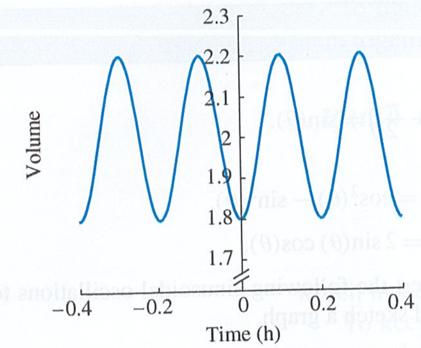
34.



35–38 ▪ Graph the following functions. Give the average, maximum, minimum, amplitude, period, and phase of each and mark them on your graph.

35. $f(x) = 3.0 + 4.0\cos\left(2\pi\frac{x-1.0}{5.0}\right)$.

36. $g(t) = 4.0 + 3.0\cos[2\pi(t-5.0)]$.

37. $h(z) = 1.0 + 5.0\cos\left(2\pi\frac{z-3.0}{4.0}\right)$.

38. $W(y) = -2.0 + 3.0\cos\left(2\pi\frac{y+0.1}{0.2}\right)$.

39–44 ▪ Oscillations are often combined with growth or decay. Plot graphs of the following functions, and describe in words what you see. Make up a biological process that might have produced the result.

39. $f(t) = 1 + t + \cos(2\pi t)$ for $0 < t < 4$.

40. $h(t) = t + 0.2\sin(2\pi t)$ for $0 < t < 4$.

41. $g(t) = e^t\cos(2\pi t)$ for $0 < t < 3$.

42. $W(t) = e^{-t}\cos(2\pi t)$ for $0 < t < 3$.

43. $H(t) = \cos(e^t)$ for $0 < t < 3$.

44. $b(t) = \cos(e^{-t})$ for $0 < t < 3$.

45–48 ▪ Sleepiness has two cycles, a circadian rhythm with a period of approximately 24 hours and an ultradian rhythm with a period of approximately 4 hours. Both have phase 0 (starting at midnight) and average 0, but the amplitude of the circadian rhythm is 1.0 sleepiness unit and the ultradian is 0.4 sleepiness unit.

45. Find the formula and sketch the graph of sleepiness over the course of a day due to the circadian rhythm.

46. Find the formula and sketch the graph of sleepiness over the course of a day due to the ultradian rhythm.

47. Sketch the graph of the two cycles combined.

48. At what time of day are you sleepiest? At what time of day are you least sleepy?

Computer Exercises

49. Consider the following functions.

$$f_1(x) = \cos\left(x - \frac{\pi}{2}\right)$$
$$f_3(x) = \frac{\cos\left(3x - \frac{\pi}{2}\right)}{3}$$
$$f_5(x) = \frac{\cos\left(5x - \frac{\pi}{2}\right)}{5}$$
$$f_7(x) = \frac{\cos\left(7x - \frac{\pi}{2}\right)}{7}.$$

a. Plot them all on one graph.

b. Plot the sum $f_1(x) + f_3(x)$.

c. Plot the sum $f_1(x) + f_3(x) + f_5(x)$.

d. Plot the sum $f_1(x) + f_3(x) + f_5(x) + f_7(x)$.

e. What does this sum look like?

f. Try to guess the pattern, and add on $f_9(x)$ and $f_{11}(x)$. This is an example of a **Fourier series**, a sum of cosine functions that add up to a **square wave** that jumps between values of −1 and 1.

50. Use a computer to cobweb and graph solutions of the following discrete-time dynamical systems. Try three different initial conditions for each. Can you make any sense of what happens? Why don't the solutions follow a sinusoidal oscillation?

a. $x_{t+1} = \cos(x_t)$.

b. $y_{t+1} = \sin(y_t)$.

c. $z_{t+1} = \sin(z_t) + \cos(z_t)$.

51. Plot the function $f(x) = \cos(2\pi\cdot 440x) + \cos(2\pi\cdot 441x)$. Describe the result. If these were sounds, what might you hear? (This corresponds to playing two notes with the same amplitude and slightly different frequencies.)

1.9 A Model of Gas Exchange in the Lung

The exchange of materials between an organism and its environment is one of the most fundamental biological processes. By following the amount of chemical step by step through the breathing process, we can derive a discrete-time dynamical system that models this process for a simplified pair of lungs. This discrete-time dynamical system describes how the outside air mixes with internal air, and takes the form of a **weighted average**. This model provides a framework we use to study more complicated biological processes such as absorption or release of chemical.

1.9.1 A Model of the Lungs

Consider a simplified breathing process. An adult male lung has a volume of about 6.0 L when full. With each breath, 0.6 L of the air in the lungs are exhaled, and replaced by 0.6 L of outside (or **ambient**) air. After exhaling, the volume of the lungs is 5.4 L, and returns to 6.0 L after inhaling (Figure 1.9.1).

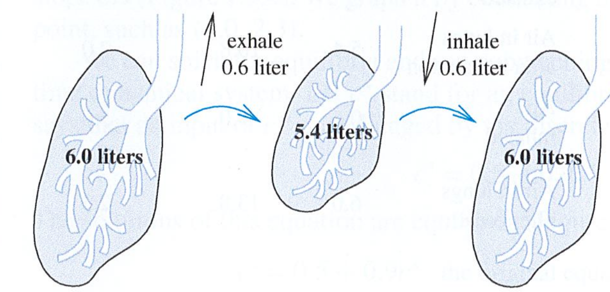


FIGURE 1.9.1
Gas exchange in the lung: the volume

Suppose further that the lung contains a particular chemical with a concentration of 2.0 mmol/L before exhaling that the lungs contain. (A mole is a convenient chemical unit indicating 6.02×10^{23} molecules, and a millimole is 6.02×10^{20} molecules.) The ambient air has a chemical concentration of 5.0 mmol/L. What is the chemical concentration after one breath?

