c. $b_{t+1} = 1.5b_t - 100$ starting from $b_0 = 201$.

d. $b_{t+1} = 1.5b_t - 100$ starting from $b_0 = 199$.

e. What happens if you run the last one for 15 steps? What is wrong with the model?

50. Compose the medication discrete-time dynamical system $M_{t+1} = 0.5M_t + 1.0$ with itself 10 times. Plot the resulting function. Use this composition to find the concentration after 10 days starting from concentrations of 1.0, 5.0, and 18.0 milligrams per liter. If the goal is to reach a stable concentration of 2.0 milligrams per liter, do you think this is a good therapy?

1.7 Expressing Solutions with Exponential Functions

The solution associated with the bacterial discrete-time dynamical system given by $b_{t+1} = 2.0b_t$ is

$$b_t = 2.0^t$$

when $b_0 = 1.0$. As a function of t, the solution is an example of an **exponential function**. To find how long it will take the population to reach a particular target value such as 100 requires solving an equation where the variable t appears in the exponent. Solving for t can be simplified by converting this function into a standard form with the base e, and working with the inverse of the exponential function, the **natural logarithm**. In this section, we will study the **laws of exponents** and the **laws of logarithms** that make this conversion convenient. We will generalize the bacterial population growth discrete-time dynamical system to include the death of some bacteria and show that the solution is again an exponential function, but with **base** equal to the **per capita reproduction** of the bacteria.

1.7.1 Bacterial Population Growth in General

The bacteria studied hitherto have doubled in number each hour. Each bacterium divided once and both "daughter" bacteria survived. Suppose instead that only a fraction σ ("sigma") of the daughters survive. Instead of 2.0 offspring per bacteria, we find an average of 2σ offspring (Figure 1.7.1). For example, if only 75% of offspring survived ($\sigma = 0.75$), there are an average of 1.5 surviving offspring per parent. Let

$$r = 2\sigma.$$

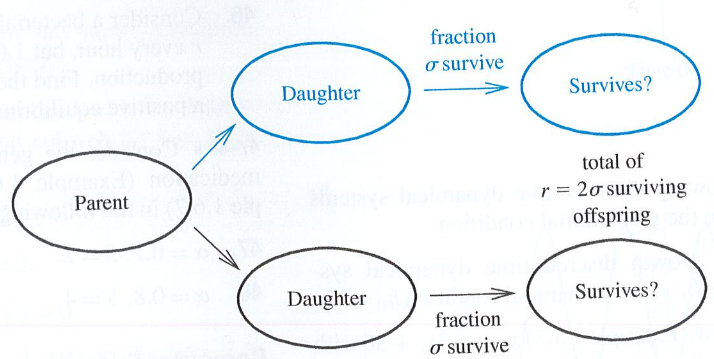

FIGURE 1.7.1
Bacterial population growth with reproduction and mortality

The new parameter r represents the number of new bacteria produced per bacterium and is called the **per capita reproduction**.

In terms of the parameter r, the discrete-time dynamical system is

$$b_{t+1} = rb_t.$$

This fundamental equation of population biology says that the population at time $t+1$ is equal to the per capita reproduction (the number of new bacteria per old bacterium)

times the population at time t (the number of old bacteria), or

$$\text{new population} = \text{per capita reproduction} \times \text{old population}.$$

Example 1.7.1 Discrete-Time Dynamical System If Most Offspring Survive

If $\sigma = 0.75$, then $r = 2 \cdot 0.75 = 1.5$. The discrete-time dynamical system is

$$b_{t+1} = 1.5b_t.$$

If $b_0 = 100$, then $b_1 = 1.5 \cdot 100 = 150$. The population increases by 50% each hour.

Example 1.7.2 Discrete-Time Dynamical System If Few Offspring Survive

If $\sigma = 0.25$, then $r = 2 \cdot 0.25 = 0.5$. The discrete-time dynamical system is

$$b_{t+1} = 0.5b_t.$$

If $b_0 = 100$, then $b_1 = 0.5 \cdot 100 = 50$. Because the value of the survival σ is so small, this population decreases by 50% each hour.

Starting from a population with b_0 bacteria, we can apply the discrete-time dynamical system repeatedly to derive a solution, much as we did in Example 1.5.11 with the particular value $r = 2$ (Figure 1.7.2). We find

$$b_1 = rb_0$$
$$b_2 = rb_1 = r^2b_0$$
$$b_3 = rb_2 = r^3b_0.$$

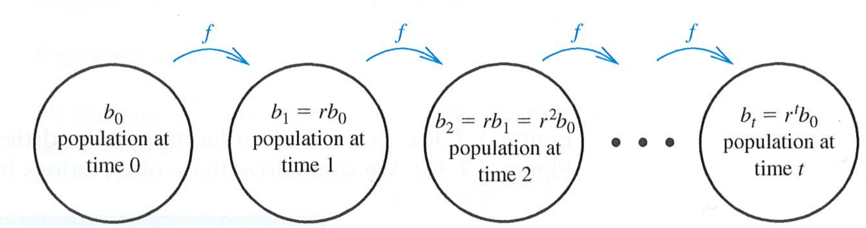

FIGURE 1.7.2
Bacterial population growth

Each hour, the initial population b_0 is multiplied by the per capita reproduction r. After t hours, the initial population b_0 has been multiplied by t factors of r. Therefore

$$b_t = r^tb_0.$$

How do these solutions behave for different values of the per capita reproduction r? Results with four values of r starting from $b_0 = 1.0$ are given in the following table.

t	$r = 2.0$	$r = 1.5$	$r = 1.0$	$r = 0.5$
0	1.0	1.0	1.0	1.0
1	2.0	1.5	1.0	0.5
2	4.0	2.25	1.0	0.25
3	8.0	3.37	1.0	0.125
4	16.0	5.06	1.0	0.0625
5	32.0	7.59	1.0	0.0312
6	64.0	11.4	1.0	0.0156
7	128.0	17.1	1.0	0.00781
8	256.0	25.6	1.0	0.00391

In the first two columns, $r > 1$ and the population increases each hour (Figure 1.7.3a and b). In the third column, $r = 1$ and the population remains the same hour after hour

