CHAPTER 3

Differentiation

3.1 AVERAGE RATES OF CHANGE

Average rate of change of a function

What is a **rate**? How do we use rates in daily life? To address these issues, work in groups and answer the following questions.

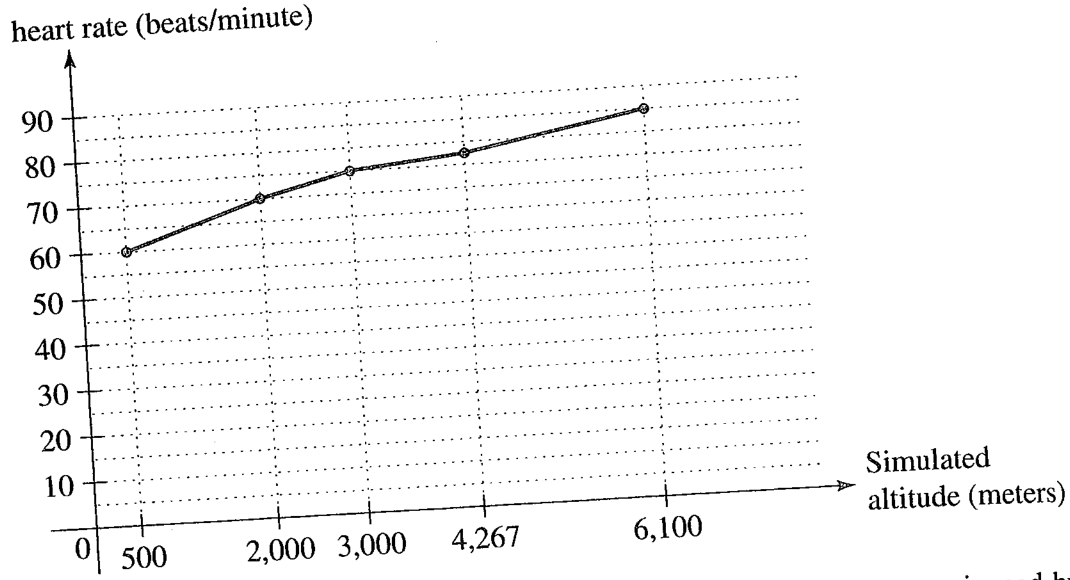
Classroom Discussion 3.1.1: Rates of Change in Real Life

1. What is a person's heart rate? How do you compute it? What units of measurement do you use for heart rate? Fill in the blanks:

$$\text{heart rate} = \frac{\text{number of }\_\_\_\_\_\_\_\_}{\text{time, measured in }\_\_\_\_\_\_\_\_}$$

2. A person's painting rate is the ratio of the area the person paints to the time it takes to paint that area. Two painters must paint a rectangular wooden fence that is 15 yards long and 5 feet tall. The first painter paints at a rate of 25 square feet per minute. The second painter paints at a rate of 20 square feet per minute. How long will it take for the two workers to finish painting both sides of the wooden fence?

H in beats per minute for a person at rest and the altitude x measured in meters is sketched here.



What is the average rate of change of H with respect to x experienced by a person in a hot air balloon as the balloon rises from 2,000 to 3,000 meters? How about from 4,267 to 6,100 meters? How do the two compare?

3. Jesse filled the gas tank of her Chevy Beretta before beginning a trip from Kansas City to Houston. After driving 199 miles on I-70 and I-35, she stopped in Wichita to pick up three friends. Since she was stopped anyway, she filled the tank again with 6 gallons of gasoline. The four friends made good time on the 363-mile stretch from Wichita to Dallas and filled the tank with 11.7 gallons of gas. The next morning, they picked up yet another friend and drove the 239 miles from Dallas to Houston, making slow progress because of heavy traffic in both cities. When they finally arrived in Houston, they put 8.2 gallons of gas in the car to fill up the tank. The rate at which a car burns fuel is expressed as the number of miles traveled divided by the number of gallons of gasoline used. What was the rate at which Jesse's car burned fuel (a) between Kansas City and Wichita, (b) between Wichita and Dallas, (c) between Dallas and Houston, and (d) on the entire trip?

4. Rachel's 3-cylinder Geo Metro accelerates from 0 to 60 miles per hour in 31 seconds. What is the average acceleration over the 31 seconds that it takes her to reach 60 miles per hour? Express your answer in miles per hour squared.

5. During the summer, Diana earns money by selling lemonade. She charges 50 cents for each cup of lemonade.
   a. Let $f(x)$ denote the amount of money (in dollars) Diana earns by selling x cups of lemonade. Determine a formula for $f(x)$ and sketch the graph of f.
   b. What is the average rate of change of f with respect to x as x increases from a to b, where $a < b$ and a, b are arbitrary whole numbers? Explain.

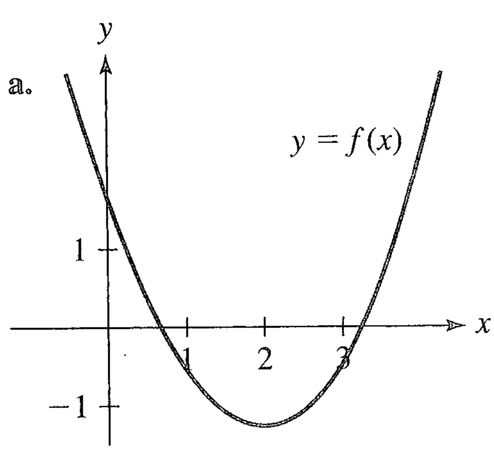
6. When selling lemonade for 50 cents per cup, Diana can sell 40 cups per day. She observes that the number of cups sold per day decreases by 2 for every 5-cent increase in the price of a cup of lemonade.

EXERCISES 3.2

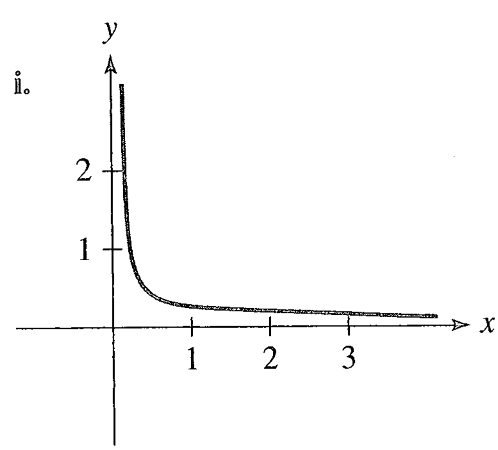
In Exercises 1–4, compute the derivative of the given function.

1. $f(x) = x^2 + 2x$
2. $g(x) = x^2 - 3x$
3. $A(s) = 2s^2 + s - 2$
4. $f(t) = -t^2 + 3t - 10$
5. Determine the equation of the tangent line to the curve $y = 2x^2 - x + 10$ at the point $(-1, 13)$.
6. Determine the equation of the tangent line to the curve $y = -3x^2 + 5x - 2$ at the point $(1, 0)$.
7. Scientists have found that radioactive carbon-14 (C14) has a half-life of 5,730 years. This means that if the amount of C14 now is α, then the amount 5,730 years from now will be $\frac{1}{2}\alpha$, the amount 11,460 years from now will be $\frac{1}{2^2}\alpha$, and so on. The amount of C14 remains constant in living organisms due to metabolic processes but decreases once the organism dies. This is the idea behind carbon dating.
   a. Make a sketch of the amount $A(t)$ of C14 in an organism t years after it has died, if the amount of C14 present while the organism was living is α. Is A increasing, decreasing, or neither? Will $A(t)$ ever be zero? As $t \to \infty$, what value does $A(t)$ approach?
   b. Use the graph you have sketched in a to answer the following questions: Is A' positive or negative? Will $A'(t)$ ever be zero? Is A' increasing, decreasing, or neither? As $t \to \infty$, what value does $A'(t)$ approach? Make a sketch of A'.
8. Match each limit a–c with the corresponding derivative from i–iii.
   a. $\lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$ i. $(x^3)'$
   b. $\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$ ii. $(\sqrt{x})'$
   c. $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ iii. $\left(\frac{1}{x}\right)'$
9. Match each function a–c with the graph of its derivative i–iii.

a.



i.



trip from Columbia to Kansas City if Columbia is halfway between Kansas City and Saint Louis?

d. After a 1-hour wait in Kansas City, the train returns to Columbia. Write the expression of the position function for the Kansas City–Columbia route.

e. Sketch the graphs of the displacement $d(t)$ and the velocity $v(t)$ from the moment the train began its trip to Saint Louis until it returned to Columbia from Kansas City.

2. The Washington Monument is 555 feet tall. The height h of a coin dropped from the top of the monument is $h(t) = -16t^2 + 555$. Here h is measured in feet from the ground, and t is measured in seconds from the moment the coin was dropped.
   a. Sketch the graph of $h(t)$.
   b. How long will it take for the coin to hit the ground?
   c. Determine the coin's velocity when it hits the ground.
   d. Sketch the graph of the velocity $v(t)$. Interpret the graph.
   e. Determine the coin's acceleration $a(t)$ at time t.

3. The height h in feet of an object fired straight up is given by $h(t) = -16t^2 + 100t$, where time t is measured in seconds from the instant the object was fired.
   a. From where was the object fired, and what was its initial velocity?
   b. What is the velocity of the object 1.2 seconds after it was fired?
   c. What is the maximum height attained by the object?
   d. During which intervals of time is the speed decreasing?
   e. During which intervals of time is the speed increasing?
   f. Compute the acceleration of the object at $t = 1$ seconds and $t = 2$ seconds.

4. A diver jumps from a diving board. The diver's height (measured in feet) at time t is $h(t) = -16t^2 + 32t + 48$, where t is the time in seconds $h(t)$.
   a. How high is the diving board?
   b. When does the diver hit the water?
   c. What is the diver's velocity at impact?

5. A car pulls out of a driveway, accelerates, and then stops for a red light. This motion takes place on a straight road. The car's displacement, in miles, from the front of the driveway is given by the function

$$d(t) = \begin{cases} 0.17t^3, & 0 \le t \le 1 \\ -0.17(t-2)^2 + 0.34, & 1 \le t \le 2, \end{cases}$$

where t is the time in minutes.
   a. Find the car's velocity $v(t)$ at time t. Graph the function $v(t)$. What can you say about $v(1)$?
   b. Find the car's acceleration $a(t)$ at time t. Graph the function $a(t)$. What can you say about $a(1)$? How do you interpret the sign of $a(t)$?

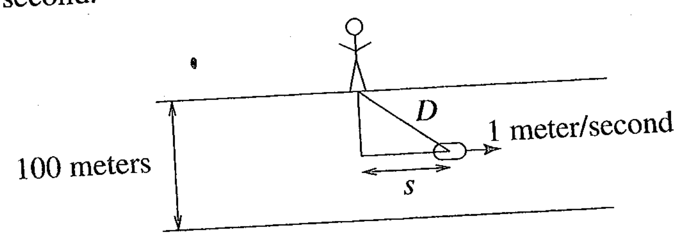
3.4 RULES FOR COMPUTING DERIVATIVES

The power rule • The constant multiple rule • The sum and difference rule • The product rule • The quotient rule

Recall that, by definition, the derivative $f'(x)$ of a given function $f(x)$ is equal to $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, whenever such a limit exists. Using this definition, you proved in

24. $f(y) = \frac{y+1}{y^2+1}$, $g(t) = \frac{1}{t}$, $t = -1$

25. A person standing on a riverbank observes an empty boat floating down the middle of the river. The river is 100 meters wide, and the water flows at a rate of 1 meter per second. Let D be the distance between the person and the boat at time t. Determine the rate of change of D with respect to time: 2 seconds prior to, 3 seconds after, and 10 minutes after the moment the boat passes in front of the person.
    Hint: You must compute $(D(s(t)))'$ at the specified times knowing that $s'(t) = 1$ meter per second.



26. On a hot summer day, a spherically shaped scoop of ice cream is left outside in a bowl. The ice cream starts melting. What is the rate of change with respect to time of this ice cream ball's radius at the moment when the radius is 3 centimeters, if the ice cream's volume is changing at a rate of -5 cubic centimeters per minute?

27. In Section 3.3, you studied the motion of the rocket launched by Adrian. Recall that the rocket's height in feet is $h(t) = -16t^2 + 128t$ when measured from the ground t seconds after launching. Diana watches the launching of the rocket from a distance of 20 feet from the launcher. Set D as the distance from Diana to the rocket at time t. Compute the rate of change of D with respect to time at $t = 2$ seconds and $t = 5$ seconds after launching.

28. A function f is even if for any x in the domain of f, $-x$ also belongs to the domain of f and if $f(-x) = f(x)$. Similarly, a function g is odd if for any x in the domain of g, $-x$ also belongs to the domain of g and if $g(-x) = -g(x)$. Use the chain rule to show that the derivative of a differentiable even function is an odd function, while the derivative of a differentiable odd function is an even function.

29. Suppose $f(x) = x^3 + \sqrt{x^2+1}$ and g, h are two functions such that $g'(x) = f(x-1), g(1) = 2, h'(x) = x^5 - 5x^3 + 7$, and $h(1) = -1$. Compute $(f \circ g)'(1)$ and $(h \circ f)'(0)$.

30. Let $f(x) = x^4 + 5x^3 - 6x$ and let $k(x) = f(x^3)$. Compute $k'(x^3)$.
    Warning: $k'(x^3) \neq (k(x^3))'$.

CHAPTER 3 REVIEW

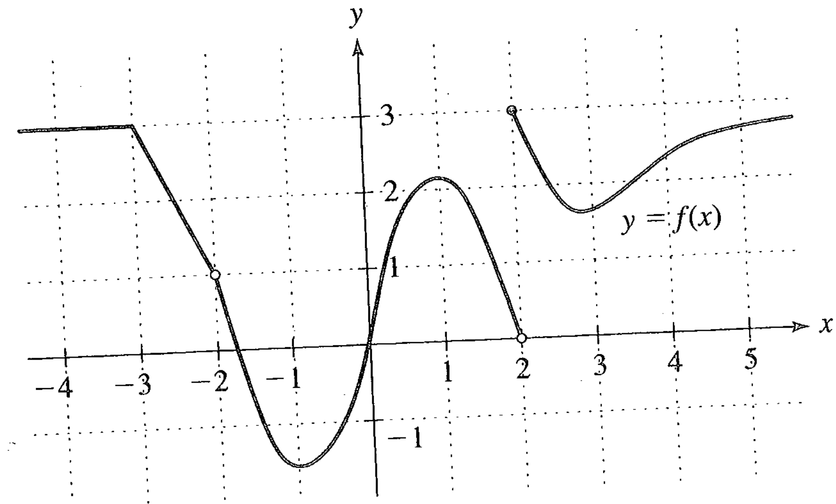
The average rate of change of a function f over an interval $[a, b]$ is by definition the value of $\frac{f(b)-f(a)}{b-a}$ (see Definition 3.1.1).