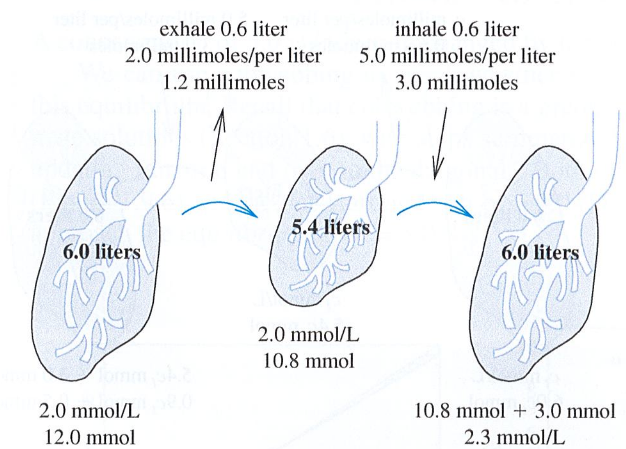


FIGURE 1.9.2
Gas exchange in the lung: the concentration

We must track three quantities through these steps: the volume (Figure 1.9.1), the total amount of chemical, and the chemical concentration (Figure 1.9.2). To find the total amount from the concentration, we use the fundamental relation

$$\text{total amount} = \text{concentration} \times \text{volume}.$$

Conversely, to find the concentration from the total amount, we rearrange the fundamental relation as

$$\text{concentration} = \frac{\text{total amount}}{\text{volume}}.$$

One basic biological assumption underlies our reasoning—that air breathed out has a concentration equal to that of the whole lung. This means that the air in the lungs is completely mixed each breath, which is not exactly true. Assuming that neither air nor chemical is produced or used while breathing, we can track through the process step by step.

Step	Volume (L)	Total Chemical (mmol)	Concentration (mmol/L)	What We Did
Air in lungs before breath	6.0	12.0	2.0	Multiplied volume of lungs (6.0) by concentration (2.0) to get 12.0.
Air exhaled	0.6	1.2	2.0	Multiplied volume exhaled (0.6) by concentration (2.0) to get 1.2.
Air in lungs after exhalation	5.4	10.8	2.0	Multiplied volume remaining (5.4) by concentration (2.0) to get 10.8.
Air inhaled	0.6	3.0	5.0	Multiplied volume inhaled (0.6) by ambient concentration (5.0) to get 3.0.
Air in lungs after breath	6.0	13.8	2.3	Found total by adding 10.8 + 3.0 = 13.8, and divided by volume (6.0) to get 2.3.

Breathing creates a discrete-time dynamical system. The original concentration of 2.0 mmol/L is updated to 2.3 mmol/L after a breath. To write the discrete-time dynamical system, we must figure out the concentration after a breath, c_{t+1}, as a function of the concentration before the breath, c_t. We follow the same steps, but replace 2.0 with c_t (Figure 1.9.3).

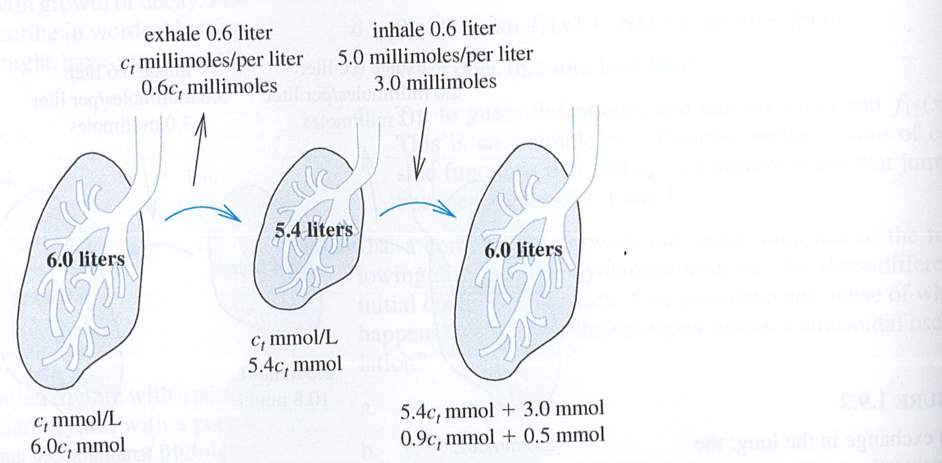


FIGURE 1.9.3
Gas exchange in the lung: finding the discrete-time dynamical system

Step	Volume (L)	Total Chemical (mmol)	Concentration (mmol/L)	What We Did
Air in lungs before breath	6.0	$6.0c_t$	c_t	Multiplied volume of lungs (6.0) by concentration (c_t) to get $6.0c_t$.
Air exhaled	0.6	$0.6c_t$	c_t	Multiplied volume exhaled (0.6) by concentration (c_t) to get $0.6c_t$.
Air in lungs after exhalation	5.4	$5.4c_t$	c_t	Multiplied volume remaining (5.4) by concentration (c_t) to get $5.4c_t$.
Air inhaled	0.6	3.0	5.0	Multiplied volume inhaled (0.6) by ambient concentration (5.0) to get 7.5.
Air in lungs after breath	6.0	$3.0 + 5.4c_t$	$0.5 + 0.9c_t$	Added inhaled chemical (3.0) to remaining chemical ($+5.4c_t$) and divided by volume (6.0) to get $0.5 + 0.9c_t$.