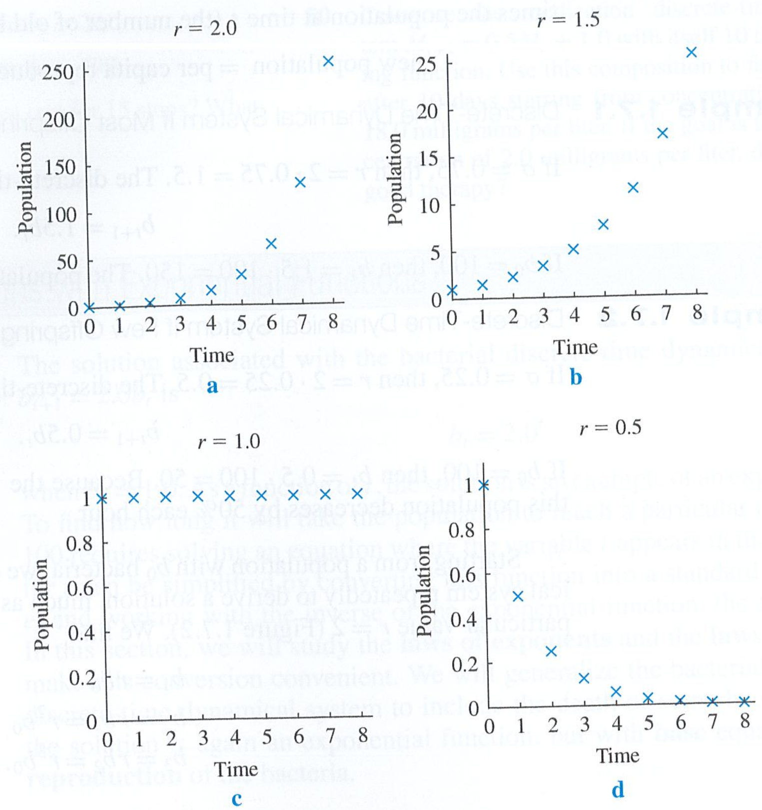

FIGURE 1.7.3
Growing and declining bacterial populations

(Figure 1.7.3c). In the final column, $r < 1$ and the population decreases each hour (Figure 1.7.3d). We summarize these observations in the following table.

Value of r	Behavior of Population
$r > 1$	population increases
$r = 1$	population remains constant
$r < 1$	population decreases

A population with $r = 1$ exactly replaces itself each generation and retains a constant size, even though the individuals in the population change. This is consistent with our finding that any value of b is an equilibrium when $r = 1$ (Example 1.6.8).

1.7.2 Laws of Exponents and Logs

In the solution $b_t = r^t b_0$, the variable t appears in the exponent, in contrast to a function like $f(t) = t^3$ where the variable t is raised to a power. For any positive number a, the **exponential function to the base a** is written

$$f(x) = a^x$$

and said to be "a to the xth power." This function takes x as input and returns as output x factors of a multiplied together. The notation generalizes that used in equations like

$$a^2 = a \cdot a.$$

The key to using exponential functions is knowing the **laws of exponents**, summarized in the table. The table also includes examples using $a = 2$ that can help in remembering when to add and when to multiply.

Laws of Exponents

	General Formula	Example with $a = 2$, $x = 2$ and $y = 3$
Law 1	$a^x \cdot a^y = a^{x+y}$	$2^2 \cdot 2^3 = 2^5 = 32$
Law 2	$(a^x)^y = a^{xy}$	$(2^2)^3 = 2^6 = 64$
Law 3	$a^{-x} = \frac{1}{a^x}$	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
Law 4	$\frac{a^y}{a^x} = a^{y-x}$	$\frac{2^3}{2^2} = 2^{3-2} = 2$
Law 5	$a^1 = a$	$2^1 = 2$
Law 6	$a^0 = 1$	$2^0 = 1$

The exponential function is defined for all values of x, including negative numbers and fractions. What does it mean to multiply half an a or -3 a's together? These expressions must be computed with the laws of exponents.

Example 1.7.3 Negative Powers

To compute a^{-3}, apply law 3 to find

$$a^{-3} = \frac{1}{a^3}.$$

For example

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8} = 0.125.$$

Negative powers are in the denominator. ▲

Example 1.7.4 Fractional Powers

To compute $a^{0.5}$, we raise this unknown quantity to the 2nd power (square it), and use law 2 to find

$$(a^{0.5})^2 = a^{0.5 \cdot 2} = a^1 = a.$$

Therefore, a to the 0.5th power is the number that, when squared, gives back a. In other words, a to the 0.5 power is the square root of a. For example

$$2^{0.5} = \sqrt{2} = 1.41421.$$ ▲

For reasons that will make sense only with a bit of calculus (Section 2.8), the base most commonly used throughout the sciences is the irrational number

$$e = 2.718281828459\ldots$$

The function

$$f(x) = e^x$$

said "e to the x" is called the **exponential function to the base e**, or simply the **exponential function** (Figure 1.7.4). Calculators and computers often abbreviate this as exp. The domain of this function consists of all numbers, and the range is all **positive** numbers.

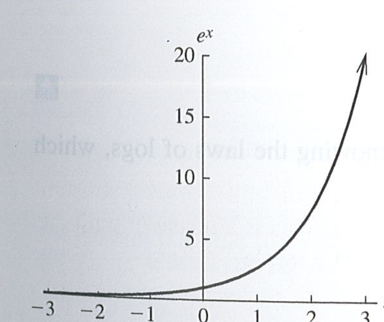

FIGURE 1.7.4
Graph of the exponential function

Example 1.7.5 Examples of the Laws of Exponents with the Base e

- $e^3 \cdot e^4 = e^{3+4} = e^7$ (law 1).
- $(e^3)^4 = e^{3 \cdot 4} = e^{12}$ (law 2).
- $e^{-2} = \frac{1}{e^2}$ (law 3).
- $\frac{e^4}{e^3} = e^{4-3} = e^1 = e$ (laws 4 and 5).

- $e^0 = 1$ (law 6).
- $e^3 + e^4$ cannot be simplified with a law of exponents.

The graph of the exponential function crosses every positive horizontal line only once, and thus passes the horizontal line test for having an inverse (see Section 1.2.4). The inverse is the natural log.

Definition 1.12 The inverse function of the exponential function e^x is called the **natural logarithm** (or natural log). The natural log of x is written $\ln(x)$. The natural logarithm has a domain consisting of all positive numbers.

From the definition of the inverse (Definition 1.6),

$$\ln(e^x) = x$$
$$e^{\ln(x)} = x$$

(Figure 1.7.5).

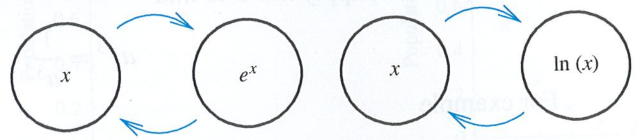

FIGURE 1.7.5
The exponential function and natural logarithm are inverses

The graph of the natural logarithm increases from "negative infinity" near $x = 0$, through 0 at $x = 1$, and rises more and more slowly as x becomes larger (Figure 1.7.6). It is impossible to compute the natural log of a negative number.