The derivative of a function f at a point x, denoted by $f'(x)$, is the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ (also called the *instantaneous rate of change of f at x*), provided it exists. A function for which $f'(x)$ exists is differentiable at x. Alternatively, f is differentiable at x provided $\lim_{y\to x} \frac{f(y)-f(x)}{y-x}$. In addition, if f is differentiable at x, then $f'(x)$ equals the slope of the tangent line to the graph of f at the point $(x, f(x))$. The

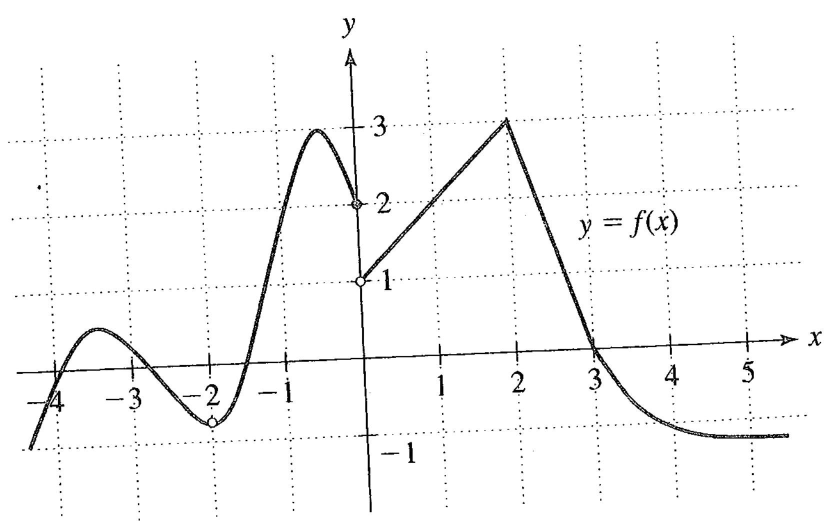
CHAPTER 3 REVIEW EXERCISES

In Exercises 1–2, compute the average rate of change of f over the interval $[a, b]$. Write the equation of the line containing the points $(a, f(a))$ and $(b, f(b))$.

1. $f(x) = \sqrt{x+2}, a = 2, b = 7$
2. $f(x) = \frac{2}{x}, a = 2, b = 3$
3. Use the graph of f to sketch the graph of its derivative.



4. Use the graph of f to sketch the graph of its derivative.

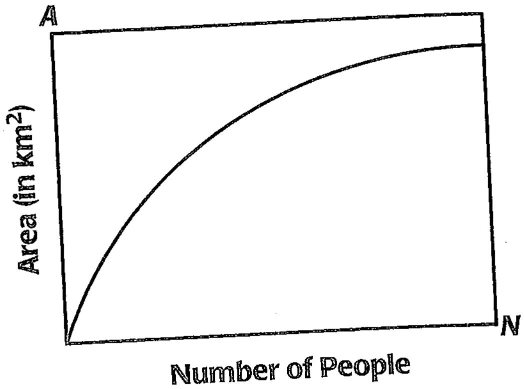


5. The velocity of an ice skater moving along a horizontal straight line is $v(t) = 2(t-3)^2 - 2$ feet per second, where t is the time in seconds.
   a. Plot the velocity of the skater for t between 0 and 8 seconds, and determine his initial speed.
   b. Determine the acceleration of the skater as a function of time, and calculate his initial acceleration.
   c. When does the skater reverse his motion?
   d. Based on the results just obtained, describe the skater's motion.
   e. At what moment is the skater's velocity minimal, and what is its velocity then?

F. HYPERBOLAS

In Section C, you worked with a feasible region that has a curved border.

That is not uncommon in real-world problems. In this section, you will look at another problem involving a curve.



In the 19th century, many adventurers traveled to North America to search for gold. A man named Dan Jackson owned some land where gold had been found. Instead of digging for the gold himself, he rented plots of land to the adventurers. The "rent" was to give Dan 50% of any gold found on the plot of land.

Dan gave each adventurer four stakes and a rope that was exactly 100 meters long. Each adventurer had to use the stakes and rope to mark off a rectangle with north-south and east-west sides.

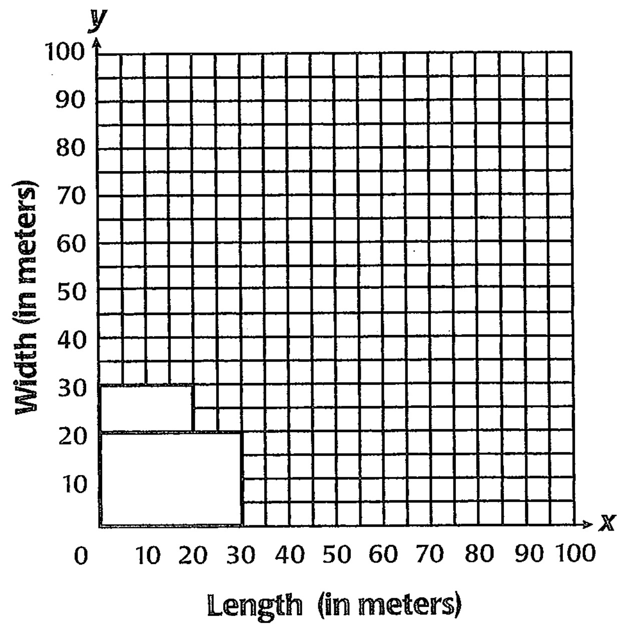
1. Did everyone get the same area to dig for gold? Explain your answer.

Get the Most Out of It

There are many different rectangles you can make that have a perimeter of 100 meters.

2. a. On graph paper, draw at least five different rectangles with perimeters of 100 meters. Draw your rectangles to scale, using one grid unit to represent five meters.

   b. Cut out the rectangles. Tape them on a graph so the lower-left corners lie on the origin. Use the same grid size as in part a, as shown at the right.



For each rectangle, only the upper-right vertex will not touch an axis.

3. You should notice that all the upper-right vertices of your rectangles seem to lie on a line. What is the equation of that line? Use x and y as the dimensions of the rectangles.

4. Calculate the area of each rectangle, and organize all of your information in a table.

One of the diggers discovered that one kind of rectangle always had the greatest area. He decided to sell the secret to other diggers.

5. What was the secret?

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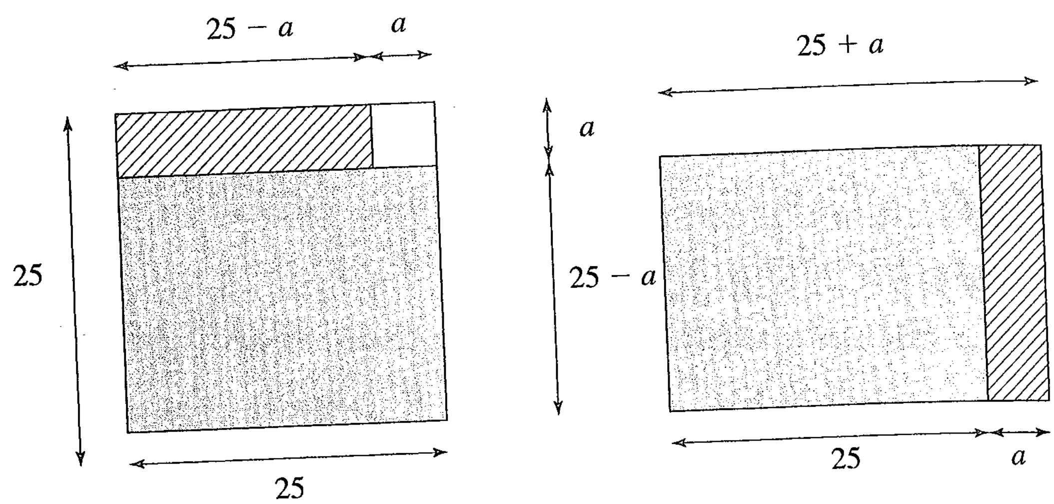
1. Does the reasoning you used for answering Question 5 constitute a complete proof? How would you show that no other rectangle with a perimeter of 100 meters will have an area larger than the rectangle you discovered when answering Question 5? Do this in two ways.

a. Give a geometric proof using the following figure.



b. Give an algebraic proof by showing that $(25 - a)(25 + a) \leq 625$.

2. Complete the following tasks to tackle Question 5 by other means.

a. Denote by x and y the dimensions of a plot of land measured in meters. Use the information about the rope's length to write an identity involving x and y. We refer to this identity as the **constraint** for x and y.

b. Let A denote the area, measured in square meters, of such a rectangle. Write A first in terms of x and y, and then use the constraint for x and y to express A as a function of x only. Call this new expression $A(x)$. What is the domain of $A(x)$?

c. Use a calculator to graph $A(x)$. What do you see? Use your calculator's zoom and trace features to find the value of x for which $A(x)$ is the largest. How does your answer compare to the answer you obtained in Problem 1?

3. Derivatives at Work

Return to the function $A(x)$ from Problem 2. You should have seen that the domain of $A(x)$ is the interval $(0, 50)$ and that the graph of A is a parabola. Use calculus to determine the coordinates of the highest point on this parabola, which corresponds to the value of x that gives the largest area.

a. What slope does the tangent line have at the highest point on the graph of A? What does this information tell you about the derivative $A'(x)$?

b. Compute $A'(x)$ and graph it. Find the value x_0 in the interval $(0, 50)$ with the property that $A'(x_0) = 0$.

c. What is the sign of $A'(x)$ to the left of x_0? How about to the right of x_0? What sort of information do you obtain about $A(x)$ from the sign of $A'(x)$? What can you say about the value of $A(x_0)$? How does this relate to Question 5 in Classroom Discussion 4.1.1? ◆

Classroom Discussion 4.1.2: Smallest Perimeter

The following exploration is from page 48 in the eighth-grade textbook *Mathematics in Context, Get the Most Out of It*. Work through tasks 6–9.

A New Plot

Once the secret was out, Dan changed his rental agreement. He still gave the adventurers four stakes, and they still marked off rectangles with north-south and east-west sides, but with a new constraint:

The rectangle had to have an *area* of 400 square meters.

Since the adventurers had to use their *own* rope to mark off the plot, they had to decide how long their rope should be.

6. a. Using graph paper, draw at least five different rectangles with areas of 400 square meters. Again, use the scale that one grid unit represents five meters. How long is the rope needed for each rectangle?

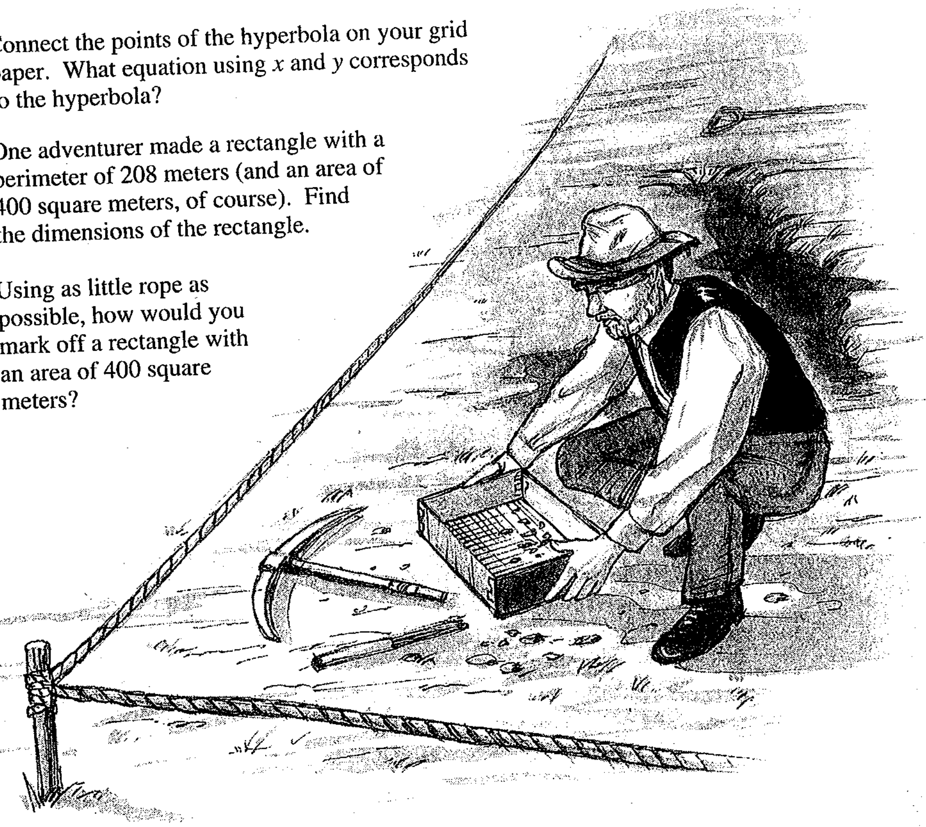
b. As you did before, cut out the rectangles and tape them on grid paper so two sides lie on the axes.

You should notice that the upper-right vertices no longer lie on a straight line, but instead on a curve called a *hyperbola*.

7. Connect the points of the hyperbola on your grid paper. What equation using x and y corresponds to the hyperbola?

8. One adventurer made a rectangle with a perimeter of 208 meters (and an area of 400 square meters, of course). Find the dimensions of the rectangle.

9. Using as little rope as possible, how would you mark off a rectangle with an area of 400 square meters?



1. Does the reasoning you used for answering Question 9 constitute a complete proof? How do you show that no other rectangle with an area of 400 square meters will have a perimeter smaller than the rectangle you discovered when answering Question 9?

   Hint: Use algebra and show that $2\left(20 + a + \frac{400}{20+a}\right) < 80$ leads to a contradiction.

2. Complete the following tasks to tackle Question 9 by other means.

   a. Denote by x and y the dimensions of a plot of land measured in meters. What is the constraint for x and y in this case?

   b. Let P denote the perimeter, measured in meters, of such a rectangle. Write P first in terms of x and y, and then use the constraint you found for x and y to express P as a function of x. Call this new expression $P(x)$. What is the domain of $P(x)$?

   c. Use a calculator to graph $P(x)$. You need to choose an appropriate window scale. What do you see? Use your calculator's zoom and trace features to find the value of x for which $P(x)$ is the smallest. How does your answer compare to the answer you obtained in Problem 1?

3. Return to the function $P(x)$ from Problem 2. Use calculus to determine the smallest value attained by $P(x)$.

   a. Compute $P'(x)$ and determine the critical points of $P(x)$.

   b. Determine the sign of $P'(x)$ on the intervals in the domain to the left and right of the critical points.

   c. Use your answer from b to determine if $P(x)$ is increasing or decreasing on each interval.

   d. Determine the smallest value of $P(x)$ and determine the point x_0 for which this value is attained. Compare $P(x_0)$ to the solutions you obtained in Problems 1 and 2. ◆

In the Classroom Discussions 4.1.1 and 4.1.2 you have solved two optimization problems using calculus. In the process, you completed several steps that are typical of any optimization problem. Let's look back and outline these steps.

Step 1: Read the problem carefully and decide which variables to use, the function involving your variables that you need to optimize, and the constraint that you have for the variables. In both examples in Classroom Discussions 4.1.1 and 4.1.2 the variables were x and y, the length and width of a plot of land. In the problem from Classroom Discussion 4.1.1, you had to maximize the area xy under the constraint $2(x + y) = 100$. In the one from Classroom Discussion 4.1.2, you had to minimize the perimeter $2x + 2y$ under the constraint $xy = 400$.

