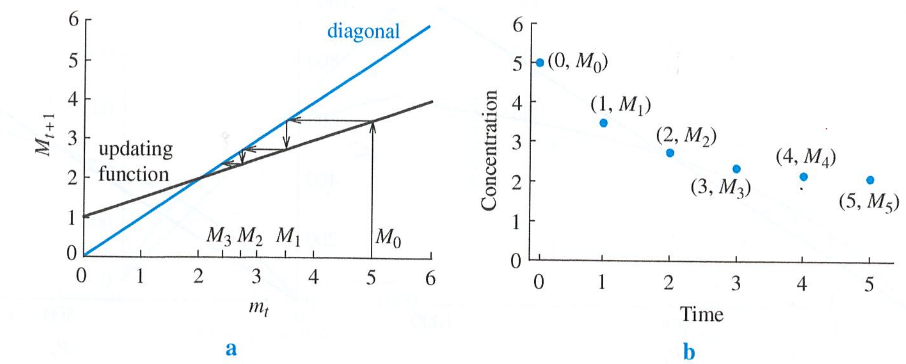


FIGURE 1.6.7

Cobweb and solution of the medication model: $M_0 = 5.0$

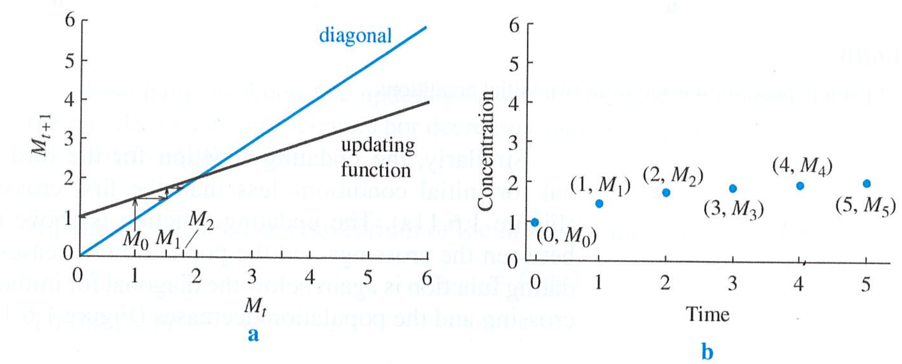


FIGURE 1.6.8

Cobweb and solution of the medication model: $M_0 = 1.0$

1.6.2 Equilibria: Graphical Approach

The points where the graph of the updating function intersects the diagonal play a special role in cobweb diagrams. These points also play an essential role in understanding the behavior of discrete-time dynamical systems. Consider the discrete-time dynamical systems plotted in Figure 1.6.9. The first describes a population of plants (denoted by P_t at time t) and the second a population of birds (denoted by B_t at time t). Each graph includes the diagonal line used in cobwebbing.

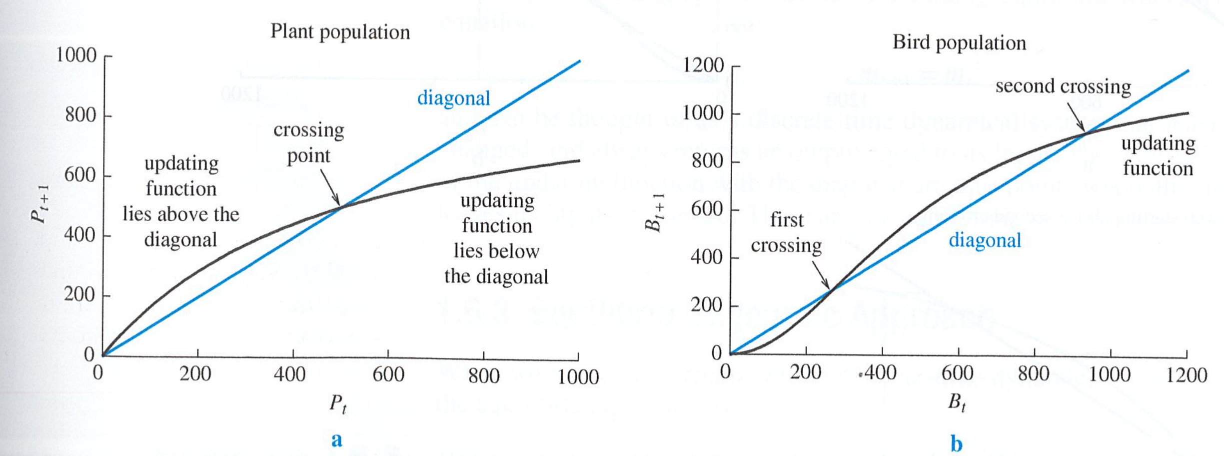


FIGURE 1.6.9

Dynamics of two populations

If we begin cobwebbing from an initial condition where the graph of the updating function lies **above** the diagonal, the population increases (Figure 1.6.10a). In contrast, if we begin cobwebbing from an initial condition where the graph of the updating function lies **below** the diagonal, the population decreases (Figure 1.6.10b). The plant population will thus increase if the initial condition lies to the left of the crossing point, and decrease if it lies to the right of the crossing point.