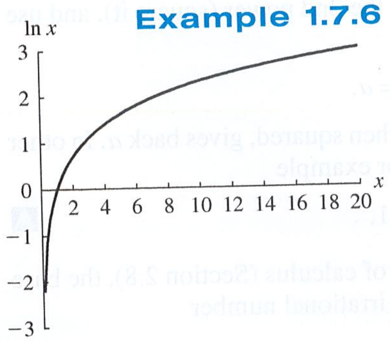

FIGURE 1.7.6
Graph of the natural logarithm

Example 1.7.6 Exponential and Logarithmic Functions

- If $\ln(100) = 4.605$, then $e^{4.605} = 100$.
- If $e^5 = 148.41$, then $\ln(148.41) = 5$.
- If $\ln(0.1) = -2.302$, then $e^{-2.302} = 0.1$.
- If $e^{-3} = 0.04979$, then $\ln(0.4979) = -3$.

The key to understanding natural logarithms is knowing the laws of logs, which are the laws of exponents in reverse.

Example 1.7.7 The Laws of Logs in Action

The Laws of Logs

Law 1	$\ln(xy) = \ln(x) + \ln(y)$
Law 2	$\ln(x^y) = y\ln(x)$
Law 3	$\ln(1/x) = -\ln(x)$
Law 4	$\ln(x/y) = \ln(x) - \ln(y)$
Law 5	$\ln(e) = 1$
Law 6	$\ln(1) = 0$

- $\ln(3) + \ln(4) = \ln(3 \cdot 4) = \ln(12)$, using law 1.
- $\ln(3^4) = 4\ln(3)$, using law 2.
- $\ln(1/3) = -\ln(3)$, using law 3.

- $\ln(4/3) = \ln(4) - \ln(3)$, using law 4.
- $\ln(3) \cdot \ln(4)$ cannot be simplified with a law of logs.

In some disciplines, people use the **exponential function with base 10**, or

$$f(x) = 10^x.$$

Its inverse is the **logarithm to the base 10**, written

$$\log_{10} x$$

and said "log base 10 of x." Just as $\ln(x) = y$ implies that $x = e^y$,

$$\log_{10} x = y$$

implies

$$x = 10^y.$$

For example, if $\log_{10} x = 2.3$, $x = 10^{2.3} = 199.5$. In most ways, the exponential function with base 10 and the log base 10 work much like the exponential function with base e and the natural logarithm. All laws of exponents and logs are the same except for law 5, which becomes

Law 5 of exponents:	$10^1 = 10$
Law 5 of logs:	$\log_{10}(10) = 1.$

The base e is more convenient for studying dynamics with calculus.

Example 1.7.8 Converting Logarithms in Base 10 to Natural Logs

Suppose $\log_{10}(x) = y$. How can we find $\ln(x)$? By the definition of $\log_{10}$,

$$x = 10^y.$$

Then

$$\begin{aligned}
\ln(x) &= \ln(10^y) && \text{taking the natural log of both sides}\\
&= y\ln(10) && \text{law of logs 2}\\
&= 2.303y && \text{because } \ln(10) = 2.303\\
&= 2.303\log_{10}(x). && \text{definition of } y
\end{aligned}$$

For instance, $\log_{10}(100) = 2$, so $\ln(100) = 2.302 \cdot 2 = 4.604$.

1.7.3 Expressing Results with Exponentials

We can use the laws of exponentials and logs to express

$$b_t = r^t b_0$$

in terms of the exponential function with base e. Because the exponential function and the natural logarithm are inverses, we can rewrite r as

$$r = e^{\ln(r)}.$$

Then, using law 2 of exponents,

$$r^t = \left(e^{\ln(r)}\right)^t = e^{\ln(r)t}.$$

The **general solution** for the discrete-time dynamical system

$$b_{t+1} = rb_t$$

with initial condition b_0 can be written in exponential notation as

$$b_t = b_0 e^{\ln(r)t}.$$

Example 1.7.9 Expressing a Solution with the Exponential Function

Consider the case $r = 2.0$ and $b_0 = 1.0$. Because $\ln(2.0) = 0.6931$, the solution is

$$b_t = 1.0e^{\ln(2.0)t} = 1.0e^{(0.6931)t}.$$

What is the value of rewriting the solution in this way? Exponential notation makes it easier to solve equations describing the future behavior of a population.

Example 1.7.10 Using a Solution Expressed with the Exponential Function: Increasing Case

When will the population that obeys $b_{t+1} = 2.0b_t$, with solution

$$b_t = 2.0^t$$

reach 100.0 million? In Example 1.7.9 we wrote this solution in exponential notation. Now we can set $b_t = 100$ and solve for t with the steps

$$e^{\ln(2.0)t} = 100 \quad \text{equation for } t$$
$$\ln(2.0)t = \ln(100) \quad \text{take the natural log of both sides}$$
$$t = \frac{\ln(100)}{\ln(2)} = 6.64. \quad \text{solve for } t$$

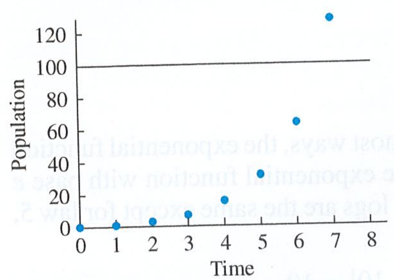

FIGURE 1.7.7
Using solutions to find times

The population will pass 100.0 million between hours 6 and 7 (Figure 1.7.7). The key step uses the natural log, the inverse of the exponential function, to remove the variable t from the exponent.

Example 1.7.11 Using a Solution Expressed with the Exponential Function: Decreasing Case

How long it will take a population with $r < 1$ to decrease to some specified value? Suppose $r = 0.7$ and $b_0 = 100.0$. The population decreases because $r < 1$. When will it reach $b_t = 2$ (Figure 1.7.8)? In exponential notation,

$$b_t = 100.0e^{\ln(0.7)t}.$$

Then $b_t = 2.0$ can be solved

$$100.0e^{\ln(0.7)t} = 2.0 \quad \text{equation for } t$$
$$e^{\ln(0.7)t} = 0.02 \quad \text{divide both sides by 100}$$
$$\ln(0.7)t = \ln(0.02) \quad \text{take the natural log}$$
$$t = \frac{\ln(0.02)}{\ln(0.7)} = 10.96. \quad \text{solve for } t$$

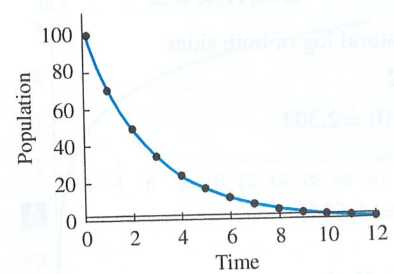

FIGURE 1.7.8
Using solutions to find times

All the negative signs cancel, and we see that this population will pass 2.0 just before hour 11.