Step 2: Use the constraint to write the function to be optimized in terms of only one variable. Let us call this function f for the purpose of our discussion. In the Classroom Discussion 4.1.1, the resulting function was $f(x) = x(50 - x)$, in the Classroom Discussion 4.1.2 the resulting function was $f(x) = 2x + \frac{800}{x}$.

Step 3: Determine the domain of f as dictated by the physical assumptions of the problem. In the examples discussed, the domains were $(0, 50)$ for Classroom Discussion 4.1.1 and $(0, \infty)$ for Classroom Discussion 4.1.2.

Step 4: Compute the derivative of f and find the critical points, i.e., the values of x in the domain of f where $f'(x) = 0$. Be careful here since the equation $f'(x) = 0$ might yield solutions that are not in the domain of f. In the first and second example, the critical points were $x = 25$ and $x = 20$, respectively.

Step 5: Determine the intervals where f' is negative and where it is positive. Use this information to decide whether f increases or decreases on these intervals. The function in the first example was increasing for $x < 25$ and decreasing for $x > 25$. In the second example, the function was decreasing for $x < 20$ and increasing for $x > 20$.

Step 6: Decide if there are any points where your function attains its maximum or minimum, as required by the problem. Evaluate the function at any such points. In the first example, the function attained its maximum at $x = 25$, with the value of this maximum being 625. In the second example, the function attained its minimum at $x = 20$, with the value of this minimum being 80.

Step 7: Interpret and check your solution. In the first example, the conclusion was that, among all the rectangles with perimeters equal to 100 meters, the one with the largest area is the square with side length 25 meters. In the second example, you proved that among all the rectangles having an area of 400 square meters, the one with the smallest perimeter is the square with side length 20 meters.

Practice Problems

1. Use calculus to prove generalizations of the examples discussed in the Classroom Discussions 4.1.1 and 4.1.2.

   a. Among all the rectangles with a fixed perimeter P, the one with the largest area is the square with side length $\frac{P}{4}$.

   b. Among all the rectangles with fixed area A, the one with the smallest perimeter is the square with side length $\sqrt{A}$.

2. Denote by x and y the dimensions of a rectangle, by A its area, and by P its perimeter. The following identity then holds:

$$\frac{P^2}{4} = (x - y)^2 + 4A.$$

Give an algebraic and a geometric proof of this identity; for the geometric proof, use the following picture. Then, utilize the preceding identity to prove, without using calculus, the statements a and b in Problem 1.

a: How does the last question in this exploration relate to the practice problems in Section 4.1?

9 Fenced In

SOLVING A PROBLEM THAT INVOLVES A PARABOLA

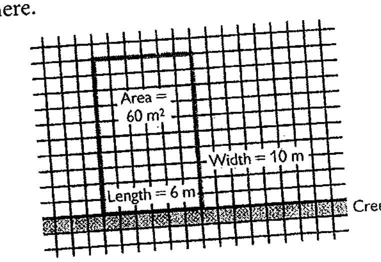
Suppose you have 32 m of fencing material. What is the largest rectangle you can fence off? And what does this problem have to do with parabolas? You will explore this situation and make a graph to describe it. This will help you see the connection between perimeters, areas, and parabolas.

Explore Rectangular Pens

How can you find a rectangle with the greatest area for a given perimeter?

A farmer has 32 m of fencing material and wants to fence off a rectangular pen for animals. One side of the pen must lie along a creek. What length along the creek results in a pen with the greatest area for the animals?

1. What is the perimeter of any pen the farmer can make?
2. Use a sheet of graph paper to help sketch all of the possible pens that have whole-number lengths. One possible pen is shown here.



3. Find the area of each pen.

Which length along the creek results in the pen with the greatest area?

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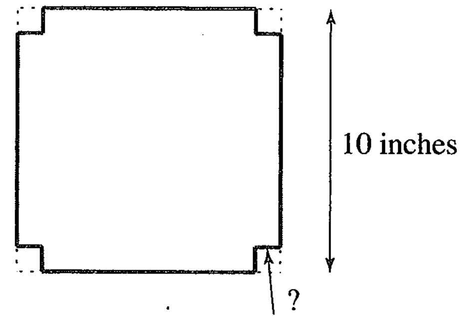
b: Now suppose there exists a wall along the creek, so no fencing is needed for that side of the pen. Use calculus to determine the dimensions of the rectangular pen with the greatest area if the wall is used for one side of the pen. ◆

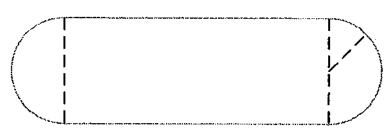
7. **Chocolate Boxes:** A candy store is closed for remodeling. The owner has purchased square pieces of thin plastic of different colors in order to make new candy

display boxes. The plastic squares have a 10-inch side length. He decides to cut out four equal squares from the corners of each piece and then form the boxes by turning up the sides. What should be the side length of the squares he cuts so that the resulting boxes will have the largest volume?

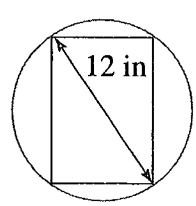


8. **Pillows:** Tracy has a home-based business in which she sews decorative pillows and sells them on eBay. Each pillow requires \$2 of materials and 2 hours of sewing. Tracy finds that if she charges \$12 per pillow, she sells 30 pillows per week. For each \$1 increase in price, she sells 5 fewer pillows per week; for each \$1 decrease in price, she sells 5 more pillows per week. Use calculus to determine what price Tracy should charge to maximize her profit.

9. **A Rollerblading Track:** A city plans to construct a new park that will have a rollerblading track that is 400π meters long. The portion enclosed by the track will consist of a rectangular playground with two semicircular ends in which to plant flowers. At what dimensions will the rectangular area in the center be the largest? For these dimensions, how much space will be available for planting flowers?



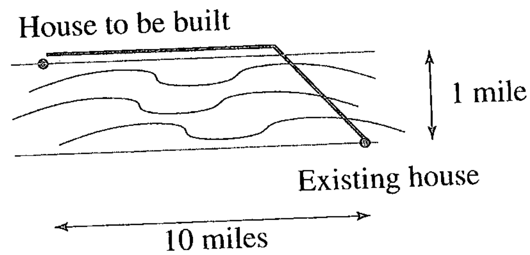
10. **Wooden Beams:** In the construction of houses, beams with rectangular cross sections are often used. These beams are cut from wooden logs. A beam's strength is directly proportional to the product of its width and the square of its height. Suppose you have a log with a circular cross section of diameter 12 inches. What are the dimensions of the cross section of the strongest beam you can cut from this log?



11. **Watermelon Crop:** A gardener must decide when to harvest and sell his watermelon crop. He estimates that there are approximately 100 pounds of ripe watermelon in his garden. Each week an additional 25 pounds of watermelon ripens, and 5 pounds go to waste. The watermelon's market price is \$0.90 per pound, but it drops \$0.10 per pound each week that passes. When should the gardener sell his watermelon crop in order to make the maximum revenue? What is the maximum revenue he can make?

12. A gardener is trying to decide what is the best time to harvest and sell his cantaloupe crop. He estimates that there are approximately 200 pounds of ripe cantaloupe in his garden. Each week an additional 60 pounds of cantaloupe ripens, and 10 pounds go to waste. The cantaloupe's market price is \$1 per pound, but it drops \$0.10 per pound each week that passes. When should the gardener sell his cantaloupe crop in order to maximize revenue? What is the maximum revenue he can make?

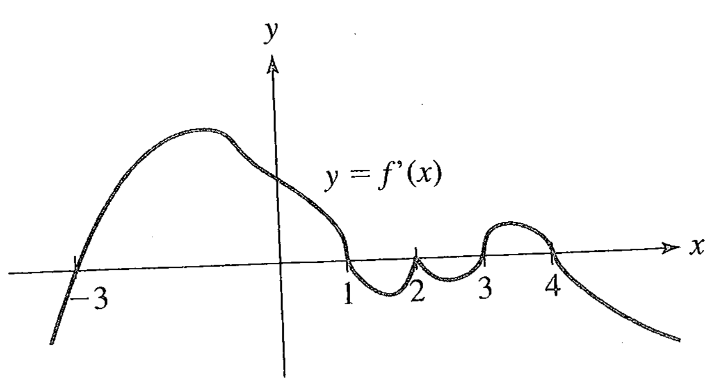
13. **Cable Lines:** A house is built along a 1-mile wide river. On the river's other side, 10 miles downstream, there is another building from which cable line will be run to the new house. The underwater cable costs twice as much per foot as the underground cable. How long should the cable line along the river be in order to minimize the cost?



14. Tod runs a chocolate store. He invests \$4 for each pound of chocolate he makes. He sells 500 pounds of chocolate each month for the price of \$12 per pound. Tod discovers that for every 10 cents he takes off the price, he sells 10 more pounds of chocolate each month. What should he charge for 1 pound of chocolate in order to maximize his profit? How many pounds of chocolate will Tod sell at that price?

15. **Inventory Costs:** A retail appliance store sells 500 refrigerators each year. It costs \$50 to store one refrigerator for a year. At each reordering, there is a fixed \$20 fee for the truck rental and an additional \$5 handling fee for each refrigerator ordered. How many times per year should the store place an order to minimize the storage, truck rental, and handling costs, if the number of refrigerators per order is constant? When modeling this problem, assume that at any time during the year, the average number of refrigerators in stock is half the number x of refrigerators ordered each time.

16. The following figure shows the graph of the derivative of a function f. Find all the points where f has a local maximum or a local minimum.



3. Consider the function $f(x) = x^5 - 5x^4$ defined on the real line. Use the following outline to sketch the graph of f.

   a. Compute the derivative of f. Find the critical points of f and the sign of f' on each subinterval determined by the critical points.
   b. Decide on which intervals f is increasing and on which intervals f is decreasing.
   c. Compute $f''(x)$ and determine the points where $f''(x) = 0$.
   d. Analyze the sign of f'' and the concavity of the graph of f. What are the inflection points for f?
   e. Determine the values of x where the graph of f crosses the x-axis. Observe that these are the values of x for which $f(x) = 0$. Compute $f(0)$.
   f. Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
   g. Use all the information you have gathered to complete the following table and to sketch the graph of f.

x	$-\infty$		0		3		4		∞
$f'(x)$		+	0	−					
f	$-\infty$	↗		↘					
$f''(x)$		−	0	−					
f		∩		∩					

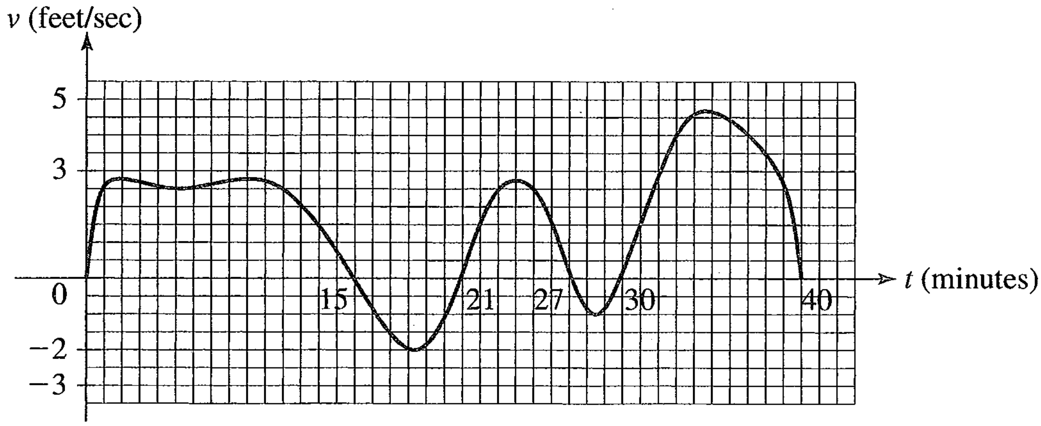
   h. Does this function have an absolute maximum?
   i. Does this function have an absolute minimum?
   j. Consider now the function $g(x) = x^5 - 5x^4$ with domain $[-1, 6]$. Use the graph of f to sketch on another system of coordinates the graph of g. Does g have an absolute maximum or an absolute minimum? ◆

EXERCISES 4.2

In Exercises 1–4, decide if the graph of the function is concave up or concave down on the specified interval.

1. $f(x) = x^2 - x$ on $(0.5, 0.75)$
2. $f(x) = 2x^3 - 4x - 1$ on $(0, 1)$
3. $f(x) = x^4 - 6x^2 + 1{,}000$ on $(-0.5, 0.5)$
4. $f(x) = \frac{1}{3}x^3 - x^2 + x + 1$ on $(4, 5)$
5. Consider an arbitrary quadratic function $f(x) = ax^2 + bx + c$ with a, b, and c fixed real numbers. How many points of inflection does this function have? What is the relationship between the value of a and the concavity of the graph of f?
6. Little Red Riding Hood visits her grandmother every weekend. To get from her house to her grandmother's, she follows a straight path that crosses a forest. One day, it took her 40 minutes to complete the trip from her house to grandmother's house. Twice she had to backtrack—once to find the apple she dropped from her

basket and once to pick a flower to bring to her grandmother. Here is the graph of Little Red Riding Hood's velocity during the trip.

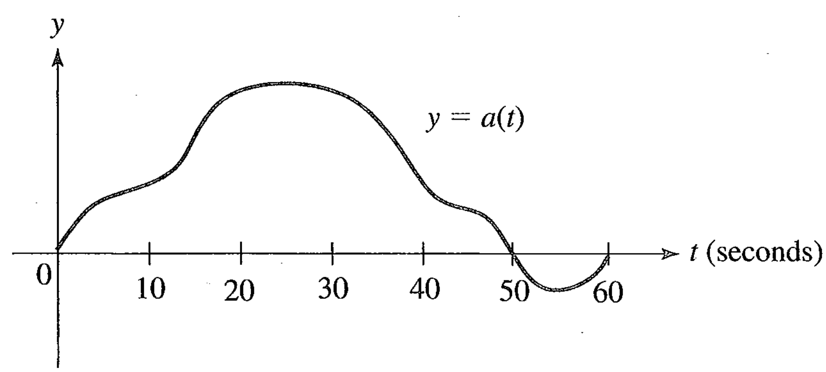


a. Let $d(t)$ denote Little Red Riding Hood's distance (in feet) from her house after t seconds. Determine the intervals of time when d is increasing.
b. At what time t in the interval $[21, 27]$ is Little Red Riding Hood farthest from her house? Explain your reasoning.
c. On which intervals is the graph of d concave up? Explain.

In Exercises 7–18, sketch the graph of the function f. To do so, follow the outline from Problem 3 in Classroom Discussion 4.2.3 and complete a similar table.