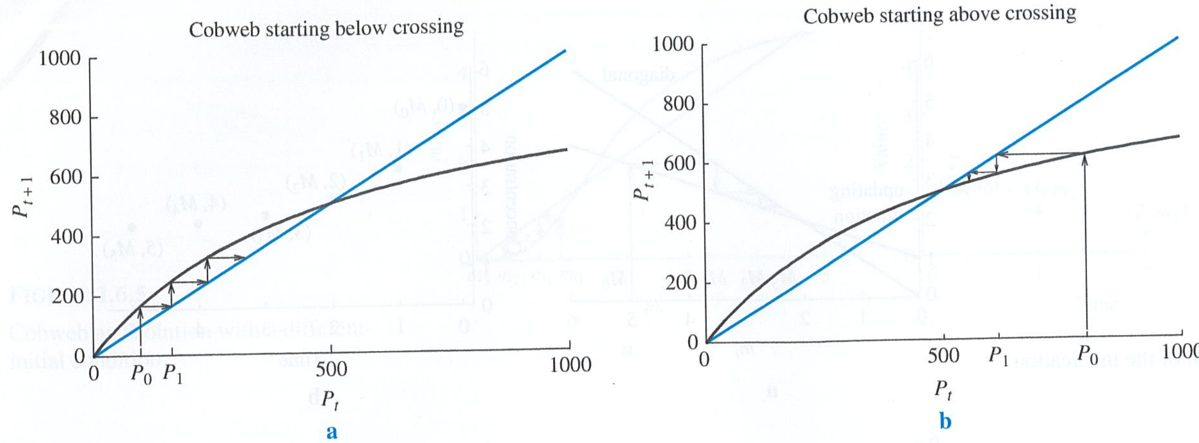


FIGURE 1.6.10
Behavior of plant population starting from two initial conditions

Similarly, the updating function for the bird population lies below the diagonal for initial conditions less than the first crossing and the population decreases (Figure 1.6.11a). The updating function is above the diagonal for initial conditions between the crossings and the population increases (Figure 1.6.11b). Finally, the updating function is again below the diagonal for initial conditions greater than the second crossing and the population decreases (Figure 1.6.11c).