Throughout the sciences, many measurements other than population sizes are described by exponential functions. In such cases, we write the measurement S as a function of t as

$$S(t) = S(0)e^{\alpha t}.$$

The parameter $S(0)$ represents the value of the measurement at time $t = 0$. The parameter α describes how the measurement changes and has dimensions of 1/time. When $\alpha > 0$ the function is increasing (Figure 1.7.9a and b). When $\alpha < 0$ the function is decreasing (Figure 1.7.9c and d). The function increases most quickly with large positive values of α, and decreases most quickly with large negative values of α.

The **doubling time** is defined as the time it takes the initial value of a growing measurement to double (Figure 1.7.10).

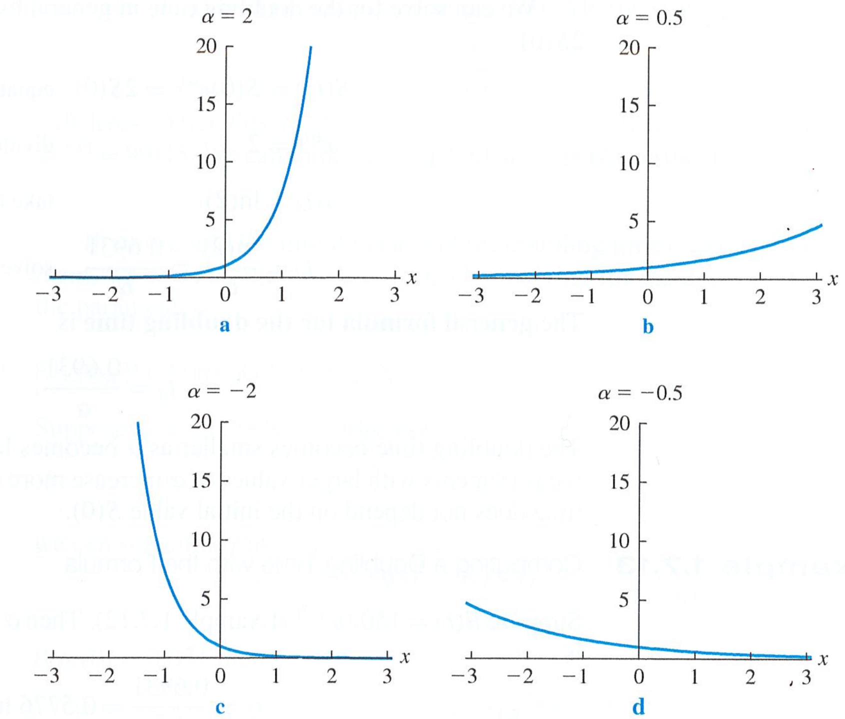

FIGURE 1.7.9
The exponential function with different parameter values in the exponent

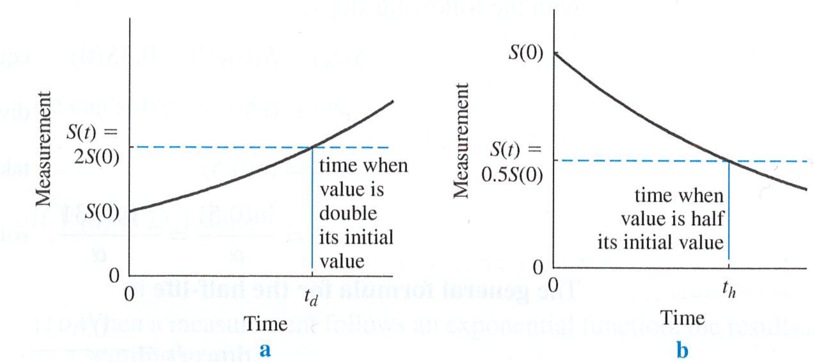

FIGURE 1.7.10
Doubling times and half-lives

Example 1.7.12 Computing a Doubling Time

Suppose

$$S(t) = 150.0e^{1.2t}$$

with t measured in hours. This measurement starts at $S(0) = 150.0$, and doubles when $S(t) = 300.0$, or

$$150.0e^{1.2t} = 300.0$$
$$e^{1.2t} = 2.0$$
$$1.2t = \ln(2.0)$$
$$t = \frac{\ln(2.0)}{1.2} = 0.5776.$$
$$S(0.5776) = 150.0e^{1.2 \cdot 0.5776} = 300.0.$$

We can solve for the doubling time in general by finding the time t_d when $S(t_d) = 2S(0)$,

$$
\begin{aligned}
S(t_d) = S(0)e^{\alpha t_d} &= 2S(0) && \text{equation for } t_d\\
e^{\alpha t_d} &= 2 && \text{divide by } S(0)\\
\alpha t_d &= \ln(2) && \text{take the natural log}\\
t_d &= \frac{\ln(2)}{\alpha} = \frac{0.6931}{\alpha}. && \text{solve for } t_d
\end{aligned}
$$

The **general formula for the doubling time** is

$$t_d = \frac{0.6931}{\alpha}.$$

The doubling time becomes smaller as α becomes larger, consistent with the fact that measurements with larger values of α increase more quickly. Importantly, the doubling time does not depend on the initial value $S(0)$.

Example 1.7.13 Computing a Doubling Time with the Formula

Suppose $S(t) = 150.0e^{1.2t}$ (Example 1.7.12). Then $\alpha = 1.2$/hour, and the doubling time is

$$t_d = \frac{0.6931}{1.2} = 0.5776 \text{ hours.}$$

When $\alpha < 0$, the measurement is decreasing, and we can ask how long it will take to become half as large. This time, denoted t_h, is called the **half-life**, and can be found with the following steps.

$$
\begin{aligned}
S(t_h) = S(0)e^{\alpha t_h} &= 0.5S(0) && \text{equation for } t_h\\
e^{\alpha t_h} &= 0.5 && \text{divide by } S(0)\\
\alpha t_h &= \ln(0.5) && \text{take the natural log}\\
t_h &= \frac{\ln(0.5)}{\alpha} = -\frac{0.6931}{\alpha}. && \text{solve for } t_h
\end{aligned}
$$

The **general formula for the half-life** is

$$t_h = -\frac{0.6931}{\alpha}.$$

The half-life becomes smaller when α grows larger in absolute value. Apply this equation only when $\alpha < 0$.