7. $f(x) = 4 - x^3$
8. $f(x) = x^3 - 8$
9. $f(x) = x^3 + 3x$
10. $f(x) = x^3 - \frac{1}{3}x + 1$
11. $f(x) = \frac{4}{3}x^3 - 4x$
12. $f(x) = x^3 + x^2$
13. $f(x) = (x - 2)^6 + 1$
14. $f(x) = (x + 1)^5 + 2$
15. $f(x) = 2x^2 - x^4 - 1$
16. $f(x) = \frac{1}{4}x^4 + 2x^2 + 1$
17. $f(x) = x^2(x^2 - 3x + 6)$
18. $f(x) = x^4 + 2x^3 + 6x^2$

19. Steven swims one pool lap in 1 minute. The graph of his acceleration is depicted here.



Let $v(t)$ be Steven's velocity at time t.
a. Is $v(t)$ increasing, decreasing, or neither on the interval $[35, 45]$? Explain.

b. Is the graph of $v(t)$ concave up or concave down on the interval $[15,20]$? Explain.
c. Is the graph of $v(t)$ concave up or concave down on the interval $[30,40]$? Explain.
d. Is $v(40) < v(20)$? Explain.
e. Is $v(55) < v(59)$? Explain.

PROJECTS AND EXTENSIONS 4.2

I. The Graph of $f(x) = x^n$

1. Follow the outline from Problem 3 in Classroom Discussion 4.2.3 and complete a similar table for each of the functions here.

   a. $f(x) = x^3$
   b. $f(x) = x^4$
   c. $f(x) = x^5$
   d. $f(x) = x^6$

2. Use the information from Problem 1 to predict the shape of the graph of $f(x) = x^n$ when n is a whole number.
3. Use calculus to show that your prediction for the graph of $f(x) = x^n$ is correct.

II. Limits at Infinity of Rational Functions

The goal of this project is to investigate the behavior of rational functions as $x \to \infty$ and $x \to -\infty$. The following approach builds on the ideas used in Problem 2 of Classroom Discussion 4.2.3.

1. Consider the rational function $f(x) = \frac{x^2+2x+3}{x^3-3x-1}$. For $x \neq 0$, we can write

$$\frac{x^2 + 2x + 3}{x^3 - 3x - 1} = \frac{x^2\left(1 + 2 \cdot \frac{1}{x} + 3 \cdot \frac{1}{x^2}\right)}{x^3\left(1 - 3 \cdot \frac{1}{x^2} - \frac{1}{x^3}\right)} = \frac{1}{x} \cdot \frac{1 + 2 \cdot \frac{1}{x} + 3 \cdot \frac{1}{x^2}}{1 - 3 \cdot \frac{1}{x^2} - \frac{1}{x^3}}.$$

   Use this expression of $f(x)$ to determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

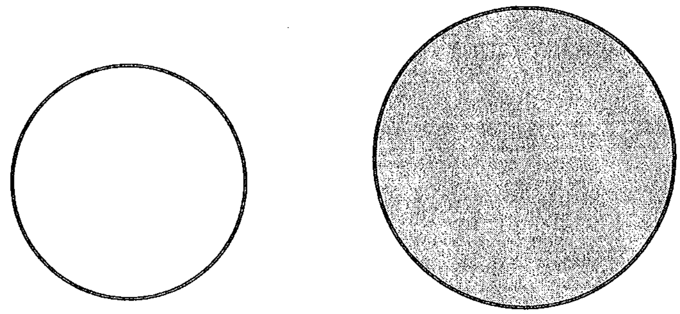
2. Let $g(x) = \frac{x^2+2x+3}{-x^3-3x-1}$. Determine $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$.
3. Let $h(x) = \frac{3x^2-4x-2}{-5x^2+x+1}$. Determine $\lim_{x\to\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$. Explain your reasoning.
4. Let $k(x) = \frac{-x^4-x^3-2}{x-4}$. Determine $\lim_{x\to\infty} k(x)$ and $\lim_{x\to-\infty} k(x)$.
5. Consider now the general case of a rational function $f(x)$; that is, $f(x) = \frac{p(x)}{q(x)}$,

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$
$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

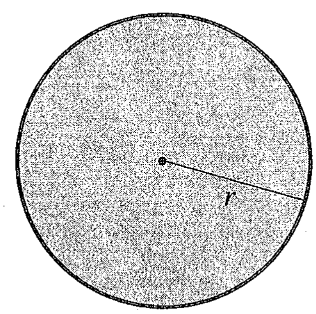
6.4 AREA OF A DISC FROM DIFFERENT POINTS OF VIEW

Computing the area of a disc in middle-school, high school, and college

In your own words, how would you define the terms *circle* and *disc*?

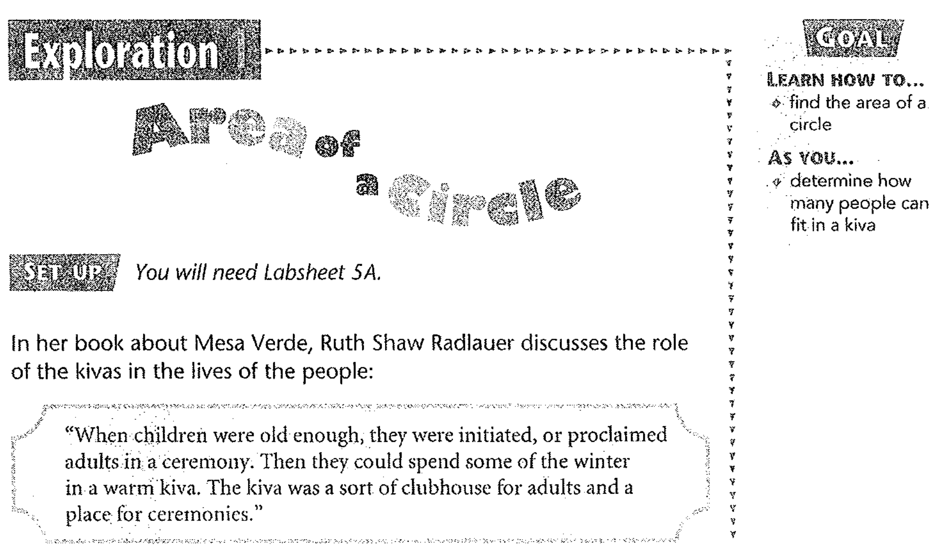


Our goal in this section is to find a formula for the area of a disc of radius $r > 0$. We present four different approaches: one for sixth-grade middle-school students based on elementary estimates, one for seventh-grade middle-school students based on dividing the disc into sectors, one for high school and college students based on approximating the circle by regular n-gons, and one for college students based on the Fundamental Theorem of Calculus.



Classroom Connection 6.4.1: An Elementary Estimate for the Area of a Disc

This exploration is taken from pages 499–500 and page 7–55 in the sixth-grade textbook *Math Thematics, Book 1.* Discuss it in small groups. ◈


Exploration 1

Area of a Circle

SET UP You will need Labsheet 5A.

In her book about Mesa Verde, Ruth Shaw Radlauer discusses the role of the kivas in the lives of the people:

"When children were old enough, they were initiated, or proclaimed adults in a ceremony. Then they could spend some of the winter in a warm kiva. The kiva was a sort of clubhouse for adults and a place for ceremonies."

GOAL

LEARN HOW TO...
- find the area of a circle

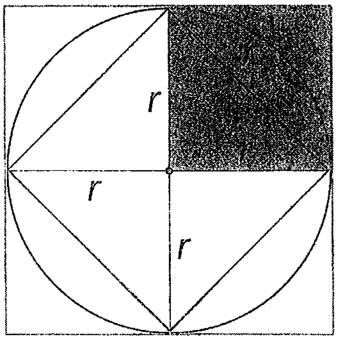
AS YOU...
- determine how many people can fit in a kiva

To find out how many people fit in a kiva, you need to find the area of the floor. The floor of a kiva is shaped like a circle.

3 **Use Labsheet 5A.** Follow the directions for *Estimating the Area of a Circle* by finding the areas of the inner and outer squares.

You can use the method in Question 3 to estimate the area of any circle with radius *r*.

4 **Try This as a Class** The diagram below can help you see the relationship between the area of the circle and *r*.



a. Use the variable *r* to write an expression for the area of a small red square.

b. How many small red squares fit in the green outer square? Use your answer to write an expression for the area of the green outer square.

c. How many small red squares fit in the blue inner square? (*Hint:* Each small red square is made up of two triangles.) Use your answer to write an expression for the area of the blue inner square.

d. Use your answers to parts (b) and (c) to write an expression that can be used to estimate the area of the circle.

5 Use your answer to Question 4(d) to estimate the area of a circle with a radius of 4 cm. How does your estimate compare with the estimate you found in Question 3?

Formula for the Area of a Circle **In Question 4, you found an expression that can be used to estimate the area of a circle. To find the actual area of a circle with radius *r*, multiply pi by *r* to the second power.**

$$A = \pi r^2$$

You can read r^2 as "r squared."

6 How does the formula above compare with the expression you found in Question 4(d)?

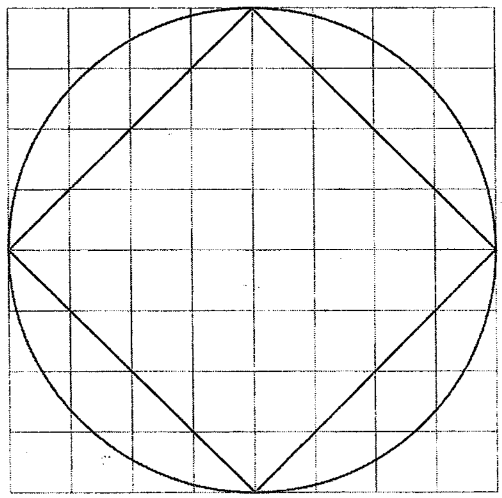
Name ______________________ Date ______________________

MODULE 7 **LABSHEET 5A**

Estimating the Area of a Circle

(Use with Question 3 on page 499.)

Directions Complete parts (a)–(d) to estimate the area of the circle in the figure. Each grid square is 1 cm by 1 cm, or 1 cm^2.



a. Is the area of the circle *greater than*, *less than*, or *equal* to the area of the outer square? the area of the inner square?

b. Find the area of the outer square. Describe the method you used.

c. Find the area of the inner square. Describe the method you used.

d. Use your results from parts (b) and (c) to estimate the area of the circle. Explain how you made your estimate.

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Classroom Connection 6.4.2: Cutting the Disc into Sectors

This exploration is taken from pages 401–403 and page 6-51 in the seventh-grade textbook *Math Thematics*, *Book 2*. Discuss it in small groups. ◈

Exploration 3

AREA OF A CIRCLE

GOAL

LEARN HOW TO...
- find the area of a circle

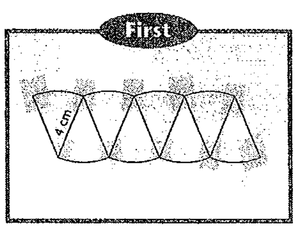
AS YOU...
- investigate kite designs

SET UP *You will need:* • *Labsheet 2A* • *scissors* • *tape* • *ruler*

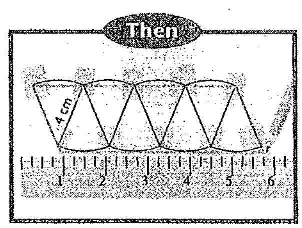
23 **Discussion** A centipede kite is made from circular pieces of silk. How can you estimate the amount of silk in a centipede kite?

In Section 1 you used rectangles to develop a formula for the area of a parallelogram. You can use the same idea to find the area of one of the circles in a centipede kite.

24 **Use Labsheet 2A.** Follow the directions below.



Cut out the *Circle*. Then cut apart the eight sectors and arrange them to form the figure shown. Tape the figure to a sheet of paper.



Use a ruler to draw segments across the top and bottom of your figure. Extend the sides of the figure to meet the bottom segment.

25 What kind of polygon is the new figure you drew in Question 24?

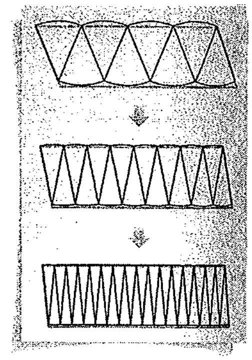
26 **Try This as a Class** Use the figure you made in Question 24.

a. **Estimation** Explain how you could use the new figure you drew to estimate the area of the circle. Then estimate the area. Do you think this is a good estimate? Why or why not?

b. How is the length of the base of the figure related to the circumference of the circle?

c. How is the height of the figure related to the radius of the circle?

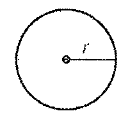
27 **Discussion** Examine the drawings shown.



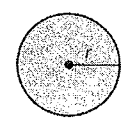
a. As a circle is cut into more and more sectors and put back together as shown, what begins to happen to its shape?

b. The area A of a parallelogram is found by multiplying the length of its base b by the height h, or $A = bh$. Use your figure to explain why this formula can be written as $A = \frac{1}{2}Cr$ to find the area of a circle, where C is the circumference and r is the length of the radius of the circle.

28 The circumference C of a circle is equal to $2\pi r$, where r is the length of the radius. Rewrite the formula $A = \frac{1}{2}Cr$ by substituting $2\pi r$ for C.



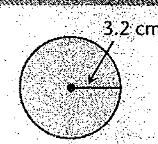
29 The area of a circle with radius r is $A = \pi r^2$. Compare πr^2 with the expression you wrote for Question 28. Do they have the same value? Explain.



▶ **You can use the equation $A = \pi r^2$ to find the area of a circle when you know the length of its radius.**

EXAMPLE

Find the area of a circle with radius 3.2 cm.
Use 3.14 for π to find the approximate area.



SAMPLE RESPONSE

Exact Area	**Approximate Area**
$A = \pi r^2$	$A = \pi r^2$
$= \pi(3.2)^2$	$= \pi(3.2)^2$
$= \pi(10.24)$	$\approx 3.14(10.24)$
$= 10.24\pi$	≈ 32.15
The exact area is 10.24π cm^2.	**The approximate area is 32.2 cm^2.**

This is an approximation because 3.14 is an approximation for π.

For Questions 30–31, use 3.14 or the π key on a calculator.

30 **Try This as a Class** Use your figure from Question 24.

a. Use $A = \pi r^2$ to write an expression that represents the exact area of the *Circle*.

b. Find the approximate area of the *Circle*.

c. If you were planning to make a circle kite with a 4 cm radius, would you use your answer from part (a) or from part (b) to order the material? Why?

d. How does the area of the circle from part (b) compare with the estimated area of the figure in Question 26(a)?

31 ✓ **CHECKPOINT** A centipede kite has 10 in. diameter circles.

a. Find the exact area of one circle.

b. Find the approximate area of one circle.

c. About how many square inches of silk were used to make all 11 circles of the kite?

✓ **QUESTION 31**

...checks that you can find the area of a circle.

HOMEWORK EXERCISES See Exs. 16–23 on p. 407.

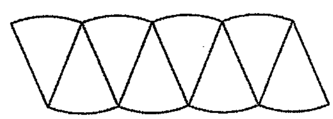
Name ______________________ Date ______________

MODULE 6 **LABSHEET 2A**

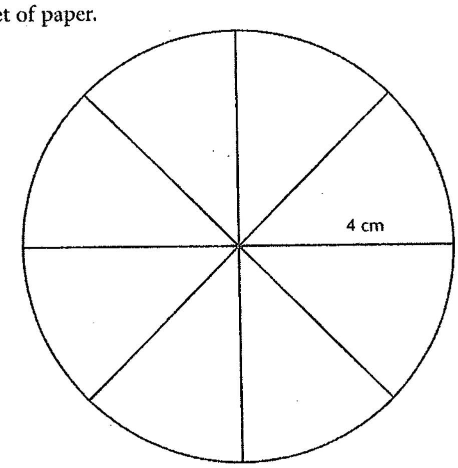
Circle (Use with Question 24 on page 402.)