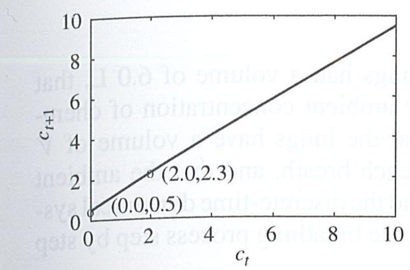


FIGURE 1.9.4
Updating function for the lung model

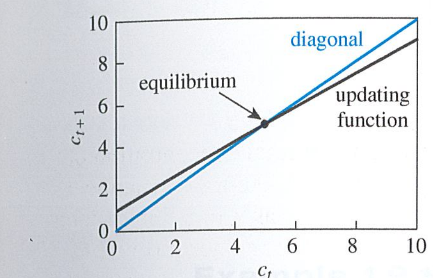


FIGURE 1.9.5
Equilibrium of the lung discrete-time dynamical system

The discrete-time dynamical system is therefore

$$c_{t+1} = 0.5 + 0.9c_t.$$

Checking, an input of $c_t = 2.0$ gives

$$c_{t+1} = 0.5 + 0.9 \cdot 2.0 = 2.3$$

as found above. The graph of the updating function is a line with y-intercept 0.5 and slope 0.9 (Figure 1.9.4). We graph it by connecting the y-intercept (0, 0.5) with another point, such as (2.0, 2.3).

We can solve for equilibria and use cobwebbing to better understand this discrete-time dynamical system. Let c^* stand for an equilibrium. The equation for equilibrium says that an input of c^* is unchanged by the discrete-time dynamical system, or

$$c^* = 0.5 + 0.9c^*.$$

The solutions of this equation are equilibria (Figure 1.9.5). To solve,

$c^* = 0.5 + 0.9c^*$	the original equation
$c^* - 0.9c^* = 0.5$	subtract $0.9c^*$ to get unknowns on one side
$0.1c^* = 0.5$	do the subtraction
$c^* = \dfrac{0.5}{0.1} = 5.0.$	divide by 0.1

The equilibrium value is 5.0 mmol/L. We can **check** this by plugging $c_t = 5.0$ into the discrete-time dynamical system, finding

$$c_{t+1} = 0.5 + 0.9 \cdot 5.0 = 5.0.$$

A concentration of 5.0 is indeed unchanged by the breathing process.

We can use cobwebbing to check whether solutions move toward or away from this equilibrium. Recall that cobwebbing is a graphical procedure for finding approximate solutions (Section 1.6), with steps summarized in the phrase "up or down to the updating function and over to the diagonal." Both the cobweb starting from $c_0 = 10.0$ (Figure 1.9.6) and the one starting from $c_0 = 0.0$ (Figure 1.9.7) produce solutions that approach the equilibrium at $c^* = 5.0$.

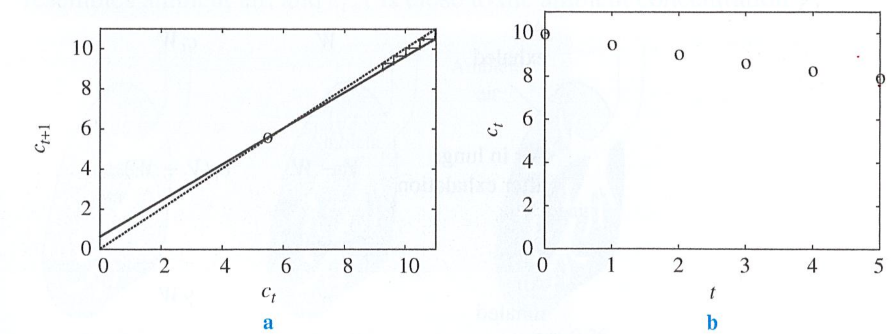


FIGURE 1.9.6
Cobweb and solution of the lung discrete-time dynamical system with $c_0 = 10.0$

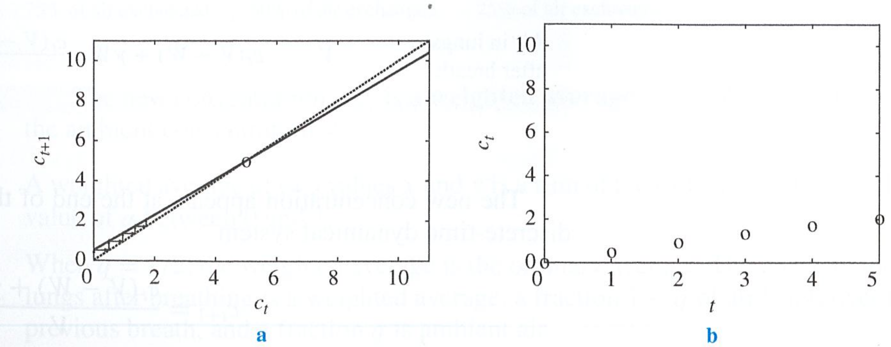


FIGURE 1.9.7
Cobweb and solution of the lung discrete-time dynamical system with $c_0 = 0.0$

1.9.2 The Lung System in General

In the previous subsection, we assumed that the lungs had a volume of 6.0 L, that 0.6 L of air were exhaled and inhaled, and that the ambient concentration of chemical was 5.0 mmol/L. Suppose, more generally, that the lungs have a volume of V liters, that W liters of air are exhaled and inhaled each breath, and that the ambient concentration of chemical is γ ("gamma"). We can find the discrete-time dynamical system giving c_{t+1} as a function of c_t by again following the breathing process step by step (Figure 1.9.8).