Example 1.7.14 Computing the Half-Life

If a measurement follows the equation

$$M(t) = 240.0e^{-2.3t},$$

with t measured in seconds, then $\alpha = -2.3$/s and the half-life is

$$t_h = \frac{-0.6931}{-2.3} = 0.3014 \text{ s.}$$

Example 1.7.15 Thinking in Half-Lives

Consider the measurement $M(t)$ given in Example 1.7.14, with a half-life of 0.3014s. To figure out how much the value will have decreased in 2.0s, we could plug into the original formula, finding

$$M(2.0) = 240.0e^{-2.3\cdot 2.0} = 2.41.$$

The value decreased by a factor of nearly 100. Alternatively, 2.0s is

$$\frac{2.0}{0.3014} = 6.636$$

half-lives. After this many half-lives, the value will have decreased by a factor of $2^{6.636} = 99.45$. We can think of using half-lives as converting the exponential to base 2.

If we are told the initial value and the doubling time or half-life of some measurement, we can find the formula. Instead of solving for the doubling time, we solve for the parameter α.

Example 1.7.16 Finding the Formula from the Doubling Time

Suppose $t_d = 26{,}200$ years. Because

$$t_d = \frac{0.6931}{\alpha},$$

we can solve for α as

$$\alpha = \frac{0.6931}{t_h} = \frac{0.6931}{26{,}200} = 2.645 \times 10^{-5}.$$

If $m(0) = 0.031$ then

$$m(t) = 0.031e^{2.645 \times 10^{-5}t}.$$

Example 1.7.17 Finding the Formula from the Half-Life

Suppose $t_h = 6.8$ years. Because

$$t_h = -\frac{0.6931}{\alpha}.$$

we can solve for α as

$$\alpha = -\frac{0.6931}{t_h} = -\frac{0.6931}{6.8} = -0.1019.$$

If $V(0) = 23.1$, then

$$V(t) = 23.1e^{-0.1019t}.$$

When a measurement follows an exponential function, the results are often plotted on a **semilog graph**.

Definition 1.13 A semilog graph plots the logarithm of the output against the input.

Example 1.7.18 A Semilog Graph of a Growing Value

Suppose

$$S(t) = 150.0e^{1.2t}$$

with t measured in hours (Example 1.7.12). To plot a semilog graph of $S(t)$ against t, we find the natural logarithm of $S(t)$.

$$
\begin{aligned}
\ln(S(t)) &= \ln(150.0e^{1.2t}) && \text{the natural logarithm of } S(t)\\
&= \ln(150.0) + \ln(e^{1.2t}) && \text{break up with law of logs 2}\\
&= 5.01 + 1.2t. && \text{evaluate } \ln(150.0) \text{ and cancel ln and exponential function}
\end{aligned}
$$

The semilog graph is a line with intercept 5.01 and slope 1.2 (Figure 1.7.11). The semilog graph is useful for exponentially growing measurements because it contracts the large range of values and converts an exponential curve into a straight line.

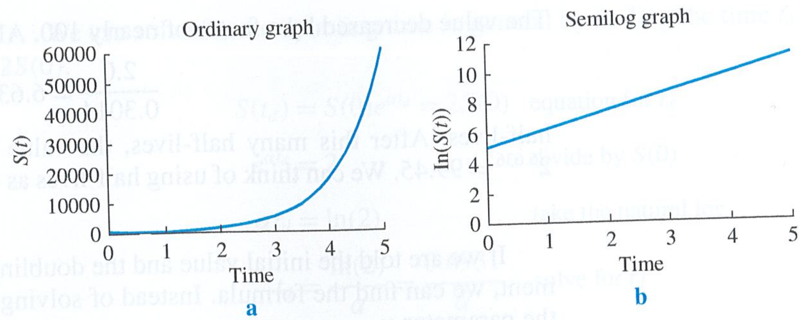

FIGURE 1.7.11
Original graph and semilog graph

Example 1.7.19 A Semilog Graph of Some Data

Suppose we are to graph the following data.

Time	Value
0	120.12
1	24.34
2	2.19
3	0.89
4	0.056
5	0.078
6	0.125
7	0.346
8	1.128

The graph of the original data is difficult to read because the large vertical scale makes the small values almost indistinguishable (Figure 1.7.12a). If we take the logarithm of the data, however, the values are much easier to compare (Figure 1.7.12b).

Time	Value	Logarithm of Value
0	120.12	4.79
1	24.34	3.19
2	2.190	0.78
3	0.89	−0.11
4	0.056	−2.88
5	0.078	−2.55
6	0.125	−2.08
7	0.346	−1.06
8	1.128	0.12

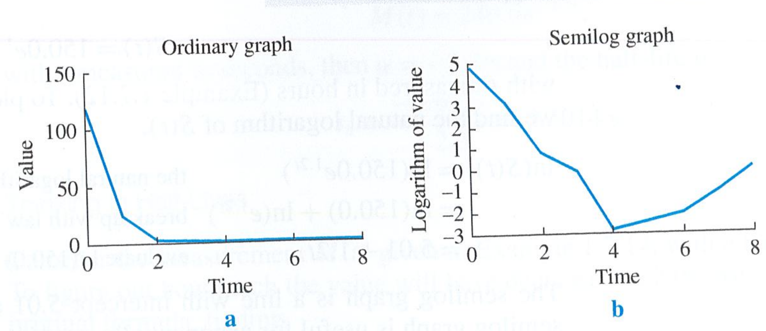

FIGURE 1.7.12
Original graph and semilog graph

The value reaches a minimum at time 4 and increases steadily thereafter.

The semilog graph is particularly useful for graphing outputs that follow an exponential function or have a wide range of values. If both the input and the output satisfy these conditions, taking the logarithm of both values can help illustrate their relationship. Such a graph is called a **double-log plot**.

Definition 1.14 A double-log graph plots the logarithm of the output against the logarithm of the input.

Example 1.7.20 Using Ordinary, Semilog, and Double-Log Graphs to Illustrate Data

Suppose

$$S(t) = 150.0e^{1.2t}$$

(Example 1.7.12) and that

$$M(t) = 13.2e^{2.0t}.$$

Values for times t from 0 up to 5 are given in the following table.