Directions

- Cut out the circle.
- Cut apart the eight sectors and arrange them to form the figure shown below.



- Tape the figure to a sheet of paper.



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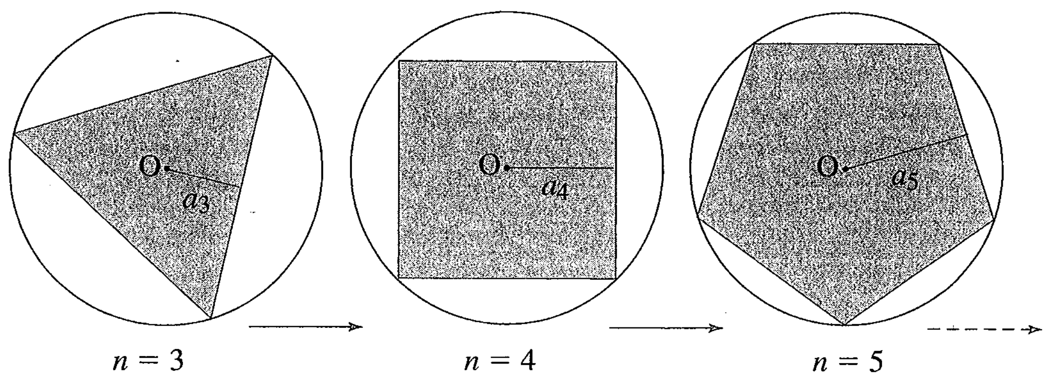
Classroom Discussion 6.4.1: Approximating the Circle by Inscribed Regular n-gons

To find the area of a disc, we use here the *method of exhaustion*, which was invented by Eudoxus (similar ideas may be found in the project "Archimedes's Computation of π" in Section 1.1). To see how the method works, use the following outline.

1. For each $n \geq 3$, let a_n and p_n denote the apothem and the perimeter of an inscribed regular n-gon, respectively. How can you express the area $\mathcal{A}_n$ inside the n-gon in terms of a_n and p_n?



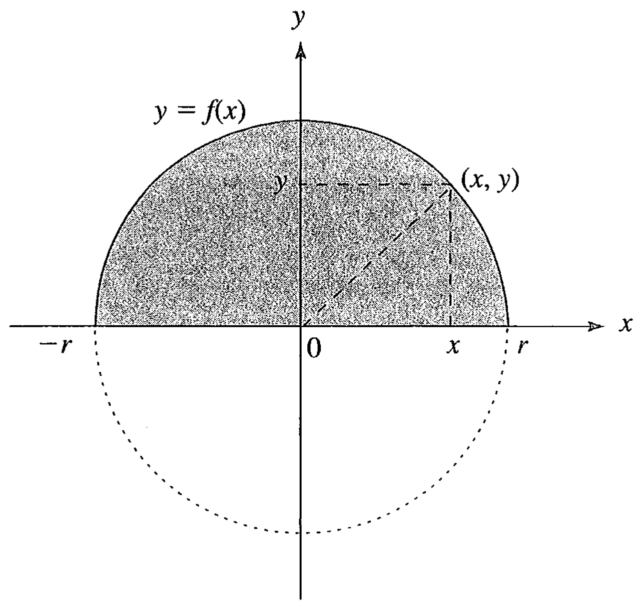
2. What happens to the n-gons as n increases? What happens to the values of a_n, p_n, and $\mathcal{A}_n$ as n increases? What can you say about their limits as $n \longrightarrow \infty$?
3. Using your findings in Problems 1 and 2, find the area $\mathcal{A}$ of the disc.
4. Why do you think this method is called the *method of exhaustion*? ◆

Historical Note: Eudoxus of Cnidus (408–355 BC; from Asia Minor [now Turkey])

Eudoxus, a contemporary of Plato, had a rich and varied academic background in mathematics, music, medicine, astronomy, theology, and meteorology. Early in his career, he developed a theory of *proportion*, which appears in Euclid's *Elements* and facilitated his early work on finding areas. Eudoxus introduced the *method of exhaustion*, which led to important developments in calculus by Archimedes and others; Eudoxus himself was the first to prove that a cone's volume is one-third the volume of a cylinder having the same base and equal height.

Classroom Discussion 6.4.2: Area of a Disc Using the Fundamental Theorem of Calculus

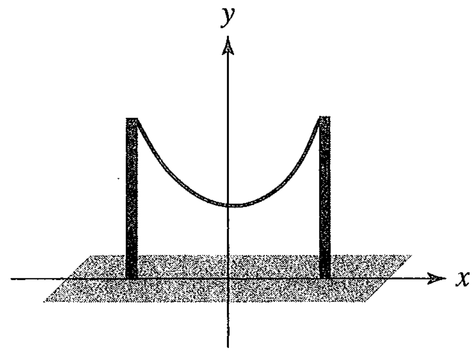
The Fundamental Theorem of Calculus allows us to find the area of a region bounded by the graph of a function and the x-axis. The disc is not such a region, but since it has a reflectional symmetry about the x-axis, its area is twice that of the upper semidisc. The upper semidisc is a region for which the Fundamental Theorem of Calculus applies.



1. Use the Pythagorean Theorem to find a relation between x and y for the point (x, y) to lie on the circle centered at the origin with radius r.
2. What additional constraint must be placed on y for the point (x, y) to lie on the upper semicircle?
3. Solve for y in your equation from Problem 1 to obtain the equation of the upper semicircle.
4. Use the Fundamental Theorem of Calculus to express the disc's area as a definite integral.
5. (T) Compute the area A by using your calculator to evaluate the definite integral.

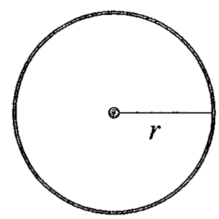
EXERCISES 6.4

1. Write a paragraph about the Fundamental Theorem of Calculus. In particular, write about your understanding of this theorem, and why you think it is (or is not) useful.
2. In Classroom Connection 6.4.1, we used the formula $A_1 = 3r^2$ to estimate the area A of a disc of radius r. In the old Babylonian civilization, the formula $A_2 = \frac{C^2}{12}$ was used for the area A inside a circle of circumference C.
   a. Express A_2 in terms of r using the classical formula for the circumference of a circle.
   b. Express the errors $A_1 - A$ and $A_2 - A$ in terms of r using the formula obtained for A in Section 6.4. Interpret the results.
   c. Evaluate the relative error in each estimate. The *relative error*, when estimating an exact value v with an approximate value v_0, is the quantity $\frac{v_0 - v}{v} = \frac{v_0}{v} - 1$.
   d. Evaluate the percentage error in each estimate. The *percentage error* in an estimate is 100% times the relative error in the estimate.



Classroom Discussion 7.1.5: Circumference of a Circle from Different Points of View

The number π is defined as $\pi = \frac{C}{2r}$, where C is the circumference of any circle and r is its radius. For more on this, see the project "The Number Called π" in Section 1.1. Suppose, however, that you do not know this relation and you want to find the circumference of a circle of radius $r > 0$. We present two different approaches: one for middle-school students based on measurements, and one for college students based on calculus.



1. Classroom Connection 7.1.1: Circumference of a Circle Using Measurements

The following exploration is taken from pages 39–41 in the eighth-grade textbook *Math Thematics*, *Book 3*. Discuss it in small groups. ◈

Section 4 Circumference and Volume

IN THIS SECTION

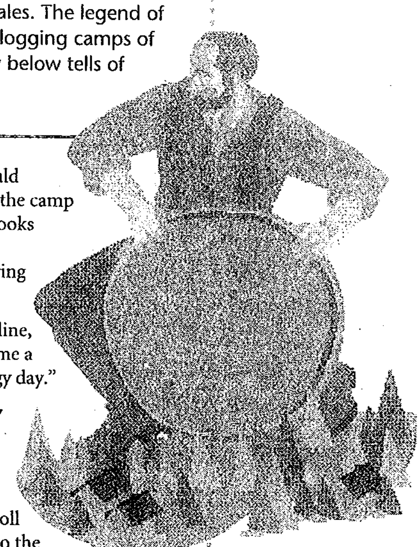
EXPLORATION 1
- Finding Circumference

EXPLORATION 2
- Finding Volume

Amazing Appetites

Setting the Stage

Amazing fiction has a way of wandering into tall tales. The legend of Paul Bunyan, a giant lumberjack, came out of the logging camps of the northern United States in the 1800s. The story below tells of some amazing appetites.



The lumber crews liked pancakes best, but they would gobble up and slurp down the pancakes so fast that the camp cooks couldn't keep up with them, even when the cooks got up twenty-six hours before daylight. The main problem was that the griddles the cooks used for frying the pancakes were too small. . . .

[Paul Bunyan] went down to the plow works at Moline, Illinois, and said, "I want you fellows here to make me a griddle so big I won't be able to see across it on a foggy day."

The men set to work. When they were finished, they had built a griddle so huge there was no train or wagon large enough to carry it.

"Let me think what to do," said Paul. "We'll have to turn the griddle up on end, like a silver dollar, and roll it up to Michigan." He hitched [Babe] the Blue Ox to the upturned griddle, and away they went. . . . A few miles from the Big Onion lumber camp, Paul unhitched Babe and let the griddle roll on by itself.

***American Tall Tales*, Adrien Stoutenberg**

Think About It

1. What amazing "facts" in the story let you know that it is a tall tale?
2. Paul Bunyan's pancake griddle was circular. According to one version of the story, its diameter was 236 ft.
   a. Sketch a circle and draw a diameter. Is this the only diameter the circle has? Explain.
   b. Would Paul Bunyan's griddle fit in any room in your school?

GOAL

LEARN HOW TO...
- find circumference
- write and evaluate expressions

AS YOU...
- investigate a claim made in a tall tale

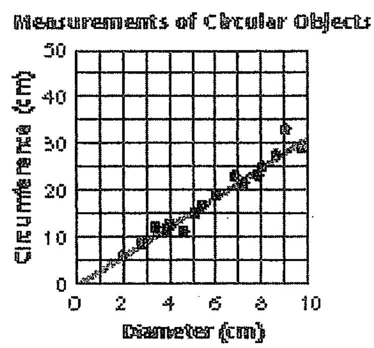
KEY TERMS
- circumference
- variable
- expression
- evaluate

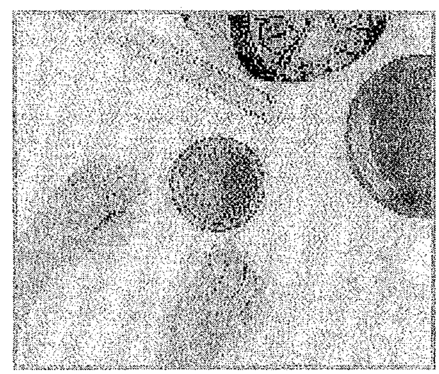
Exploration

Finding Circumference

SET UP *You will need:* • *metric ruler* • *coin* • *calculator*

One version of the story says that 3 miles from camp Paul gave the griddle a push and it rolled to the spot where he wanted it. Is this an amazing feat? One measurement that can help you answer this question is the griddle's circumference. The circumference of a circle is the distance around it.





3 Some students measured the diameter and the circumference of several circular objects. Then they made a scatter plot and drew a fitted line. Does the line seem to fit the data? Explain.

4 a. Copy the table. Then use the fitted line on page 40 to estimate the circumferences of circles with the given diameters.

b. Calculate the ratio of the *circumference* to the *diameter*. Round to the nearest tenth.

Diameter (cm)	Circumference (cm)	$\frac{\text{Circumference}}{\text{diameter}}$
11	?	?
1	?	?
8	?	?
14	?	?
4.5	?	?

c. **Discussion** The circumference of a circle appears to be about how many times its diameter?

5 Estimate the circumference of Paul Bunyan's pancake griddle. Explain how you made your estimate.

▶ **Using Variables In mathematics, symbols are often used instead of words to express relationships. A variable is a symbol used to represent a quantity that is unknown or that can change.**

EXAMPLE

Words: The **circumference** of a circle is about 3 times the **diameter**.

Symbols: $C \approx 3 \cdot d$

$C \approx 3d$

The symbol ≈ means approximately equal to.

Multiplication with variables can be written without a multiplication symbol.

▶ **C and $3d$ are *expressions*. An expression is a mathematical phrase that can contain numbers, variables, and operation symbols. Other examples of expressions are 8, lw, $5 + n$, $9x^2$, and $\frac{a}{2}$.**

6 **Discussion** The exact relationship between the circumference and the diameter of a circle is given below. Read π as *pi*.

$$\frac{C}{d} = \pi, \text{ or } C = \pi d$$

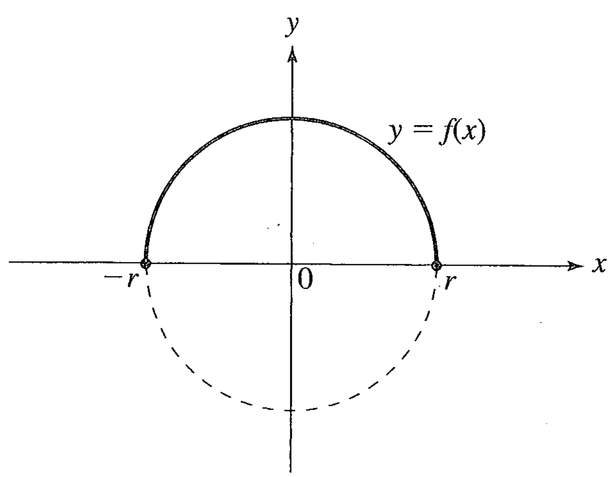
a. Calculator Press the [π] key on a calculator. What number appears?

b. π is actually a letter from the Greek alphabet. Does this mean that it is a variable? Explain.

c. Which is a closer approximation for π, 3.14 or $\frac{22}{7}$? Explain.

2. Circumference of a Circle Using Calculus

Notice that since the circle has reflectional symmetry about the x-axis, its circumference is twice the length of the upper semicircle. Because the upper semicircle is the graph of a function, Theorem 7.1.1 applies.



a. Under what conditions does a point (x, y) lie on the upper semicircle?

b. What is the equation of the upper semicircle?

c. Find the circumference $\mathcal{L}$ of the circle using Theorem 7.1.1. To answer this question, follow these steps:

   i. Evaluate the derivative $y'(x)$ using the square-root rule.

   ii. Evaluate and simplify the expression of $\sqrt{1 + [y'(x)]^2}$ as much as possible.

   iii. Use Theorem 7.1.1 to compute $\mathcal{L}$. You may use your calculator to integrate if needed. ◆

EXERCISES 7.1

In Exercises 1–4, do the following tasks: (a) Sketch the graph $y = f(x)$. (b) Find approximate values $\mathcal{L}_n$ for the length of the given graph using approximations of the graph by polygonal paths with n line segments for $n = 2, 4, \ldots, 20$ as in Classroom Discussion 7.1.3. Tabulate your results. (c) What is the value in your table that best approximates the length $\mathcal{L}$? Explain. (d) Use integral calculus to compute the length $\mathcal{L}$ of the given graph. You may use your calculator to compute the definite integral corresponding to the length $\mathcal{L}$, but you must first evaluate the integrand by hand.