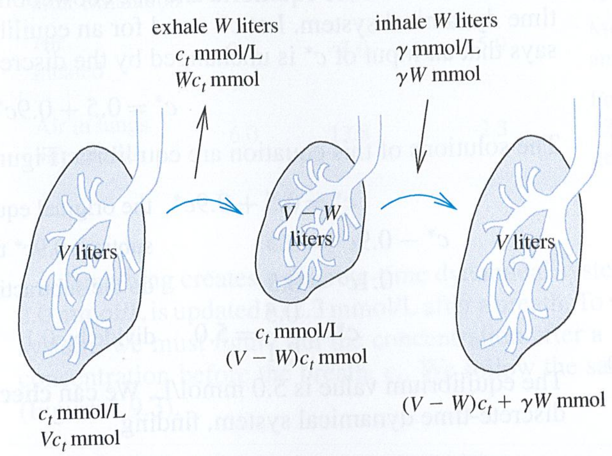


FIGURE 1.9.8
Gas exchange in the lungs: general case

Step	Volume (L)	Total Chemical (mmol)	Concentration (mmol/L)	What We Did
Air in lungs before breath	V	$c_t V$	c_t	Multiplied volume of lungs (V) by concentration (c_t) to get $c_t V$.
Air exhaled	W	$c_t W$	c_t	Multiplied volume exhaled (W) by concentration (c_t) to get $c_t W$.
Air in lungs after exhalation	$V - W$	$c_t(V - W)$	c_t	Multiplied volume remaining ($V - W$) by concentration (c_t) to get $c_t(V - W)$.
Air inhaled	W	γW	γ	Multiplied volume inhaled (W) by ambient concentration (γ) to get γW.
Air in lungs after breath	V	$c_t(V - W) + \gamma W$	$\frac{c_t(V - W) + \gamma W}{V}$	Found total by adding $c_t(V - W)$ to γW and divided by volume (V).

The new concentration appears at the end of the last line of the table, giving the discrete-time dynamical system

$$c_{t+1} = \frac{c_t(V - W) + \gamma W}{V}.$$

This equation can be simplified by multiplying out the first term and dividing out the V,

$$\begin{aligned} c_{t+1} &= \frac{c_t(V - W) + \gamma W}{V} \\ &= \frac{c_t V - c_t W + \gamma W}{V} \\ &= c_t - c_t \frac{W}{V} + \gamma \frac{W}{V}. \end{aligned}$$

The two values W and V appear only as the ratio $\frac{W}{V}$, which is the fraction of the total volume exchanged each breath. For example, when $W = 0.6$ L and $V = 6.0$ L, $\frac{W}{V} = 0.1$, meaning that 10% of air is exhaled each breath. We define a new parameter

$$q = \frac{W}{V} = \text{fraction of air exchanged}$$

to represent this quantity. We can then write the discrete-time dynamical system as

$$c_{t+1} = c_t - c_t q + \gamma q$$

or, after combining terms with c_t, as **the general lung discrete-time dynamical system**,

$$c_{t+1} = (1 - q)c_t + q\gamma. \qquad (1.9.1)$$

Example 1.9.1 Finding the Discrete-Time Dynamical System with Specific Parameter Values

In the original example, $W = 0.6$ and $V = 6.0$, giving $q = \frac{W}{V} = 0.1$. Using $\gamma = 5.0$, the general equation matches our original discrete-time dynamical system because

$$c_{t+1} = (1 - 0.1)c_t + 0.1 \cdot 5.0 = 0.9c_t + 0.5.$$

After a breath, the air in the lungs is a mix of old air and ambient air (Figure 1.9.9). The fraction $1 - q$ is old air that remains in the lungs, and the remaining fraction q is ambient air. If $q = 0.5$, half of the air in the lungs after a breath came from outside, and c_{t+1} is the average of the previous concentration and the ambient concentration. If q is small, little of the internal air is replaced with ambient air and c_{t+1} is close to c_t. If q is near 1, most of the internal air is replaced with ambient air. The air in the lungs then resembles ambient air, and c_{t+1} is close to the ambient concentration γ.

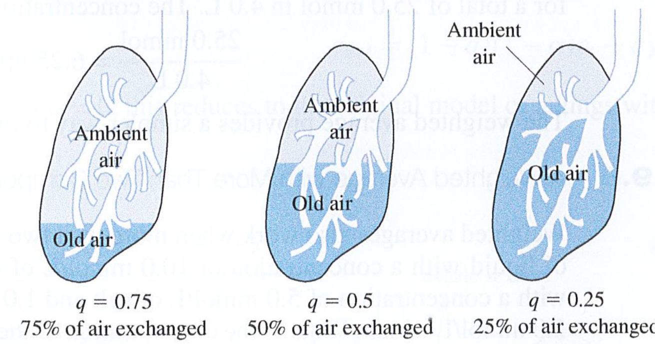


FIGURE 1.9.9
Effects of different values of q

The new concentration c_{t+1} is a **weighted average** of the old concentration c_t and the ambient concentration γ.

Definition 1.16 A weighted average of two values x and y is a sum of the form $qx + (1 - q)y$ for some value of q between 0 and 1.

When $q = 1/2$, the weighted average is the ordinary average. The concentration in the lungs after breathing is a weighted average: a fraction $1 - q$ of air is left over from the previous breath, and a fraction q is ambient air.

Example 1.9.2 A Weighted Average

Suppose $x = 2$ and $y = 5$. Then the weighted average that places a weight $q = 0.8$ on x and $1 - q = 0.2$ on y is

$$qx + (1 - q)y = 0.8 \cdot 2 + 0.2 \cdot 5 = 2.6.$$

Less weight is placed on y, and the weighted average is closer to x.

Example 1.9.3 A Contrasting Weighted Average

Suppose $x = 2$ and $y = 5$, as in Example 1.9.2. The weighted average that places a weight $q = 0.2$ on x and $1 - q = 0.8$ on y is

$$qx + (1 - q)y = 0.2 \cdot 2 + 0.8 \cdot 5 = 4.4.$$

More weight is placed on y, and the weighted average is closer to y.