Time	S	M	ln(S)	ln(M)
0	150.0	13.20	5.011	2.58
1	498.0	97.54	6.211	4.58
2	1,653.0	720.70	7.411	6.58
3	5,490.0	5,325.00	8.611	8.58
4	18,230.0	39,350.00	9.811	10.58
5	60,510.0.0	290,700.00	11.010	12.58

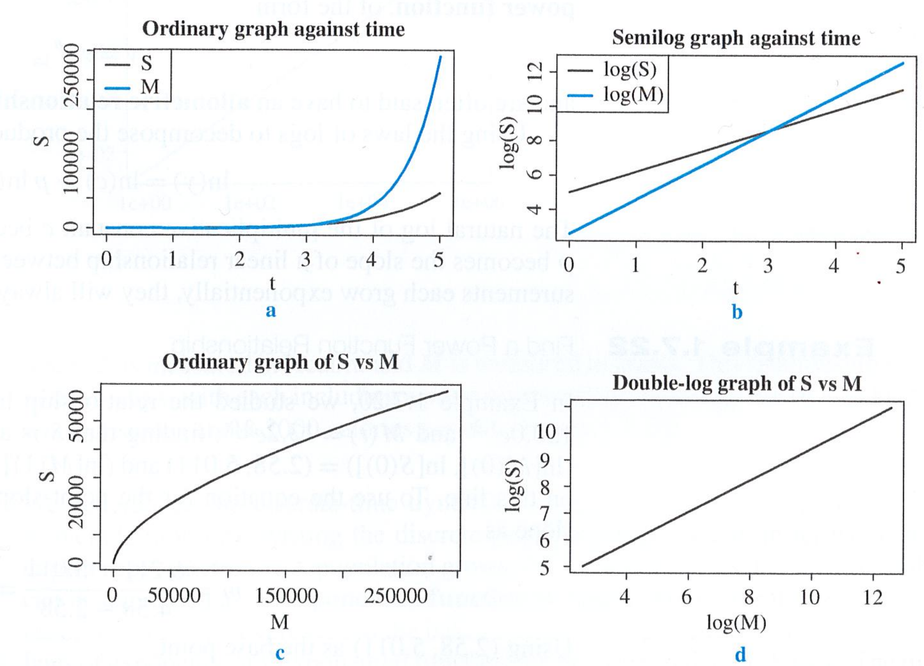

FIGURE 1.7.13
Presenting data with semilog and double-log graphs

To illustrate the behavior of S and M as functions of time, we can use either ordinary or semi-log graphs (Figure 1.7.13a,b). If instead we wish to show the relationship between these two measurements, the double-log plot can be useful (Figure 1.7.13c,d). The curved relationship caused by the differences in the growth rates of the two measurements becomes a linear relationship on a double-log graph.

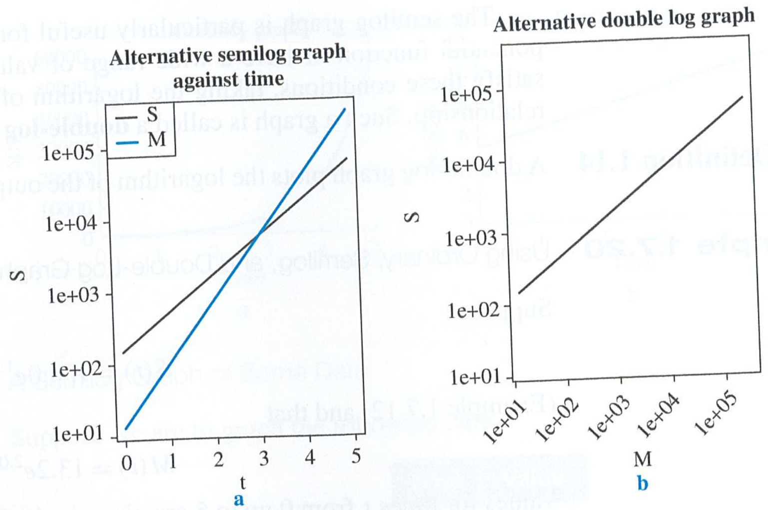

FIGURE 1.7.14
Transforming the axes on semilog and double-log graphs

Example 1.7.21 Alternative Way to Present Semilog and Double-Log Graphs

Instead of transforming the measurements by taking the natural log, we can transform the axes themselves. The values on the axis are not distributed on a linear scale as in the ordinary graph where the values 10, 20, 30 and so forth are evenly spaced. Instead, the powers of 10 such as 10, 100, and 1000 are evenly spaced, providing a way to display data that take on a larger range of values without changing the values themselves (Figure 1.7.14).

Many relationships in biology are linear on double-log graphs even though the underlying measurements do not grow exponentially. Such measurements follow a **power function**, of the form

$$y = cx^p$$

and are often said to have an **allometric relationship**.

Using the laws of logs to decompose the product and the power, we find that

$$\ln(y) = \ln(c) + p\ln(x).$$

The natural log of the multiplicative constant c becomes the intercept and the power p becomes the slope of a linear relationship between $\ln(x)$ and $\ln(y)$. When two measurements each grow exponentially, they will always be related by a power function.

Example 1.7.22 Find a Power Function Relationship

In Example 1.7.20, we studied the relationship between the measurements $S(t) = 150.0e^{1.2t}$ and $M(t) = 13.2e^{2.0t}$, finding that S is a linear function of M. The points $(\ln[M(0)], \ln[S(0)]) = (2.58, 5.011)$ and $(\ln[M(1)], \ln[S(1)]) = (4.58, 6.211)$ must lie on this line. To use the equation for the point-slope form of a line, we first find the slope as

$$m = \frac{6.211 - 5.011}{4.58 - 2.58} = 0.6.$$

Using (2.58, 5.011) as the base point,

$$\ln[S(t)] = 0.6(\ln[M(t)] - 2.58) + 5.011$$
$$= 0.6\ln[M(t)] + 3.468.$$

Example 1.7.23 The Allometric Relationship Between Surface Area and Volume

A sphere with radius r has surface area $S = 4\pi r^2$ and volume $V = \frac{4}{3}\pi r^3$ (Table 1.2). Because the power of the radius in the surface area is 2/3 that in volume, we can write

$$S = cV^{2/3}$$

where the constant for a sphere is $(36\pi)^{1/3}$. This allometric relationship holds for any given shape, but with a different value of the constant c.