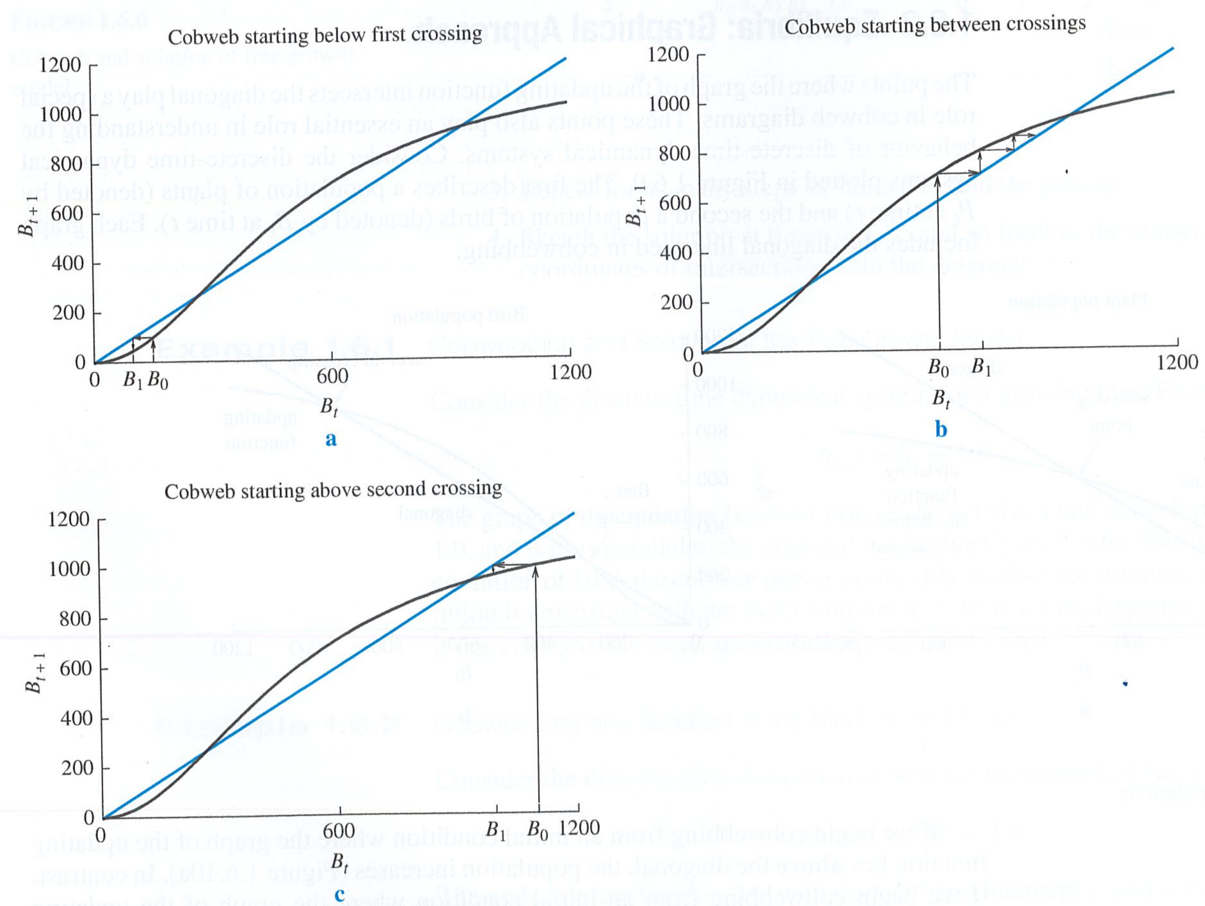


FIGURE 1.6.11
Behavior of bird population starting from three initial conditions

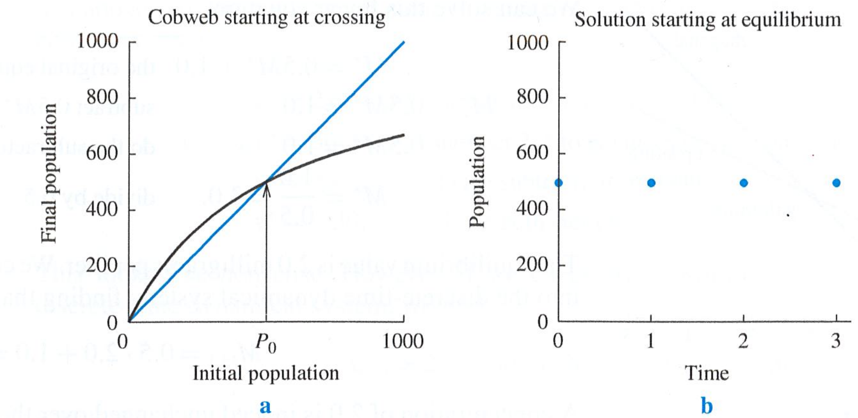


FIGURE 1.6.12
Behavior of plant population starting from an equilibrium

What happens where the updating function crosses the diagonal? At these points, the population neither increases nor decreases, and thus remains the same. Such a point is called an **equilibrium**.

Definition 1.11 A point m^* is called an equilibrium of the discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

if $f(m^*) = m^*$.

This definition says that the discrete-time dynamical system leaves m^* unchanged. These points can be found graphically by looking for intersections of the graph of the updating function with the diagonal line.

When there is more than one equilibrium, they are called **equilibria**. The plant population has two equilibria, one at $P = 0$ and one at $P = 50$. If we start cobwebbing from an initial condition exactly equal to an equilibrium, not much happens. The cobweb goes up to the crossing point and gets stuck there (Figure 1.6.12a). The solution is a horizontal sequence of dots (Figure 1.6.12b).

Why does the graphical method for finding equilibria work? The diagonal has equation

$$m_{t+1} = m_t$$

and can be thought of as a discrete-time dynamical system that leaves **all** inputs unchanged, and always returns an output equal to its input. The intersections of the graph of the updating function with the diagonal are thus points where the updating function leaves its input unchanged. These are the equilibria.

1.6.3 Equilibria: Algebraic Approach

When we know the formula for the discrete-time dynamical system, we can solve for the equilibria algebraically.