Example 1.9.4 An Ordinary Average

Suppose $x = 2$ and $y = 5$, as in Examples 1.9.2 and 1.9.3. The ordinary average places equal weight $q = 0.5$ on x and $1 - q = 0.5$ on y, and is equal to

$$qx + (1 - q)y = 0.5 \cdot 2 + 0.5 \cdot 5 = 3.5.$$

This value is exactly in the middle between x and y.

Example 1.9.5 The Weighted Average Applied to Liquids

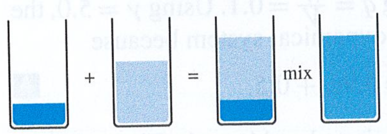


FIGURE 1.9.10
Mixing liquids as a weighted average

Suppose 1.0 L of liquid with a concentration of 10.0 mmol/L of salt are mixed with 3.0 L of liquid with a concentration of 5.0 mmol/L of salt (Figure 1.9.10). What is the concentration of the resulting mixture? We can think of this as a weighted average. The 4.0 L of the mixture contains 1.0 L of the high-salt solution (or a fraction of 0.25) and 3.0 L of the low-salt solution (or a fraction of 0.75). The resulting concentration is the weighted average

$$0.25 \cdot 10.0 \text{ mmol/L} + 0.75 \cdot 5.0 \text{ mmol/L} = 6.25 \text{ mmol/L}.$$

We could work this out more explicitly by computing the total amount of salt and the total volume. There are 10.0 mmol of salt from the first solution and 15.0 mmol from the second (multiplying the concentration of 5.0 mmol/L by the volume of 3.0 L), for a total of 25.0 mmol in 4.0 L. The concentration is

$$\frac{25.0 \text{ mmol}}{4.0 \text{ L}} = 6.25 \text{ mmol/L}.$$

The weighted average provides a simpler way to find this answer.

Example 1.9.6 A Weighted Average with More Than Two Components

Weighted averages also work when more than two solutions are mixed. Suppose 1.0 L of liquid with a concentration of 10.0 mmol/L of salt are mixed with 3.0 L of liquid with a concentration of 5.0 mmol/L of salt and 1.0 L of liquid with a concentration of 2.0 mmol/L of salt. What is the concentration of the resulting mixture? In this case, the 5.0 L of the mixture are composed of 20% (or 0.20) of the high-salt concentration solution, and 60% (or 0.60) of the medium salt concentration solution and 20% (or 0.20) of the low-salt concentration solution. The resulting concentration is the weighted average

$$0.20 \cdot 10.0 \text{ mmol/L} + 0.60 \cdot 5.0 \text{ mmol/L} + 0.20 \cdot 2.0 \text{ mmol/L} = 5.4 \text{ mmol/L}.$$

The Equilibrium of the Lung Discrete-Time Dynamical System The general discrete-time dynamical system for the lung model is

$$c_{t+1} = (1 - q)c_t + q\gamma.$$

Following the steps for finding equilibria gives

$c^* = (1 - q)c^* + q\gamma$	the equation for the equilibrium
$c^* - (1 - q)c^* - q\gamma = 0$	move everything to one side
$c^* - c^* + qc^* - q\gamma = 0$	multiply c^* through $(1 - q)$
$qc^* - q\gamma = 0$	do the subtraction
$q(c^* - \gamma) = 0$	factor out the q
$q = 0$ or $c^* - \gamma = 0$	set both factors to 0
$q = 0$ or $c^* = \gamma$	solve each term

The key algebraic step comes after factoring. Remember that the product of two terms (like q and $c^* - \gamma$) can equal 0 only if one of the terms is equal to 0.

What do these results mean? The first case, $q = 0$, occurs when no air is exchanged. Because lungs that are exchanging no air are, there is no expression for c^* in this case, **any** value of c_t is an equilibrium. This make sense because a lung that is exchanging no air is, in fact, at equilibrium. The second case is more interesting. It says that the equilibrium value of the concentration is equal to the ambient concentration. Exchanging air with the outside world has no effect when the inside and the outside match. Doing the calculation in general explains why the equilibrium of 5.0 mmol/L found in Subsection 1.9.1 had to match the ambient concentration of 5.0 mmol/L.

1.9.3 Lung Dynamics with Absorption

Our model of chemical dynamics in the lungs ignored any absorption of the chemical by the body. We can now consider the dynamics of oxygen, which is of course absorbed by blood. How will this change the discrete-time dynamical system and the resulting solution and equilibrium?

We can use the weighted average to derive the discrete-time dynamical system including absorption. Suppose that a fraction q of air is exchanged each breath, that ambient air has a concentration of γ, and that a fraction α of chemical is absorbed before breathing out (Figure 1.9.11). After absorption, the concentration in the lungs is $(1 - \alpha)c_t$. Mixing produces a weighted average with a fraction $1 - q$ of this old air and a fraction q of ambient air, giving the discrete-time dynamical system

$$c_{t+1} = (1 - q)(1 - \alpha)c_t + q\gamma.$$

If $\alpha = 0$, this reduces to the original model of a lungs without absorption.