1. $f(x) = x^3, \quad 0 \le x \le 1$
2. $f(x) = 1/x, \ 1 \le x \le 3$
3. $f(x) = \sqrt{x}, \ 0 \le x \le 4$
4. $f(x) = e^x, \ -1 \le x \le 1$

In Exercises 5–10, sketch the graph $y = f(x)$ where $a \le x \le b$, then write down a formula for its length. Do not compute the integral but simplify the integrand as much as possible.

5. $f(x) = 1 - x^2, \ a = 0, \ b = 1$
6. $f(x) = x^3 - x, \ a = -5, \ b = 2$
7. $f(x) = 2x^2 + x, \ a = -3, \ b = 3$
8. $f(x) = x^2 e^x, \ a = -1, \ b = 2$
9. $f(x) = x(x^3 - 4), \ a = 1, \ b = 2$
10. $f(x) = x/(x + 1), \ a = 0, \ b = 3$

In Exercises 11–14, find a function whose graph has length equal to the given definite integral.

11. $\int_0^2 \sqrt{1 + x^2}\, dx$

12. $\int_0^{\pi} \sqrt{1 + e^{2x}}\, dx$

13. $\int_0^5 \sqrt{x^2 + 2x + 2}\, dx$

14. $\int_1^4 x^{-2}\sqrt{x^4 + 1}\, dx$

15. If the graph $y = f(x)$ has length $\mathcal{L}$, what is the length of the graph $y = f(x) + \alpha$? Here, $a \le x \le b$ and α is a constant that is given. Answer this question by giving two different arguments, one based on geometry only and the other on Theorem 7.1.1.

16. A rope will be hung between two poles of the same height and 50 meters apart. The rope takes the shape of the catenary $y = 25/2(e^{x/25} + e^{-x/25})$, $-25 \le x \le 25$. Find the length of this rope.

17. Sketch the graph $y = \frac{1}{3}(x^2 + 2)^{3/2}$ for $0 \le x \le 2$, then compute its length by hand. You may use your calculator only to evaluate derivatives or to check your final answer.

18. Sketch the curve given by the equation $x^{2/3} + y^{2/3} = 1$, then compute its length by hand. You may use your calculator only to evaluate derivatives or to check your final answer.

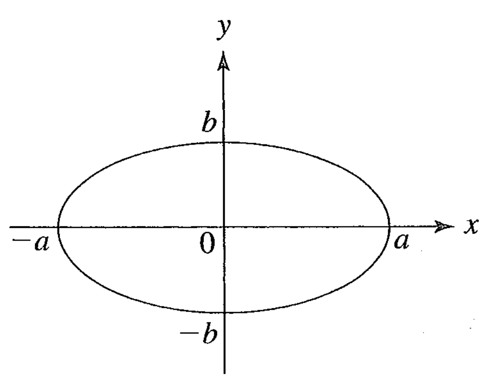
19. Circumference of an Ellipse

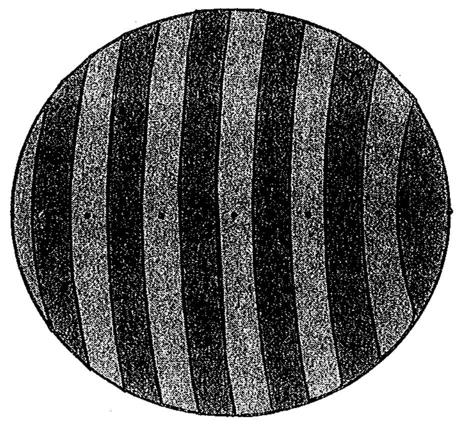
a. Recall the definition and the Cartesian equation of an ellipse.

b. Can you adapt the method used in Classroom Connection 7.1.1 to the case of an ellipse?

c. Use Theorem 7.1.1 to represent the circumference of an ellipse as an integral.

d. Can you evaluate the integral in Problem c by hand? By using a calculator? Compare with evaluating the integral in the case of a circle.





a. Do the twelve spherical bands each have the same area? Justify your answer.
b. Does your conclusion hold true if you replace twelve by an arbitrary whole number? Explain.

7.3 VOLUMES OF SOLIDS

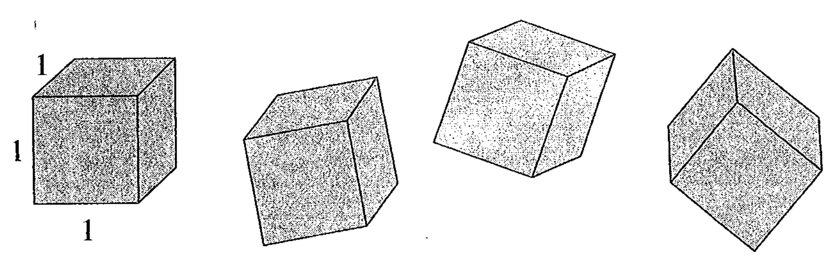
Volumes of polyhedrons and cylinders ∘ Volumes of solids ∘ Computing volumes of cones in middle-school and college ∘ Computing volumes of pyramids in middle-school and college ∘ Computing volumes of spheres in middle-school, high school, and college

You have certainly heard and used the word *volume* on many occasions. Can you think of some real-life examples involving volumes? Using your own words, can you define the meaning of *volume*?

Our objective in this section is to develop an approach for measuring the volume of a solid, and to find an explicit formula for computing it. We use this formula to evaluate the volumes of cones, pyramids, and spheres.

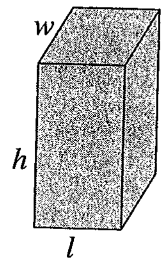
Classroom Discussion 7.3.1: Volumes of Polyhedrons and Cylinders

Let us start with the simplest solid: a unit cube; that is, a cube whose length, height, and width are all equal to 1 unit. We define its volume to be 1 *cubic unit*. This definition should make sense to you since all unit cubes occupy the same amount of space.

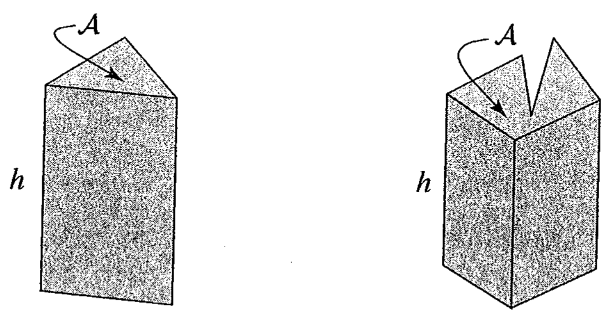


Thus, from now on, 1 cubic unit is the volume of any unit cube. Based only on this definition, explain how to compute the volumes of the solids described in Problems 1–5. At each step, you can use any results that you have proved earlier. Work in small groups.

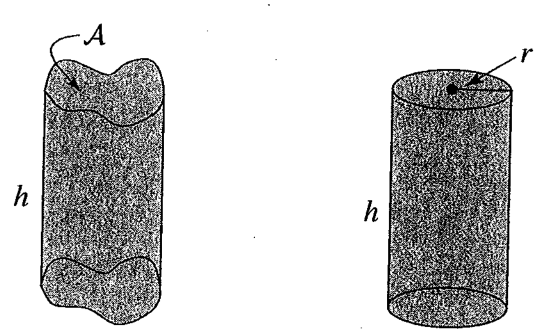
1. A right rectangular prism with length ℓ, width w, and height h, where ℓ, w, and h are whole numbers.
2. A right rectangular prism with length ℓ, width w, and height h, where ℓ, w, or h is not a whole number; they are all rational numbers.
3. A right rectangular prism with length ℓ, width w, and height h, where ℓ, w, or h is an irrational number.



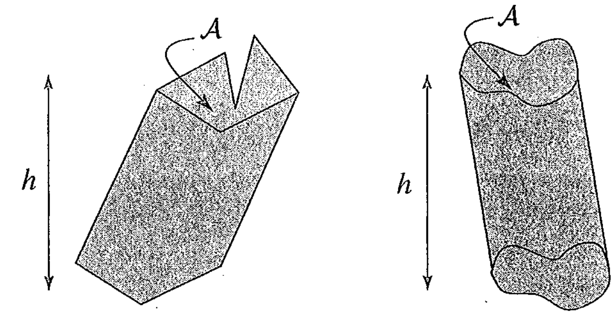
4. A right triangular prism whose base has an area $\mathcal{A}$ and whose height is h. Then, deduce a formula for the volume of an arbitrary right prism whose base has an area $\mathcal{A}$ and whose height is h.



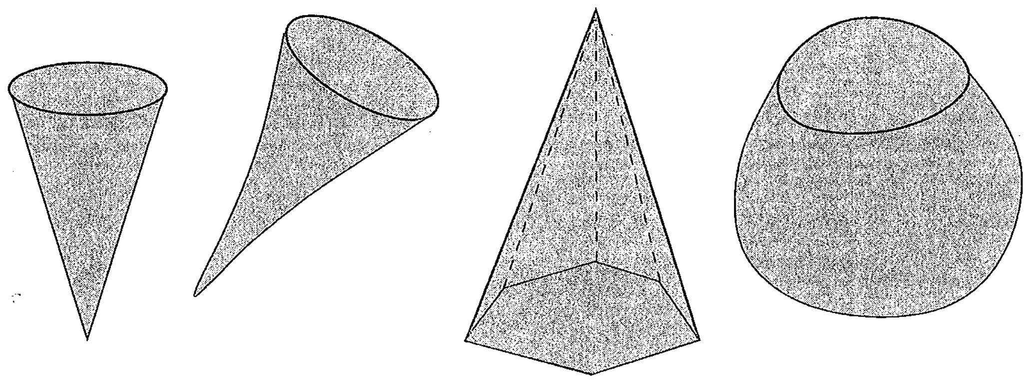
5. A generalized right cylinder whose height is h and whose base has an area $\mathcal{A}$. Then, deduce the volume of a right cylinder with radius r and height h.



6. What would your answers be for Problems 4 and 5 if you replace the word *right* with the word *oblique* in each case? Hint: Imagine that each solid is a stack of sheets of paper.



7. Can you find the volume of the following solids using the same techniques as in Problems 1–6?



8. A cylindrical drinking cup has a height of 12 cm and a radius of 4 cm. The cup is filled with water, then the water is poured into an empty container, filling it completely. What is the volume of the container? What conclusion can you draw from this? How practical is this procedure for measuring the volume of arbitrary solids? ◈

Since solids have, in general, arbitrary shapes, an obvious question is: How can the volume of a solid with an arbitrary shape be measured?

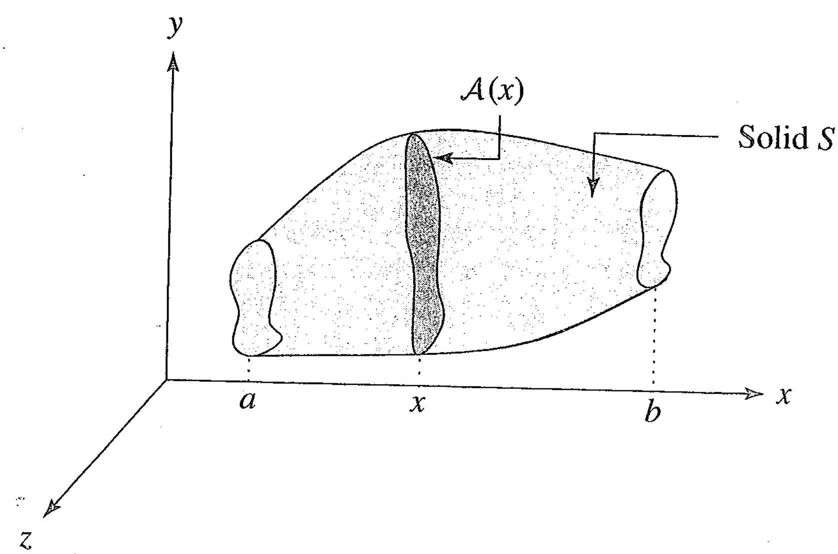
One way to answer this question is to approximate the solid by shapes whose volumes are known, such as prisms, cylinders, etc. This leads to an approximation for the volume of the solid of interest. The smaller the gap between the given solid and the approximating shapes, the better the estimate will be for its volume. Thus, if the approximating shapes are chosen to approach the shape of the given solid, then their volumes will approach its volume. The exact value of the volume can then be obtained by passing to the limit. Note the analogy between this strategy and the one that uses Riemann sums to compute the areas of irregular shapes.

Another way to answer the question is to try to find the exact volume of a given solid using techniques similar to those employed in the proof of the Fundamental Theorem of Calculus. In the following Classroom Discussion, we describe this second approach in full detail.

Classroom Discussion 7.3.2: Formula for the Volumes of Solids

This Classroom Discussion's goal is to find, if possible, a formula for computing the volume $\mathcal{V}$ of the following solid S.

Fix $a \leq x \leq b$. The **cross-section** of the solid S corresponding to x is the plane region that contains all the points in S whose first coordinates are equal to x.

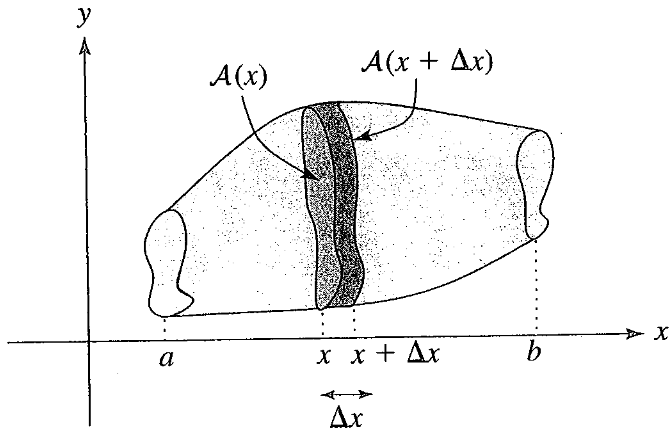


1. Describe the cross-sections of the solids listed here. In each case, consider different positions of the solid with respect to the x-axis.

   a. A rectangular prism
   b. A cylinder
   c. A cone
   d. A pentagonal pyramid
   e. A sphere
   f. A hemisphere

Now, return to the solid S. Let $\mathcal{A}(x)$ be the area of the cross-section corresponding to x, and let $\mathcal{V}(x)$ be the volume-so-far function; that is, $\mathcal{V}(x)$ is the volume of the portion of the solid S that lies between the cross-sections corresponding to a and x. To familiarize yourself with the function $\mathcal{V}(x)$, answer the two following questions.