Example 1.6.3 The Equilibrium of the Medication Discrete-Time Dynamical System

Recall the discrete-time dynamical system for medication

$$M_{t+1} = 0.5M_t + 1.0$$

(Figure 1.6.13, Example 1.5.4). Let M^* stand for an equilibrium. The equation for equilibrium says that M^* is unchanged by the discrete-time dynamical system, or

$$M^* = 0.5M^* + 1.0.$$

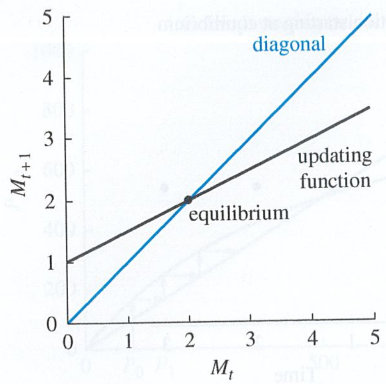


FIGURE 1.6.13
Equilibrium of the medication discrete-time dynamical system

We can solve this linear equation.

$$\begin{aligned} M^* &= 0.5M^* + 1.0 && \text{the original equation} \\ M^* - 0.5M^* &= 1.0 && \text{subtract } 0.5M^* \text{ to get unknowns on one side} \\ 0.5M^* &= 1.0 && \text{do the subtraction} \\ M^* &= \frac{1.0}{0.5} = 2.0. && \text{divide by 0.5} \end{aligned}$$

The equilibrium value is 2.0 milligrams per liter. We can check this by plugging $M_t = 2.0$ into the discrete-time dynamical system, finding that

$$M_{t+1} = 0.5 \cdot 2.0 + 1.0 = 2.0.$$

A concentration of 2.0 is indeed unchanged over the course of a day. Furthermore, we have seen that solutions approach this equilibrium (Examples 1.5.14 and 1.5.15).

Example 1.6.4 The Equilibrium of the Bacterial Discrete-Time Dynamical System

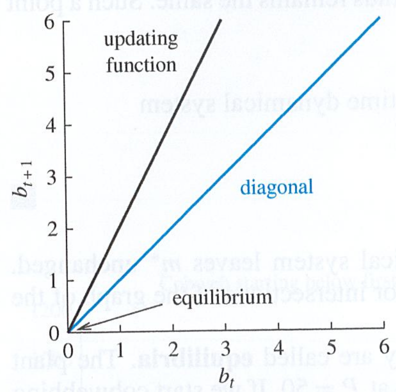


FIGURE 1.6.14
Equilibrium of the bacterial discrete-time dynamical system

To find the equilibria for the bacterial population discrete-time dynamical system

$$b_{t+1} = 2b_t$$

(Figure 1.6.14, Example 1.5.1), we write the equation for equilibria,

$$b^* = 2b^*.$$

We then solve this equation

$$\begin{aligned} b^* &= 2b^* && \text{the original equation} \\ b^* - b^* &= 2b^* - b^* && \text{subtract } b^* \text{ from both sides} \\ 0 &= b^*. && \text{do the subtraction} \end{aligned}$$

Consistent with our picture, the only equilibrium is at $b_t = 0$. The only number that remains the same after doubling is 0.

Example 1.6.5 A Discrete-Time Dynamical System with No Equilibrium

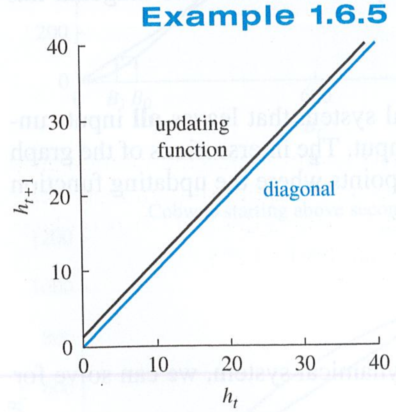


FIGURE 1.6.15
A discrete-time dynamical system with no equilibrium

The updating function for a growing tree following the discrete-time dynamical system

$$h_{t+1} = h_t + 1.0$$

has a graph that is parallel to the diagonal (Figure 1.6.15). To solve for the equilibria, we try

$$\begin{aligned} h^* &= h^* + 1 && \text{the equation for the equilibrium} \\ h^* - h^* &= 1 && \text{subtract } h^* \text{ to get unknowns on one side} \\ 0 &= 1. && \text{do the subtraction} \end{aligned}$$

This looks bad. The graph of the updating function and the graph of the diagonal do not intersect because they are parallel lines. Something that grows 1.0 m per year cannot remain unchanged.