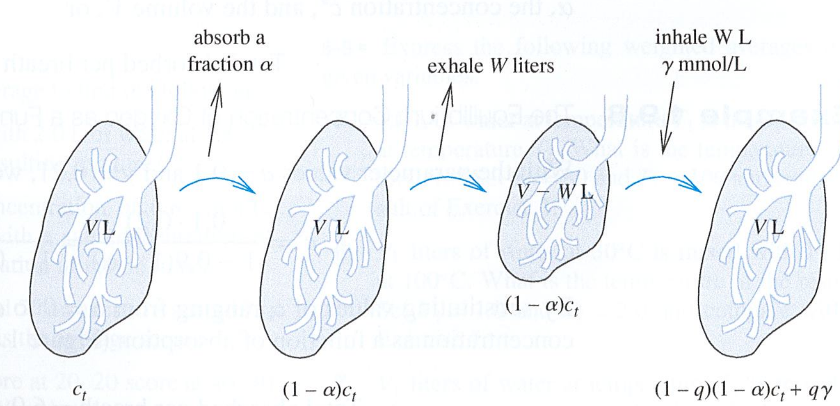


FIGURE 1.9.11
Dynamics of a lungs with absorption

Example 1.9.7 Absorption of Oxygen by the Lung

Consider again a lungs that has a volume of 6.0 L and that replaces 0.6 L each breath with ambient air (as in Figure 1.9.2). Suppose that we are tracking oxygen, with an ambient concentration of 21%, and assume that 30% of the oxygen in the lungs is absorbed each breath. Our parameter values are

$$q = 0.1$$
$$\alpha = 0.3$$
$$\gamma = 0.21.$$

The discrete-time dynamical system is then

$$c_{t+1} = 0.9 \cdot 0.7c_t + 0.1 \cdot 0.21 = 0.63c_t + 0.021.$$

The equilibrium concentration in the lungs then solves

$$c^* = 0.63c^* + 0.021$$
$$0.37c^* = 0.021$$
$$c^* = 0.057.$$

The equilibrium concentration of oxygen in the lungs, which is equal to the concentration of oxygen in the air breathed out, would be about 5.7%, or roughly one fourth of the ambient concentration.

As a result of absorption, the equilibrium concentration will be lower than the ambient concentration. By solving for the equilibrium of the system in general, we can investigate how the equilibrium depends on the fraction absorbed. To find the equilibrium, we solve

$$c^* = (1-q)(1-\alpha)c^* + q\gamma$$
$$c^* - (1-q)(1-\alpha)c^* = q\gamma$$
$$c^*(1 - (1-q)(1-\alpha)) = q\gamma$$
$$c^* = \frac{q\gamma}{1 - (1-q)(1-\alpha)}.$$

As a check, if we substitute $\alpha = 0$, we find

$$c^* = \frac{q\gamma}{1-(1-q)} = \frac{q\gamma}{q} = \gamma,$$

matching the result without absorption.

The total oxygen absorbed per breath will be the product of the fraction absorbed α, the concentration c^*, and the volume V, or

$$\text{Total absorbed per breath} = \alpha c^* V. \qquad (1.9.2)$$

Example 1.9.8 The Equilibrium Concentration of Oxygen as a Function of α

With the parameter values $q = 0.1$ and $\gamma = 0.21$, we find

$$c^* = \frac{0.1 \cdot 0.21}{1 - 0.9(1-\alpha)} = \frac{0.021}{1 - 0.9(1-\alpha)}.$$

By substituting values of α ranging from $\alpha = 0$ to $\alpha = 1$, we can plot the equilibrium concentration as a function of absorption (Figure 1.9.12). With $V = 6.0$,

$$\text{Total absorbed per breath} = 6.0\frac{0.021\alpha}{1 - 0.9(1-\alpha)}.$$

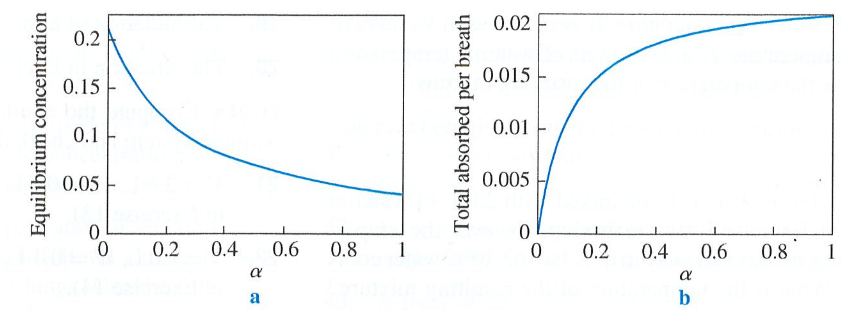


FIGURE 1.9.12
Equilibrium and absorption as a function of α

Example 1.9.9 Finding α from the Equilibrium Concentration of Oxygen

The actual oxygen concentration in exhaled air is approximately 15%, although this values varies depending on activity level. What fraction of oxygen is in fact absorbed? We can find this by solving for the value of α that produces $c^* = 0.15$.

$$0.15 = \frac{0.021}{1 - 0.9(1-\alpha)}$$
$$0.15(1 - 0.9(1-\alpha)) = 0.021$$
$$0.15(0.1 + 0.9\alpha) = 0.021$$
$$0.1 + 0.9\alpha = \frac{0.021}{0.15} = 0.14$$
$$0.9\alpha = 0.04$$
$$\alpha = 0.044.$$

Rather surprisingly, the lungs absorbs less than 5% of the available oxygen, leading to exhaled air that has nearly 30% less oxygen than ambient air.

Summary Starting from an understanding of how a lungs exchanges air, we derived a discrete-time dynamical system for the concentration of a chemical in the lungs. The discrete-time dynamical system can be described as a **weighted average** of the internal concentration and the **ambient concentration.** The equilibrium is equal to the ambient concentration, and cobwebbing diagrams indicate that solutions approach this equilibrium. Including absorption produces a slightly more complicated model with an equilibrium that is less than the ambient concentration. We used this model to investigate the dynamics of oxygen in the lungs.

1.9 Exercises

Mathematical Techniques

1–4 ▪ Use the idea of the weighted average to find the following.