2. What are the values of $\mathcal{V}(a)$, $\mathcal{V}(\frac{2a+b}{3})$, $\mathcal{V}(\frac{a+b}{2})$, and $\mathcal{V}(b)$?
3. Is the function $\mathcal{V}(x)$ increasing, decreasing, or neither? Explain.



Let x increase by a small amount Δx; then the volume $\mathcal{V}(x)$ increases by an amount $\Delta\mathcal{V}$ corresponding to the additional portion of the solid S.

4. If Δx is very small, how might you view this additional portion?
5. Express the corresponding approximation for $\Delta\mathcal{V}$ in terms of $\mathcal{A}(x)$ and Δx.
6. How would you approximate the rate of change $\dfrac{\Delta\mathcal{V}}{\Delta x}$?
7. Show that the derivative of $\mathcal{V}$ satisfies $\mathcal{V}'(x) = \mathcal{A}(x)$.
8. Assuming that the function $\mathcal{A}(x)$ has an antiderivative on the interval $[a, b]$, express $\mathcal{V}(x)$ as a definite integral.
9. Deduce a formula for the overall volume $\mathcal{V}$. ◆

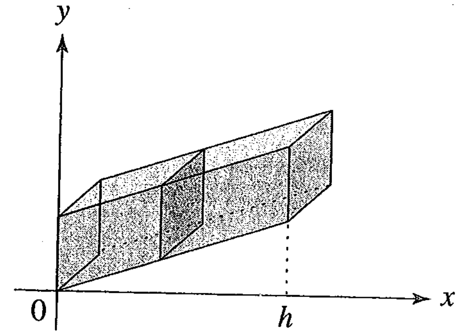
Theorem 7.3.1. *For each fixed $a \leq x \leq b$, let $\mathcal{A}(x)$ denote the area of the cross-section corresponding to x of a given solid. If the function $\mathcal{A}(x)$ has an antiderivative on the interval $[a, b]$, then the solid's volume is given by the formula*

$$\mathcal{V} = \int_a^b \mathcal{A}(x)\,dx.$$

Historical Note: Johannes Kepler (1571–1630; from the Holy Roman Empire [now Germany])

Kepler is chiefly remembered for his work on astronomy: the first cosmological model of the solar system; his three laws of planetary motion; and his calculation of astronomical tables with extreme precision, which helped to establish the truth of heliocentric astronomy. In mathematics, Kepler studied close packing of spheres, and he gave the first proof of how logarithms work. Noticing the crude way in which volumes of wine casks were measured, Kepler was inspired to study volumes of solids of revolution, using ideas of Archimedes; the resulting treatise was later developed by Cavalieri and is part of the ancestry of infinitesimal calculus.

EXAMPLE Using calculus, find the volume of a prism whose height is h and whose base has area A; see the following figure. Then, compare your result with the formula obtained in Classroom Discussion 7.3.1.



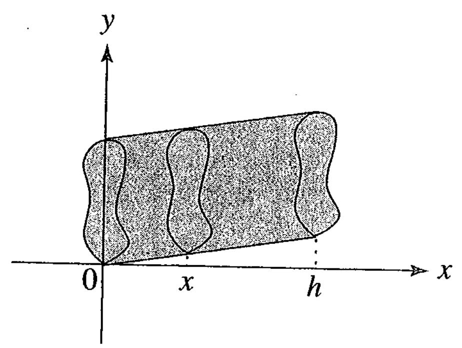
Solution The prism's cross-sections are polygons that are congruent to the base, so for each $0 \leq x \leq h$, the cross section corresponding to x has area $\mathcal{A}(x) = A$. The prism's volume is given by Theorem 7.3.1:

$$\mathcal{V} = \int_0^h \mathcal{A}(x)\,dx = \int_0^h A\,dx = A\int_0^h 1\,dx = A\,x\big|_0^h = A(h - 0) = Ah.$$

This formula is the same as the one obtained earlier. ■

Practice Problem

Using calculus, find the volume of a generalized cylinder whose base has area A. Compare your result with the formula obtained in Classroom Discussion 7.3.1.



Classroom Discussion 7.3.3: Volumes of Cones

The goal of this Classroom Discussion is to find the volume of a cone with height $h > 0$ and radius $r > 0$. We present two different approaches: one for middle-school students based on comparing the volumes of cones and cylinders and one for college students based on calculus.

1. Classroom Connection 7.3.1: Comparing the Volumes of Cones and Cylinders

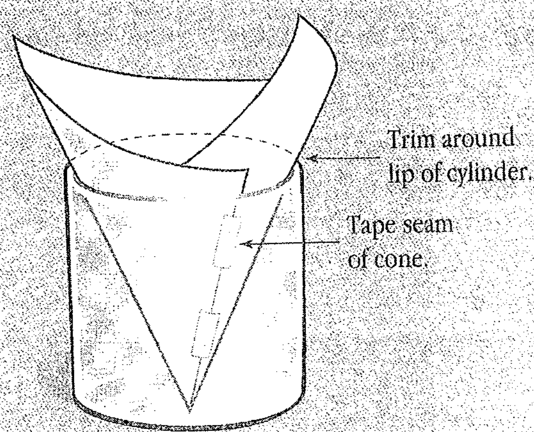
The following exploration is taken from pages 49–50 in the seventh-grade textbook *Connected Mathematics, Filling and Wrapping*. Discuss it in small groups. ◆

5.2 Comparing Cones and Cylinders

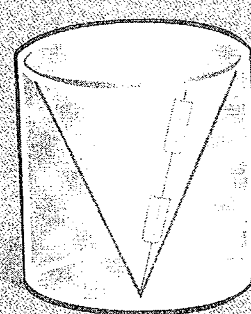
In the last problem, you discovered the relationship between the volume of a sphere and the volume of a cylinder. In this problem, you will look for a relationship between the volume of a cone and the volume of a cylinder.

Problem 5.2

- Roll a piece of stiff paper into a cone shape so that the tip touches the bottom of your cylinder.



- Tape the cone shape along the seam and trim it to form a cone with the same height as the cylinder.

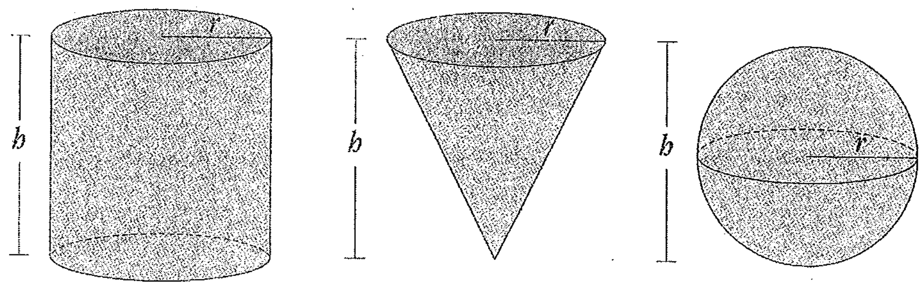


- Fill the cone to the top with sand or rice, and empty the contents into the cylinder. Repeat this as many times as needed to completely fill the cylinder.

What is the relationship between the volume of the cone and the volume of the cylinder?

Problem 5.2 Follow-Up

If a cone, a cylinder, and a sphere have the same radius and the same height, what is the relationship between the volumes of the three shapes?

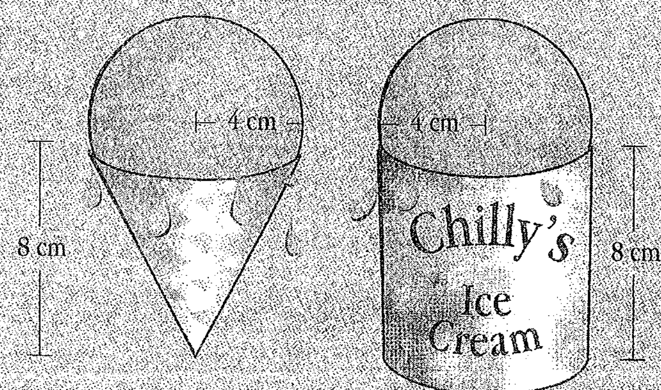


5.3 Melting Ice Cream

Olga and Serge buy ice cream from Chilly's Ice Cream Parlor. They think about buying an ice cream cone to bring back to Olga's little sister but decide the ice cream would melt before they got back home. Serge wonders, "If the ice cream all melts into the cone, will it fill the cone?"

Problem 5.3

Olga gets a scoop of ice cream in a cone, and Serge gets a scoop in a cylindrical cup. Each container has a height of 8 centimeters and a radius of 4 centimeters, and each scoop of ice cream is a sphere with a radius of 4 centimeters.



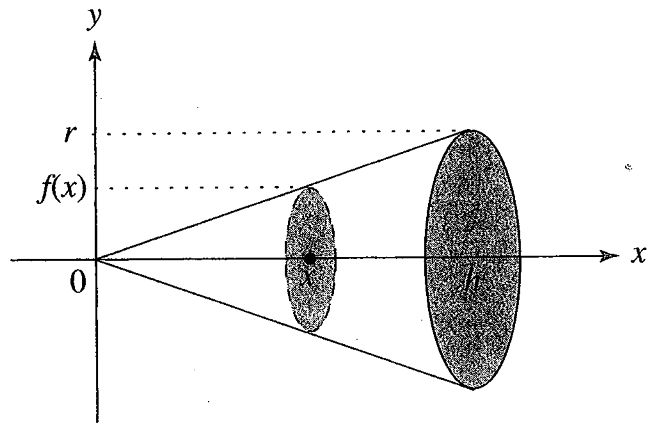
A. If Serge allows his ice cream to melt, will it fill his cup exactly? Explain.

B. If Olga allows her ice cream to melt, will it fill her cone exactly? Explain.

Problem 5.3 Follow-Up

How many scoops of ice cream of the size above can be packed into each container?

2. Volumes of Cones Using Calculus



a. Explain why, when computing a cone's volume, the cone can be assumed to be right without loss of generality.

b. Describe the cross-sections of the cone in the preceding figure and compute their areas.

c. Use Theorem 7.3.1 to determine the volume of the cone. Compare this value with the volume of a cylinder that has the same radius and height as the cone. ◆

Classroom Discussion 7.3.4: Volumes of Pyramids

The goal is to find the volume of a pyramid whose height is $h > 0$ and whose base has an area A_0. We present three different approaches: one for seventh-grade students based on combining pyramids to form prisms, one for eighth-grade students based on comparing the volumes of pyramids and prisms, and one for college students based on calculus.

1. Classroom Connection 7.3.2: Combining Pyramids to Form Prisms

The following exploration is taken from pages 544–545 and page 8-38 in the seventh-grade textbook *Math Thematics, Book 2*. Discuss it in small groups. ◆

GOAL

LEARN HOW TO...
- recognize a pyramid
- find the volume of a pyramid

AS YOU...
- build rectangular prisms using pyramid blocks

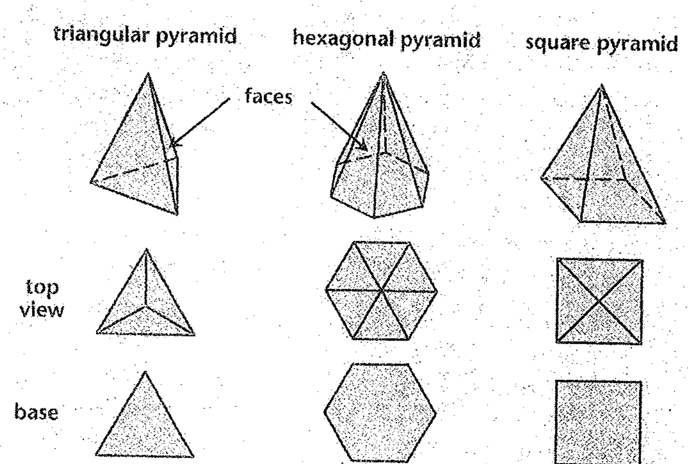
KEY TERM
- pyramid

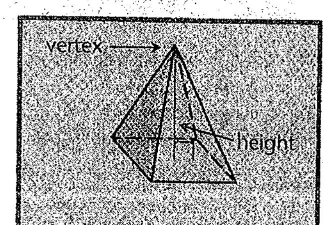
Exploration 2

VOLUME OF A PYRAMID

SET UP *You will need:* • *Labsheet 2A* • *scissors* • *tape* • *metric ruler*

A city skyline usually contains many different types of space figures. To organize the space of a building efficiently, architects need to consider the amount of space or volume a building will enclose. In this exploration you'll see how the volume of a pyramid relates to the volume of a rectangular prism. Several types of pyramids are shown below.



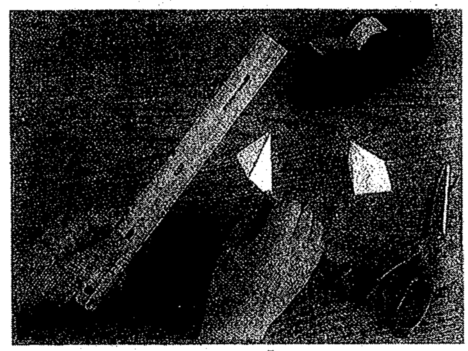


The *height* is the perpendicular distance from the base to the vertex opposite it.

9 How are the three pyramids above alike? How are they different?

10 **Use Labsheet 2A.** Cut out the *Pyramid Nets.* Crease each net along the dashed lines. Then fold each net and tape the edges together to form two pyramids.

For Questions 11–17, use the pyramids you made in Question 10.



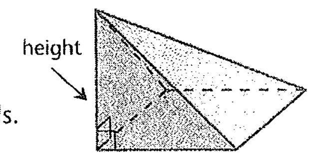
11 The two pyramids you made are identical. Look at one of the pyramids.

a. What type of pyramid did you form?

b. Not including the base, how many faces does the pyramid have?

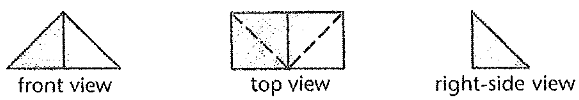
c. Which faces of the pyramid are congruent? Explain.

12 Measure the height of one of your pyramids to the nearest centimeter.



13 The dashed lines below represent folds.

a. Use your two pyramids to make a figure with the views shown.



b. Use your two pyramids to make a figure with the front and back views shown. Then draw the top view of the figure.

14 Work with a partner.

a. Make a cube by putting three of your pyramids together.

b. Write an expression to represent the volume of the cube. Use B for the area of the base of the cube and h for the height of the cube.

c. Compare the area of the base and the height of a single pyramid with the area of the base and the height of the cube you formed.

d. Suppose you know the volume of a cube formed by putting together three identical pyramids as in part (a). How can you find the volume of each pyramid?