Example 1.6.6 A Biologically Unrealistic Equilibrium

The graph of the updating function associated with a mite population that follows the discrete-time dynamical system

$$x_{t+1} = 2x_t + 30$$

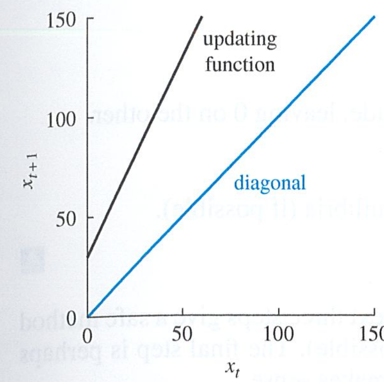


FIGURE 1.6.16
The discrete-time dynamical system for mites

lies above the diagonal for all positive values of x_t (Figure 1.6.16). To solve for the equilibria, try

$$\begin{aligned} x^* &= 2x^* + 30 && \text{the equation for the equilibrium} \\ x^* - 2x^* &= 30 && \text{subtract } 2x^* \text{ to get unknowns on one side} \\ -x^* &= 30 && \text{do the subtraction} \\ x^* &= -30. && \text{divide both sides by } -1 \end{aligned}$$

This looks like nonsense. However, if we check by substituting $x_t = -30$ into the discrete-time dynamical system, we find

$$x_{t+1} = 2 \cdot (-30) + 30 = -30,$$

which is indeed equal to x_t.

Although there is a **mathematical** equilibrium, there is no **biological** equilibrium. If we extend the graph to include biologically meaningless negative values, we see that the graph of the updating function does intersect the diagonal (Figure 1.6.17).

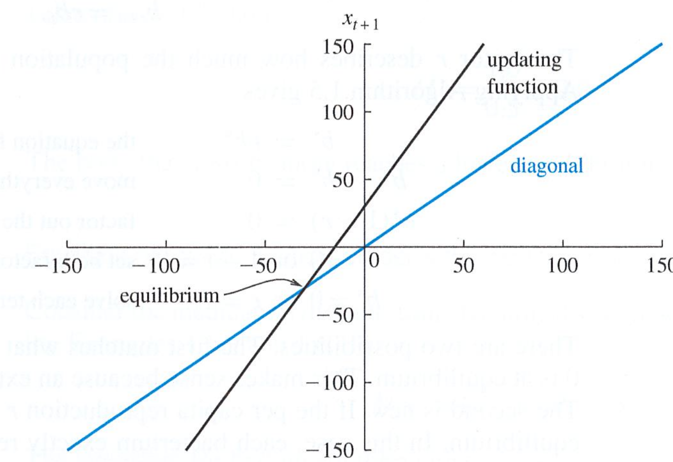


FIGURE 1.6.17
Extending the discrete-time dynamical system for mites to include a negative domain

Example 1.6.7 The Loading Dose for a Medication

In the discrete-time dynamical system for medication (Example 1.6.3) the solution only slowly increases to its equilibrium at $M^* = 2$. To reach this equilibrium immediately, the dose on the first day can be increased. Rather than giving a dose of 1.0 as on later days, suppose a larger dose of 2.0 were given. Then, on this first day,

$$M_1 = 0.5M_0 + 2.0.$$

Because this is the first dose given, $M_0 = 0$, meaning that $M_1 = 2.0$, equal to the equilibrium. Returning thereafter to the normal dose of 1.0, the system

$$M_{t+1} = 0.5M_t + 1.0$$

will remain at equilibrium. This **loading dose** avoids the slow build-up to the desired concentration.

Algebra Involving Parameters Studying the general form of a discrete-time dynamical system, using parameters instead of numbers, can simplify the algebra and make results easier to understand. When we work with parameters, however, we must be more careful with the algebra.

Algorithm 1.5 Solving for Equilibria

1. Write the equation for the equilibrium.
2. Use subtraction to move all the terms to one side, leaving 0 on the other.
3. Factor (if possible).
4. Set each factor equal to 0 and solve for the equilibria (if possible).
5. Think about the results.

As always, we begin by setting up the problem. The next three steps give a safe method to do the algebra (although the algebra may be impossible). The final step is perhaps the most important. A result is worthwhile only if it makes sense.