1. 1.0 L of water at 30°C is mixed with 2.0 L of water at 100°C. What is the temperature of the resulting mixture?
2. 2.0 ml of water with a salt concentration of 0.85 mol/L, is mixed with 5.0 ml of water with a salt concentration of 0.70 mol/L. What is the concentration of the mixture?
3. In a class of 52 students, 20 scored 50 on a test, 18 scored 75, and the rest scored 100. What was the average score?
4. In a class of 100 students, 10 score at 20, 20 score at 40, 30 score at 60, and 40 score at 80. What is the average score in the class?

5–8 ▪ Express the following weighted averages in terms of the given variables.

5. 1.0 L of water at temperature T_1 is mixed with 2.0 L of water at temperature T_2. What is the temperature of the resulting mixture? Set $T_1 = 30$ and $T_2 = 100$ and compare with the result of Exercise 1.
6. V_1 liters of water at 30°C is mixed with V_2 liters of water at 100°C. What is the temperature of the resulting mixture? Set $V_1 = 1.0$ and $V_2 = 2.0$ and compare with the result of Exercise 1.
7. V_1 liters of water at temperature T_1 is mixed with V_2 liters of water at temperature T_2. What is the temperature of the resulting mixture?

8. V_1 liters of water at temperature T_1 is mixed with V_2 liters of water at temperature T_2 and V_3 liters of water at temperature T_3. What is the temperature of the resulting mixture?

9–12 ▪ The following are similar to examples of weighted averages with absorption.

9. 1.0 L of water at 30°C is to be mixed with 2.0 L of water at 100°C, as in Exercise 1. Before mixing, however, the temperature of each moves half-way to 0 °C (so the 30°C water cools to 15°C). What is the temperature of the resulting mixture? Is this half the temperature of the result in Exercise 1?

10. 2.0 ml of water with a salt concentration of 0.85 mol/L, is to be mixed with 5.0 ml of water with a salt concentration of 0.70 mol/L, as in Exercise 2. Before mixing, however, evaporation leads the each concentration of each component to double. What is the concentration of the mixture? Is it exactly twice the concentration found in Exercise 2?

11. In a class of 52 students, 20 scored 50 on a test, 18 scored 75, and the rest scored 100. The professor suspects cheating, however, and deducts 10 from each score. What is the average score after the deduction? Is it exactly 10 less than the average found in Exercise 3?

12. In a class of 100 students, 10 score at 20, 20 score at 40, 30 score at 60, and 40 score at 80 as in Exercise 4. Because students did so poorly, the professor moves each score half way up toward 100 (so the students with 20 are moved up to 60). What is the average score in the class? Is the new average the old average moved half way to 100?

Applications

13–16 ▪ Suppose that the volume of the lungs is V, the amount breathed in and out is W, and the ambient concentration is γ mmol/L. For each of the given sets of parameter values and the given initial condition, find the following:

a. The amount of chemical in the lungs before breathing

b. The amount of chemical breathed out

c. The amount of chemical in the lungs after breathing out

d. The amount of chemical breathed in

e. The amount of chemical in the lungs after breathing in

f. The concentration of chemical in the lungs after breathing in

g. Compare this result with the result of using the general lungs discrete-time dynamical system (equation 1.9.1). Remember that $q = W/V$.

13. $V = 2.0$ L, $W = 0.5$ L, $\gamma = 5.0$ mmol/L, $c_0 = 1.0$ mmol/L.

14. $V = 1.0$ L, $W = 0.1$ L, $\gamma = 8.0$ mmol/L, $c_0 = 4.0$ mmol/L.

15. $V = 1.0$ L, $W = 0.9$ L, $\gamma = 5.0$ mmol/L, $c_0 = 9.0$ mmol/L.

16. $V = 10.0$ L, $W = 0.2$ L, $\gamma = 1.0$ mmol/L, $c_0 = 9.0$ mmol/L.

17–20 ▪ Find and graph the updating function in the following cases. Cobweb for three steps starting from the points indicated in the earlier problems. Sketch the solutions.

17. The situation in Exercise 13.

18. The situation in Exercise 14.

19. The situation in Exercise 15.

20. The situation in Exercise 16.

21–24 ▪ Compute the equilibrium of the lungs discrete-time dynamical system and check that $c^* = \gamma$.

21. $V = 2.0$ L, $W = 0.5$ L, $\gamma = 5.0$ mmol/L, $c_0 = 1.0$ mmol/L (as in Exercise 13).

22. $V = 1.0$ L, $W = 0.1$ L, $\gamma = 8.0$ mmol/L, $c_0 = 4.0$ mmol/L (as in Exercise 14).

23. $V = 1.0$ L, $W = 0.9$ L, $\gamma = 5.0$ mmol/L, $c_0 = 9.0$ mmol/L (as in Exercise 15).

24. $V = 10.0$ L, $W = 0.2$ L, $\gamma = 1.0$ mmol/L, $c_0 = 9.0$ mmol/L (as in Exercise 16).

25–26 ▪ Compare the equilibrium and total amount absorbed per breath for different values of q. Use an ambient concentration of $\gamma = 0.21$ and a volume of $V = 6.0$L.

25. Suppose $q = 0.4$ and $\alpha = 0.1$. Why is the equilibrium concentration higher than with $q = 0.2$ even though the person is breathing more?

26. Suppose $q = 0.1$ and $\alpha = 0.05$. Think of this as a person gasping for breath. Why is the concentration nearly the same as in Example 1.9.9? Does this mean that gasping for breath is OK?

27–30 ▪ The following problems investigate absorption that is not proportional to the concentration in the lungs. Assume $\gamma = 0.21$ and $q = 0.1$, and find the equilibrium concentration.