Example 1.7.24 The Allometric Relationship of Energy Use and Body Mass

A famous allometric relationship concerns the link between metabolic rate, the amount of energy an organism uses, and body mass. For mammals, this takes roughly the form

$$E = 0.018M^{0.75}$$

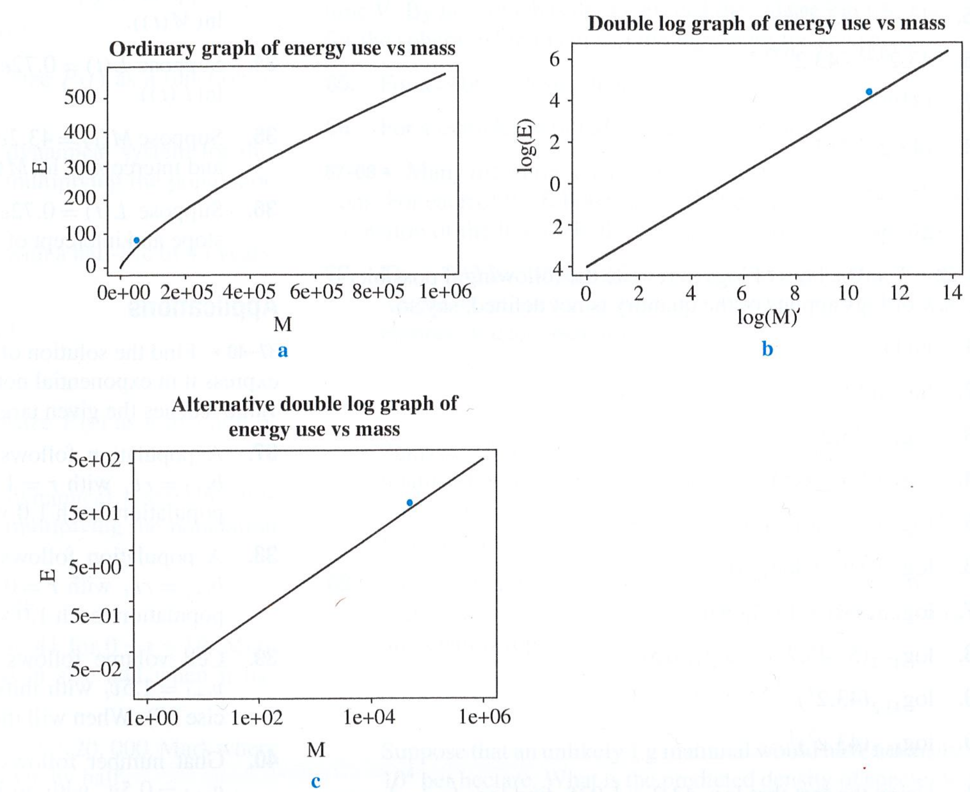

FIGURE 1.7.15
Energy use and body mass

where E is measured in Kcal/hr and M is measured in grams. This relationship is shown in three ways, with each including a point representing the energy use of a 75 kg person who consumes about 2000 calories per day (Figure 1.7.15).

Summary We generalized the discrete-time dynamical system for bacterial population growth to include mortality, writing the discrete-time dynamical system in terms of the **per capita reproduction** r. A population grows if $r > 1$ and declines if $r < 1$. The solution can be expressed as an **exponential function to base r**. For convenience, exponential functions are often expressed to the base e, often the **exponential function**. Using the laws of exponents, any exponential function can be expressed to the base e. The inverse of the exponential function is the **natural logarithm** or natural log. This function can be used to solve equations involving the exponential function, including finding **doubling times** and **half-lives**. Measurements that cover a large range of positive values can be conveniently displayed on a **semilog graph**, which reduces the range, and produces a linear graph if the measurements follow an exponential function. Double-log graphs help to display data where both the input and output cover a large range of values, and are linear when the two measurements are both exponential functions of time.

1.7 Exercises

Mathematical Techniques

1–10 ■ Use the laws of exponents to rewrite the following (if possible). If no law of exponents applies, say so.

1. 43.2^0
2. 43.2^1
3. 43.2^{-1}
4. $43.2^{-0.5}+43.2^{0.5}$
5. $43.2^{7.2}/43.2^{6.2}$
6. $43.2^{0.23}\cdot 43.2^{0.77}$
7. $(3^4)^{0.5}$
8. $(43.2^{-1/8})^{16}$
9. $2^{2^3}\cdot 2^{2^2}$
10. $4^2\cdot 2^4$

11–22 ■ Use the laws of logs to rewrite the following if possible. If no law of logs applies or the quantity is not defined, say so.

11. $\ln(1)$
12. $\ln(-6.5)$
13. $\log_{43.2}43.2$
14. $\log_{10}(3.5+6.5)$
15. $\log_{10}(5)+\log_{10}(20)$
16. $\log_{10}(0.5)+\log_{10}(0.2)$
17. $\log_{10}(500)-\log_{10}(50)$
18. $\log_{43.2}(5\cdot 43.2^2)-\log_{43.2}(5)$
19. $\log_{43.2}(43.2^7)$
20. $\log_{43.2}(43.2^7)^4$
21. Using the fact that $\log_7 43.2=1.935$, find $\log_7\left(\frac{1}{43.2}\right)$.
22. Using the fact that $\log_7 43.2=1.935$, find $\log_7(43.2)^3$.

23–26 ■ Solve the following equations for x and check your answer.

23. $7e^{3x}=21$.
24. $4e^{2x+1}=20$.
25. $4e^{-2x+1}=7e^{3x}$.
26. $4e^{2x+3}=7e^{3x-2}$.

27–30 ■ Sketch graphs of the following exponential functions. For each, find the value of x where it is equal to 7.0. For the increasing functions, find the doubling time, and for the decreasing functions, find the half-life. For what value of x is the value of the function 3.5? For what value of x is the value of the function 14.0?

27. e^{2x}
28. e^{-3x}
29. $5e^{0.2x}$
30. $0.1e^{-0.2x}$

31–32 ■ Sketch graphs of the following updating functions over the given range and mark the equilibria.

31. $h(z)=e^{-z}$ for $0\le z\le 2$.
32. $F(x)=\ln(x)+1$ for $0\le x\le 2$. (Although this cannot be solved algebraically, you can guess the answer.)

33–36 ■ Find the equations of the lines after transforming the variables to create semilog or double-log plots.

33. Suppose $M(t)=43.2e^{5.1t}$. Find the slope and intercept of $\ln(M(t))$.
34. Suppose $L(t)=0.72e^{-2.34t}$. Find the slope and intercept of $\ln(L(t))$.
35. Suppose $M(t)=43.2e^{5.1t}$ and $S(t)=18.2e^{4.3t}$. Find the slope and intercept of $\ln(M(t))$ as a function of $\ln(S(t))$.
36. Suppose $L(t)=0.72e^{-2.34t}$ and $K(t)=4.23e^{0.91t}$. Find the slope and intercept of $\ln(L(t))$ as a function of $\ln(K(t))$.