Example 1.6.8 Finding Equilibria of the Bacterial Model in General

Consider the bacterial discrete-time dynamical system where the factor of 2.0 has been replaced with a general **per capita reproduction** of r,

$$b_{t+1} = rb_t.$$

The factor r describes how much the population grows (or declines) in one hour. Applying Algorithm 1.5 gives

$b^* = rb^*$	the equation for the equilibrium
$b^* - rb^* = 0$	move everything to one side
$b^*(1-r) = 0$	factor out the common factor of b^*
$b^* = 0$ or $1 - r = 0$	set both factors to 0
$b^* = 0$ or $r = 1$.	solve each term

There are two possibilities. The first matches what we found earlier; a population of 0 is at equilibrium. This makes sense because an extinct population cannot reproduce. The second is new. If the per capita reproduction r is exactly 1, any value of b_t is an equilibrium. In this case, each bacterium exactly replaces itself. The population size will remain the same no matter what its size, even though the individual bacteria are reproducing and dying.

Example 1.6.9 Equilibria of the Medication Model with a Dosage Parameter

Consider the medication discrete-time dynamical system with the parameter S

$$M_{t+1} = 0.5M_t + S,$$

where S represents the daily dosage. The algorithm for finding equilibria gives

$M^* = 0.5M^* + S$	the equation for the equilibrium
$M^* - 0.5M^* - S = 0$	move everything to one side
$0.5M^* - S = 0$	simplify
$M^* = 2.0S.$	nothing to factor, solve for M^*

The equilibrium value is proportional to S, the daily dosage.

Example 1.6.10 Equilibria of the Medication Model with Absorption

Consider the medication discrete-time dynamical system with parameter α

$$M_{t+1} = (1-\alpha)M_t + 1.0,$$

where the parameter α represents the fraction of existing medication absorbed by the body during a given day. For example, if $\alpha = 0.1$, 10% of the medication is absorbed by the body and 90% remains.

$M^* = (1-\alpha)M^* + 1.0$	the equation for the equilibrium
$M^* - (1-\alpha)M^* - 1.0 = 0$	move everything to one side
$M^* - M^* + \alpha M^* - 1.0 = 0$	distribute negative sign through quantity
$\alpha M^* - 1.0 = 0$	cancel $M^* - M^*$
$M^* = \frac{1.0}{\alpha}.$	solve for M^*

The equilibrium value is proportional to the reciprocal of α and is thus larger when the fraction absorbed is larger. If $\alpha = 0.1$, the equilibrium is

$$M^* = \frac{1.0}{0.1} = 10.0.$$

In contrast, if the body absorbs 50% of the medication each day, leading to a larger value of $\alpha = 0.5$, then

$$M^* = \frac{1.0}{0.5} = 2.0.$$

The body that absorbs more reaches a lower equilibrium.

Example 1.6.11 Equilibria of the Medication Model with Two Parameters

Consider the medication discrete-time dynamical system with two parameters (extending Examples 1.6.9 and 1.6.10),

$$M_{t+1} = (1-\alpha)M_t + S.$$

The algorithm for finding equilibria gives

$M^* = (1-\alpha)M^* + S$	the equation for the equilibrium
$M^* - (1-\alpha)M^* - S = 0$	move everything to one side
$M^* - M^* + \alpha M^* - S = 0$	distribute negative sign through quantity
$\alpha M^* - S = 0$	cancel $M^* - M^*$
$M^* = \frac{S}{\alpha}.$	solve for M^*

The equilibrium value is larger if S is larger or if α is smaller. It makes sense because the equilibrium concentration can be increased in two ways: by increasing the dosage or by decreasing the fraction absorbed.

Summary We have developed a graphical technique to estimate solutions called **cobwebbing**. By examining the diagrams used for cobwebbing, we found that intersections of the graph of the updating function with the diagonal line play a special role. These **equilibria** are points that are unchanged by the discrete-time dynamical system. Algebraically, we find equilibria by solving the equation that describes such points. We can often solve for equilibria in general, without substituting numerical values for the parameters. Solving the equations in this way can help clarify the underlying biological process.

1.6 Exercises

Mathematical Techniques

1–2 ▪ The following steps are used to build a cobweb diagram. Follow them for the given discrete-time dynamical system based on bacterial populations.

a. Graph the updating function.

b. Use your graph of the updating function to find the point (b_0, b_1).

c. Reflect it off the diagonal to find the point (b_1, b_1).

d. Use the graph of the updating function to find (b_1, b_2).

e. Reflect off the diagonal to find the point (b_2, b_2).

f. Use the graph of the updating function to find (b_2, b_3).

g. Sketch the solution as a function of time.