27. Oxygen concentration is reduced by 2% each breath (that is, if the concentration before absorption were 18%, it would be 16% after absorption). Find the discrete-time dynamical system and the equilibrium. Are there values of c_t for which the system does not make sense?

28. Oxygen concentration is reduced by 1% each breath. Find the discrete-time dynamical system and the equilibrium. Are there values of c_t for which the system does not make sense?

29. The amount absorbed is $0.2(c_t - 0.05)$ if $c_t \geq 0.05$. This models a case where the only oxygen available is that in excess of the concentration in the blood, which is roughly 5%.

30. The amount absorbed is $0.1(c_t - 0.05)$ if $c_t \geq 0.05$.

31–32 ▪ Find the value of the parameter that produces an exhaled concentration of exactly 0.15. Assume $\gamma = 0.21$ and $q = 0.1$.

31. Oxygen concentration is reduced by an amount A (generalizing the case in Exercises 27 and 28). How does the amount absorbed with this value of A compare with the amount of oxygen absorbed in Example 1.9.9?

32. The amount absorbed is $\alpha(c_t - 0.05)$ (generalizing the case where only available oxygen is absorbed in Exercises 29 and 30). How does the amount absorbed with this value of α compare with the amount of oxygen absorbed in Example 1.9.9?

33–34 ▪ The following problems investigate production of carbon dioxide by the lungs. Suppose that the concentration increases by an amount S before the air is exchanged. Assume an ambient concentration of carbon dioxide of $\gamma = 0.0004$ and $q = 0.1$.

33. Suppose $S = 0.001$. Write the discrete-time dynamical system and find its equilibrium. Compare the equilibrium with the ambient concentration.

34. The actual concentration of carbon dioxide in exhaled air is about 0.04, or 100 times the ambient concentration. Find the value of S that gives this as the equilibrium.

35–36 ▪ A bacterial population that has per capita production $r < 1$ but that is supplemented each generation follows a discrete-time dynamical system much like that of the lungs. Use the following steps to build the discrete-time dynamical system.

a. Starting from 3.0×10^6 bacteria, find the number after reproduction.

b. Find the number after the new bacteria are added.

c. Find the discrete-time dynamical system.

35. A population of bacteria has per capita production $r = 0.6$, and 1.0×10^6 bacteria are added each generation.

36. A population of bacteria has per capita production $r = 0.2$, and 5.0×10^6 bacteria are added each generation.

37–40 ▪ Find the equilibrium population of bacteria in the following cases with supplementation.

37. A population of bacteria has per capita production $r = 0.6$, and 1.0×10^6 bacteria are added each generation (as in Exercise 35).

38. A population of bacteria has per capita production $r = 0.2$, and 5.0×10^6 bacteria are added each generation (as in Exercise 36).

39. A population of bacteria has per capita production $r = 0.5$, and S bacteria are added each generation. What happens to the equilibrium when S is large? Does this make biological sense?

40. A population of bacteria has per capita production $r < 1$, and 1.0×10^6 bacteria are added each generation. What happens to the equilibrium if $r = 0$? What happens if r is close to 1? Do these results make biological sense?

41–44 ▪ Lakes receive water from streams each year and lose water to outflowing streams and evaporation. The following values are based on the Great Salt Lake in Utah. The lake receives $3.0 \times 10^6 m^3$ of water per year with salinity of 1 part per thousand (concentration 0.001). The lake contains $3.3 \times 10^7 m^3$ of water and starts with no salinity. Assume that the water that flows out has a concentration equal to that of the entire lake. Compute the discrete-time dynamical system by finding (a) the total salt before the inflow, (b) total water, (c) total salt and salt concentration after inflow, and (d) total water, total salt, and salt concentration after outflow or evaporation,

41. There is no evaporation, and $3.0 \times 10^6 m^3$ of water flows out each year.

42. $1.5 \times 10^6 m^3$ of water flows out each year, and $1.5 \times 10^6 m^3$ evaporates. No salt is lost through evaporation.

43. A total of $3.0 \times 10^6 m^3$ of water evaporates, and there is no outflow.

44. Assume instead that $2.0 \times 10^6 m^3$ of water evaporates and there is no outflow. The volume of this lake is increasing.

45–48 ▪ Find the equilibrium concentration of salt in a lake in the following cases. Describe the result in words by comparing the equilibrium salt level with the salt level of the water flowing in.

45. The situation described in Exercise 41.

46. The situation described in Exercise 42.

47. The situation described in Exercise 43.

48. The situation described in Exercise 44.

49–50 ▪ A lab is growing and harvesting a culture of valuable bacteria described by the discrete-time dynamical system

$$b_{t+1} = rb_t - h.$$

The bacteria have per capita production r, and h bacteria are harvested each generation.

49. Suppose that $r = 1.5$ and $h = 1.0 \times 10^6$ bacteria. Sketch the updating function, and find the equilibrium both algebraically and graphically.

50. Without setting r and h to particular values, find the equilibrium algebraically. Does the equilibrium get larger when h gets larger? Does it get larger when r gets larger? If the answers seem odd (as they should), look at a cobweb diagram to try to figure out why.

Computer Exercise

51. Investigate which factor is most important in absorbing oxygen at the maximum rate: the volume V of the lungs, the amount exchanged W, or the fraction absorbed α, using Equation 1.9.2. If an athlete could train to increase one of these values, which would be the most effective?

1.10 An Example of Nonlinear Dynamics

The discrete-time dynamical systems we have studied in detail (bacterial populations, tree height, mite populations, and the lung) are said to be **linear** because the updating function is linear. We now derive a model of two competing bacterial populations that leads naturally to a discrete-time dynamical system that is not linear. **Nonlinear dynamical systems** can have much more complicated behavior than a linear system. For example, they may have more than one equilibrium. By comparing the two equilibria