Applications

37–40 ■ Find the solution of each discrete-time dynamical system, express it in exponential notation, and solve for the time when the value reaches the given target. Sketch a graph of the solution.

37. A population follows the discrete-time dynamical system $b_{t+1}=rb_t$ with $r=1.5$ and $b_0=1.0\times 10^6$. When will the population reach 1.0×10^7?
38. A population follows the discrete-time dynamical system $b_{t+1}=rb_t$ with $r=0.7$ and $b_0=5.0\times 10^5$. When will the population reach 1.0×10^5?
39. Cell volume follows the discrete-time dynamical system $v_{t+1}=1.5v_t$ with initial volume of 1350 μm^3 (as in Exercise 37). When will the volume reach 3250 μm^3?
40. Gnat number follows the discrete-time dynamical system $n_{t+1}=0.5n_t$ with an initial population of 5.5×10^4. When will the population reach 1.5×10^3?

41–44 ■ Suppose the size of an organism at time t is given by

$$S(t)=S_0e^{\alpha t}$$

where S_0 is the initial size. Find the time it takes for the organism to double or quadruple in size in the following circumstances.

41. $S_0=1.0$ cm and $\alpha=1.0$/day.
42. $S_0=2.0$ cm and $\alpha=1.0$/day.
43. $S_0=2.0$ cm and $\alpha=0.1$/hour.
44. $S_0=2.0$ cm and $\alpha=0.0$/hour

45–48 ■ The amount of carbon-14 (14_c) left t years after the death of an organism is given by

$$Q(t)=Q_0e^{-0.000122t}$$

where Q_0 is the amount left at the time of death. Suppose $Q_0=6.0\times 10^{10}$ 14_c atoms.

45. How much is left after 50,000 years? What fraction is this of the original amount?
46. How much is left after 100,000 years? What fraction is this of the original amount?
47. Find the half-life of 14_c.
48. About how many half-lives will occur in 50,000 years? Roughly what fraction will be left? How does this compare with the answer of Exercise 45?

49–52 ■ Suppose a population has a doubling time of 24 years and an initial size of 500.

49. What is the population in 48 years?
50. What is the population in 12 years?
51. Find the equation for population size $P(t)$ as a function of time.
52. Find the one-year discrete-time dynamical system for this population (figure out the factor multiplying the population in one year).

53–56 ■ Suppose a population is dying with a half-life of 43 years. The initial size is 1600.

53. How long will it take to reach 200?
54. Find the population in 86 years.
55. Find the equation for population size $P(t)$ as a function of time.
56. Find the one year discrete-time dynamical system for this population (figure out the factor multiplying the population in one year).

57–60 ■ Plot semilog graphs of the values.

57. The growing organism in Exercise 41 for $0\le t\le 10$. Mark where the organism has doubled in size and when it has quadrupled in size.
58. The carbon-14 in Exercise 45 for $0\le t\le 20,000$. Mark where the amount of carbon has gone down by half.
59. The population in Exercise 49 for $0\le t\le 100$. Mark where the population has doubled.
60. The population in Exercise 53 for $0\le t\le 100$. Mark where the population has gone down by half.

61–64 ■ The following pairs of measurements can be described by ordinary, semilog, and double-log graphs.

a. Graph each measurement as a function of time on both ordinary and semilog graphs.

b. Graph the second measurement as a function of the first on both ordinary and double-log graphs.

61. The antler size $A(t)$ in centimeters of an elk increases with age t in years according to $A(t)=53.2e^{0.17t}$ and its shoulder height $L(t)$ increases according to $L(t)=88.5e^{0.1t}$.
62. Suppose a population of viruses in an infected person grows according to $V(t)=2.0e^{2.0t}$ and that the immune response (described by the number of antibodies) increases according to $I(t)=0.01e^{3.0t}$ during the first week of an infection. When will the number of antibodies equal the number of viruses?
63. The growth of a fly in an egg can be described allometrically (see H. F. Nijhout and D. E. Wheeler, 1996). During growth, two **imaginal disks** (the first later becomes the wing and the second becomes the haltere) expand according to $S_1(t)=0.007e^{0.1t}$ and $S_2(t)=0.007e^{0.4t}$ where size is measured in mm^3 and time is measured in days. Development takes about 5 days.
64. While the imaginal disks are growing (Exercise 63), the yolk of the egg is shrinking according to $Y(t)=4.0e^{-1.2t}$. Create graphs comparing $S_1(t)$ and $Y(t)$.

65–66 ■ For each of the given shapes, find the constant c in the power relationship $S=cV^{2/3}$ between the surface area S and volume V. By how much is does c exceed the value $(36\pi)^{1/3}=4.836$ for the sphere (which is in fact the minimum for any shape).

65. For a cube with side length w.
66. For a cylinder with radius r and height $3r$.

67–68 ■ Many measurements in biology are related by power functions. For each of the following, graph the second measurement as a function of the first on both ordinary and double-log graphs.

67. The $-3/2$ law of self-thinning in plants argues that the mean weight W of surviving trees in a stand increases while their number N decreases, related by

$$W=cN^{-3/2}.$$

Suppose 10^4 trees start out with mass of 0.001 kg. Graph the relationship, and find how heavy the trees would be when only 100 remain alive, and again when only 1 remains alive. Is the total mass larger or smaller than when it started?

68. Suppose that the population density D of a species of mammal is a decreasing function of its body mass M according to the relationship

$$D=cM^{-3/4}.$$

Suppose that an unlikely 1 g mammal would have a density of 10^4 per hectare. What is the predicted density of species with mass of 1000 g? A species with a mass of 100 kg? According to the metabolic scaling law (Example 1.7.24), which species will use the most energy?

Computer Exercises

69. Use your computer to find the following. Plot the graphs to check.

a. The doubling time of $S_1(t)=3.4e^{0.2t}$.

b. The doubling time of $S_2(t)=0.2e^{3.4t}$.

c. The half-life of $H_1(t)=3.4e^{-0.2t}$.

d. The half-life of $H_2(t)=0.2e^{-3.4t}$.

70. Solve for the times when the following hold. Plot the graphs to check your answer.

a. $S_1(t)=S_2(t)$ with S_1 and S_2 from the previous problem.

b. $H_1(t)=2H_2(t)$ with H_1 and H_2 from the previous problem.