1. The discrete-time dynamical system $b_{t+1} = 2.0b_t$ with $b_0 = 1.0$.

2. The discrete-time dynamical system $n_{t+1} = 0.5n_t$ with $n_0 = 1.0$.

3–6 ▪ Cobweb the following discrete-time dynamical systems for three steps starting from the given initial condition. Compare with the solution found earlier.

3. $v_{t+1} = 1.5v_t$, starting from $v_0 = 1220\mu m^3$ (as in Section 1.5, Exercise 5).

4. $l_{t+1} = l_t - 1.7$, starting from $l_0 = 13.1$ cm (as in Section 1.5, Exercise 6).

5. $n_{t+1} = 0.5n_t$, starting from $n_0 = 1200$ (as in Section 1.5, Exercise 7).

6. $M_{t+1} = 0.75M_t + 2.0$ starting from the initial condition $M_0 = 16.0$ (as in Section 1.5, Exercise 8).

7–12 ▪ Graph the updating functions associated with the following discrete-time dynamical systems, and cobweb for five steps starting from the given initial condition.

7. $x_{t+1} = 2x_t - 1$, starting from $x_0 = 2$.

8. $z_{t+1} = 0.9z_t + 1$, starting from $z_0 = 3$.

9. $w_{t+1} = -0.5w_t + 3$, starting from $w_0 = 0$.

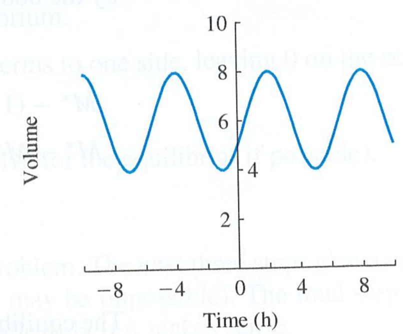
10. $x_{t+1} = 4 - x_t$, starting from $x_0 = 1$ (as in Section 1.5, Exercise 25).

11. $x_{t+1} = \dfrac{x_t}{1 + x_t}$, starting from $x_0 = 1$ (as in Section 1.5, Exercise 23).

12. $x_{t+1} = \dfrac{x_t}{x_t - 1}$ for $x_t > 1$, starting from $x_0 = 3$ (as in Section 1.5, Exercise 26).

13–16 ▪ Find the equilibria of the following discrete-time dynamical system from the graphs of their updating functions Label the coordinates of the equilibria.

13.



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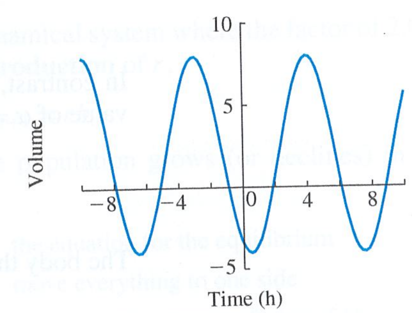

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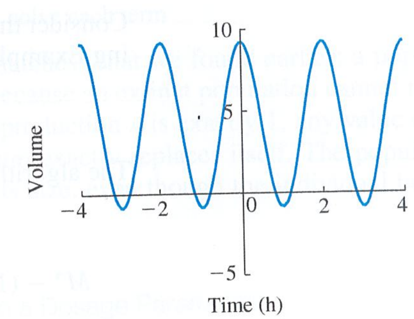
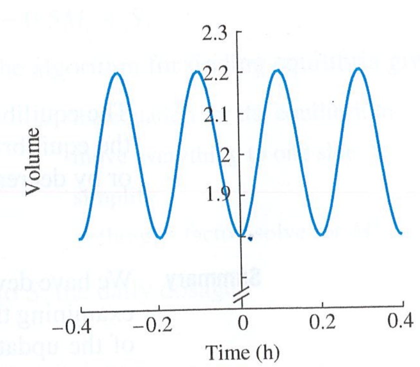


16.



17–22 ▪ Sketch graphs of the following updating functions over the given range and mark the equilibria. Find the equilibria algebraically if possible.

17. $f(x) = x^2$ for $0 \le x \le 2$.

18. $g(y) = y^2 - 1$ for $0 \le y \le 2$.

19–22 ▪ Graph the following discrete-time dynamical systems. Solve for the equilibria algebraically, and identify equilibria and the regions where the updating function lies above the diagonal on your graph.

19. $c_{t+1} = 0.5c_t + 8.0$, for $0 \le c_t \le 30$.

20. $b_{t+1} = 3b_t$, for $0 \le b_t \le 10$.

21. $b_{t+1} = 0.3b_t$, for $0 \le b_t \le 10$.

22. $b_{t+1} = 2.0b_t - 5.0$, for $0 \le b_t \le 10$.

23–30 ▪ Find the equilibria of the following discrete-time dynamical systems. Compare with the results of your cobweb diagram from the earlier problem.

23. $v_{t+1} = 1.5v_t$ (as in Section 1.5, Exercise 5).

24. $l_{t+1} = l_t - 1.7$ (as in Section 1.5, Exercise 6).

25. $x_{t+1} = 2x_t - 1$ (as in Exercise 7).

26. $z_{t+1} = 0.9z_t + 1$ (as in Exercise 8).

27. $w_{t+1} = -0.5w_t + 3$ (as in Exercise 9).

28. $x_{t+1} = 4 - x_t$ (as in Exercise 10).

29. $x_{t+1} = \dfrac{x_t}{1 + x_t}$ (as in Exercise 11).

30. $x_{t+1} = \dfrac{x_t}{x_t - 1}$ for $x_t > 1$ (as in Exercise 12).

31–34 ▪ Find the equilibria of the following discrete-time dynamical systems that include parameters. Identify values of the parameter for which there is no equilibrium, for which the equilibrium is negative, and for which there is more than one equilibrium.

31. $w_{t+1} = aw_t + 3$.

32. $x_{t+1} = b - x_t$.

33. $x_{t+1} = \dfrac{ax_t}{1 + x_t}$.

34. $x_{t+1} = \dfrac{x_t}{x_t - K}$.

Applications

35–40 ▪ Cobweb the following discrete-time dynamical systems for five steps starting from the given initial condition.

35. An alternative tree growth discrete-time dynamical system with form $h_{t+1} = h_t + 5.0$ with initial condition $h_0 = 10$.

36. The lizard-mite system (Example 1.5.3) $x_{t+1} = 2x_t + 30$ with initial condition $x_0 = 0$.

37. The model defined in Section 1.5, Exercise 37 starting from an initial volume of 1420.

38. The model defined in Section 1.5, Exercise 38 starting from an initial mass of 13.1.

39. The model defined in Section 1.5, Exercise 39 starting from an initial population of 800.

40. The model defined in Section 1.5, Exercise 40 starting from an initial yield of 20.

41–42 ▪ Reconsider the data describing the levels of a medication in the blood of two patients over the course of several days (measured in mg per liter), used in Section 1.5, Exercises 53 and 54.

Day	Medication Level in Patient 1	Medication Level in Patient 2
0	20.0	0.0
1	16.0	2.0
2	13.0	3.2
3	10.75	3.92

41. For the first patient, graph the updating function and cobweb starting from the initial condition on day 0. Find the equilibrium.

42. For the second patient, graph the updating function and cobweb starting from the initial condition on day 0. Find the equilibrium.

43–44 ▪ Cobweb and find the equilibrium of the following discrete-time dynamical system.

43. Consider a bacterial population that doubles every hour, but 1.0×10^6 individuals are removed after reproduction (Section 1.5, Exercise 57). Cobweb starting from $b_0 = 3.0 \times 10^6$ bacteria.

44. Consider a bacterial population that doubles every hour, but 1.0×10^6 individuals are removed before reproduction (Section 1.5, Exercise 58). Cobweb starting from $b_0 = 3.0 \times 10^6$ bacteria.

45–46 ▪ Consider the following general models for bacterial populations with harvest.

45. Consider a bacterial population that doubles every hour, but h individuals are removed after reproduction. Find the equilibrium. Does it make sense?

46. Consider a bacterial population that increases by a factor of r every hour, but 1.0×10^6 individuals are removed after reproduction. Find the equilibrium. What values of r produce a positive equilibrium?

47–48 ▪ Consider the general model $M_{t+1} = (1 - \alpha)M_t + S$ for medication (Example 1.6.11). Find the loading dose (Example 1.6.7) in the following cases.

47. $\alpha = 0.2$, $S = 2$.

48. $\alpha = 0.8$, $S = 4$.

Computer Exercises

49. Use your computer (it may have a special feature for this) to find and graph the first 10 points on the solutions of the following discrete-time dynamical systems. The first two describe populations with reproduction and immigration of 100 individuals per generation, and the last two describe populations that have 100 individuals harvested or removed each generation.

a. $b_{t+1} = 0.5b_t + 100$ starting from $b_0 = 100$.

b. $b_{t+1} = 1.5b_t + 100$ starting from $b_0 = 100$.