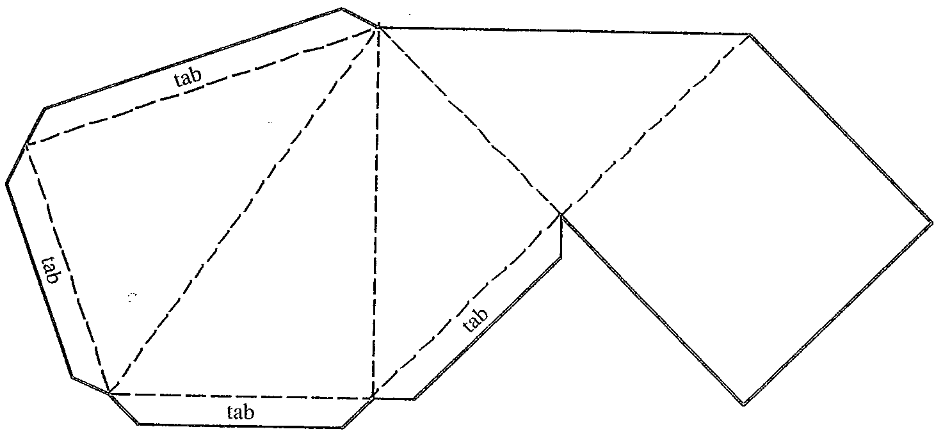
e. Use your answers to parts (b)–(d) to write an expression for the volume of a pyramid with height h and base area B.

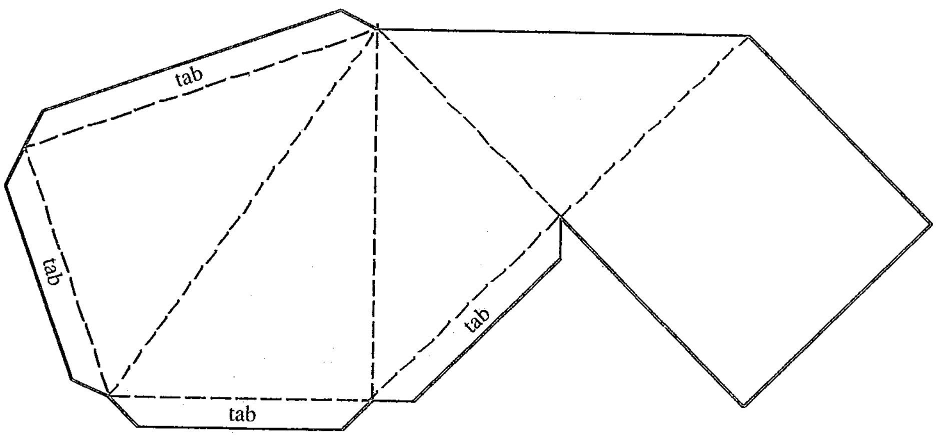
Name ______________________________ Date __________

MODULE 8 **LABSHEET 2A**

Pyramid Nets (Use with Question 10 on page 544.)

Directions Cut out each pyramid net. Crease each net along the dashed lines. Then fold each net, tuck in the tabs, and tape the edges together to form two pyramids.





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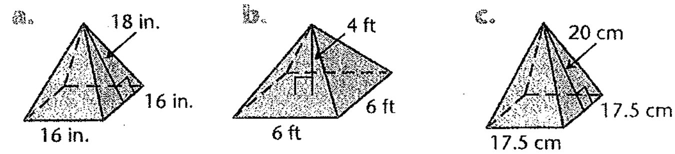
2. Classroom Connection 7.3.3: Comparing the Volumes of Pyramids and Prisms

The following exploration is taken from pages 422–423 and page 6-55 in the eighth-grade textbook *Math Thematics*, *Book 3*. Discuss it in small groups. ◆

✓ QUESTION 10

...checks that you can find the surface area of a pyramid.

10 ✓ CHECKPOINT Find the surface area of each regular pyramid.



HOMEWORK EXERCISES ▶ See Exs. 1–8 on p. 427.

GOAL

LEARN HOW TO...

- find volumes of prisms, pyramids, and cones

AS YOU...

- look at models of block pyramids and prisms

KEY TERM

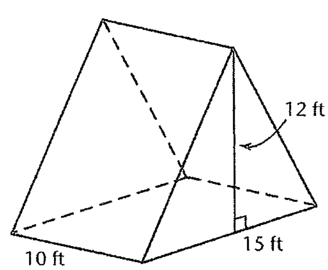
- cone

Exploration 2

Volumes of PRISMS, PYRAMIDS, and CONES

SET UP *You will need Labsheet 3B.*

Winter *mat houses* were once used by people living in the Plateau region of the northwestern United States. These homes were usually occupied from mid-October to mid-March. A mat house was built roughly in the shape of a triangular prism.

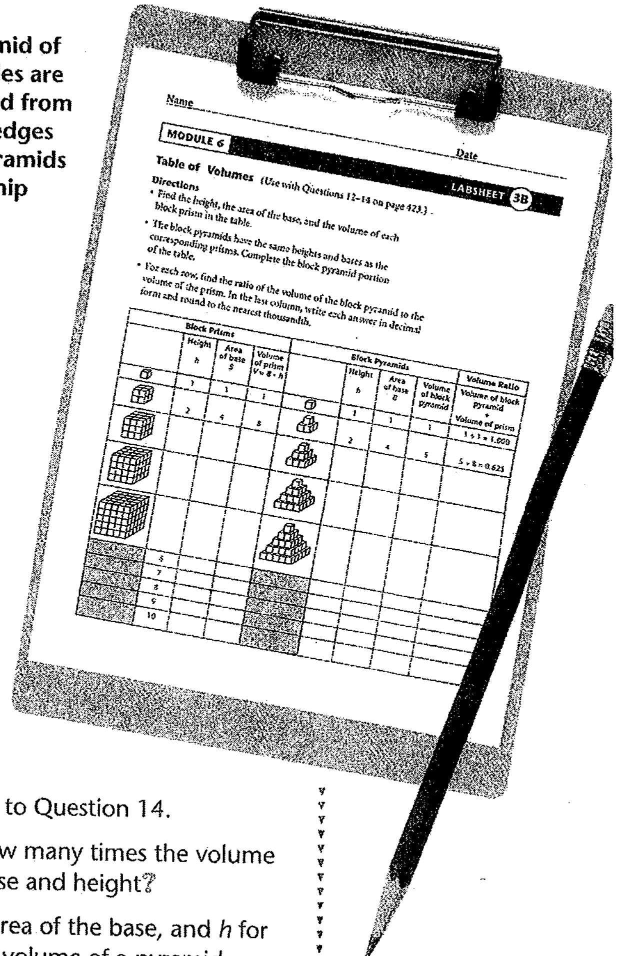


11 In Module 1 you learned how to use the formula

Volume = area of base × height, or $V = Bh$

to find the volume of a cylinder or a rectangular prism. This formula can also be used to find volumes of prisms with other bases. How can you use this formula to find the volume of the triangular prism above?

▶ **Volume of a Pyramid** The Great Pyramid of Giza is made from blocks of stone. The sides are jagged, but they look smooth when viewed from afar, as if the block pyramid had straight edges and flat faces. Now you'll look at block pyramids and block prisms to discover the relationship between the volume of a pyramid and the volume of a prism with the same base and height.



Use Labsheet 3B for Questions 12–14.

12 Follow the directions on the labsheet to complete the *Table of Volumes.*

13 Look at the volume ratio column of the table. What patterns do you notice in the values?

14 **Discussion** If the last column of the table were continued, the ratio for the 50th entry would be 0.343 and the 100th entry would be 0.338. What "nice" fraction do the ratios appear to be approaching?

15 **Try This as a Class** Use your answer to Question 14.

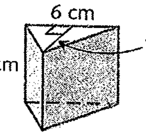
a. The volume of a prism is about how many times the volume of a pyramid that has the same base and height?

b. Using V for the volume, B for the area of the base, and h for the height, write a formula for the volume of a pyramid.

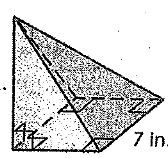
c. Use your formula from part (b) to find the volume of a pyramid that has the same base and height as the triangular prism on page 422.

16 ✓ **CHECKPOINT** Find the volume of each space figure. Round decimal answers to the nearest tenth.

a.



b.



✓ **QUESTION 16**

...checks that you can find volumes of prisms and pyramids.

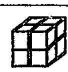
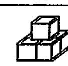
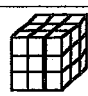
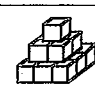
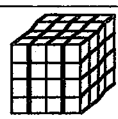
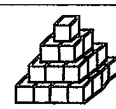
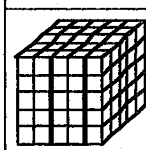
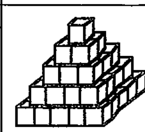
Name ______________________________ Date ____________

MODULE 6 **LABSHEET 3B**

Table of Volumes (Use with Questions 12–14 on page 423.)

Directions

- Find the height, the area of the base, and the volume of each block prism in the table.
- The block pyramids have the same heights and bases as the corresponding prisms. Complete the block pyramid portion of the table.
- For each row, find the ratio of the volume of the block pyramid to the volume of the prism. In the last column, write each answer in decimal form and round to the nearest thousandth.

Block Prisms				**Block Pyramids**				**Volume Ratio**
	Height h	Area of base B	Volume of prism $V = B + h$		Height h	Area of base B	Volume of block pyramid	Volume of block pyramid ÷ Volume of prism
	1	1	1		1	1	1	1 ÷ 1 = 1.000
	2	4	8		2	4	5	5 ÷ 8 = 0.625
								
								
								
	6							
	7							
	8							
	9							
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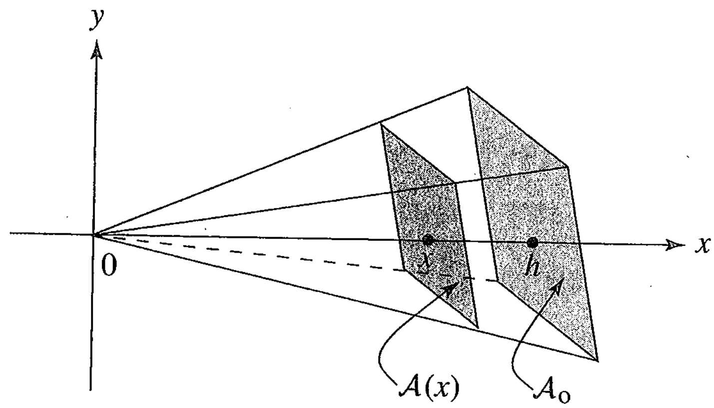
3. Volumes of Pyramids Using Calculus



a. Explain why, when computing a pyramid's volume, the pyramid can be assumed to be right without loss of generality.

b. Using similarity, express the area $\mathcal{A}(x)$ of the cross-section corresponding to x in terms of x, the height h, and the base's area $\mathcal{A}_0$.

c. Use Theorem 7.3.1 to determine the volume of the given pyramid. Compare this value with the volume of a prism that has the same base area and height as the pyramid. ◈

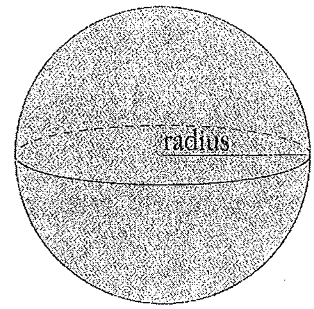
Classroom Discussion 7.3.5: Volumes of Spheres

The goal is to find the volume of a sphere of radius $r > 0$. We present three different approaches: one for middle-school students based on comparing the volumes of spheres and cylinders, one for high school students based on approximating the sphere with tiny "pyramids," and one for college students based on calculus.

1. Classroom Connection 7.3.4: Volumes of Spheres and Cylinders

The following exploration is taken from pages 47–48 in the seventh-grade textbook *Connected Mathematics, Filling and Wrapping*. Discuss it in small groups. ◈

Although spheres may differ in size, they are all the same shape. We can describe a sphere by giving its radius.



In this investigation, you will explore ways to determine the volume of cones and spheres.

5.1 Comparing Spheres and Cylinders

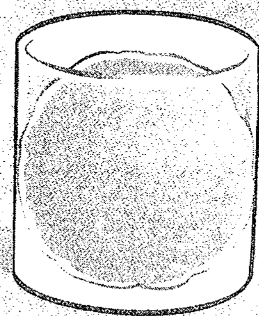
In this problem, you will make a sphere and a cylinder with the same radius and height and then compare their volumes. (The "height" of a sphere is just its diameter.) You can use the relationship you observe to help you develop a method for finding the volume of a sphere.

Did you know?

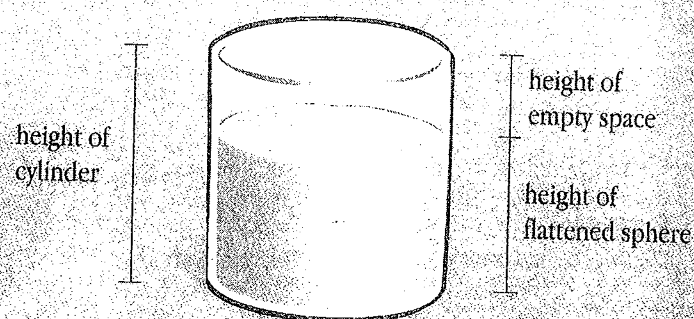
The Earth is nearly a sphere. You may have heard that, until Christopher Columbus's voyage in 1492, most people believed the Earth was flat. Actually, as early as the fourth century B.C., scientists in Greece and Egypt had figured out that the Earth was round. They observed the shadow of the Earth as it passed across the Moon during a lunar eclipse. It was clear that the shadow was round. Combining this observation with evidence gathered from observing constellations, these scientists concluded that the Earth was indeed spherical. In fact, in the third century B.C., Eratosthenes, a scientist from Alexandria, Egypt, was actually able to estimate the circumference of the Earth.

Problem 5.1

- Using modeling dough, make a sphere with a diameter between 2 inches and 3.5 inches.
- Using a strip of transparent plastic, make a cylinder with an open top and bottom that fits snugly around your sphere. Trim the height of the cylinder to match the height of the sphere. Tape the cylinder together so that it remains rigid.



- Now, flatten the sphere so that it fits snugly in the bottom of the cylinder. Mark the height of the flattened sphere on the cylinder.



A. Measure and record the height of the cylinder, the height of the empty space, and the height of the flattened sphere.

B. What is the relationship between the volume of the sphere and the volume of the cylinder?

Remove the modeling dough from the cylinder, and save the cylinder for the next problem.

Problem 5.1 Follow-Up

Compare your results with the results of a group that made a larger or smaller sphere. Did the other group find the same relationship between the volume of the sphere and the volume of the cylinder?

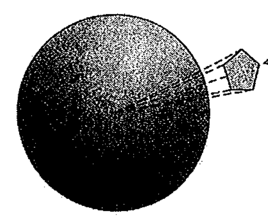
2. Classroom Connection 7.3.5: Volume of a Sphere Using Tiny "Pyramids"

The following exploration is taken from pages 546–547 in the textbook *Discovering Geometry*, third edition, by Michael Serra. In this exploration, the sphere's volume is known, and it is used to find its surface area. However, a similar reasoning works to find the sphere's volume if its surface area is known. Replace Steps 3 and 4 in the exploration with Step 3 here, and discuss it in small groups.

Step 3: The sphere's surface area is $S = 4\pi r^2$. Use the equation in Step 2 to find the sphere's volume V. ◆

Investigation
The Formula for the Surface Area of a Sphere

In this investigation you'll visualize a sphere's surface covered by tiny shapes that are nearly flat. So the surface area, S, of the sphere is the sum of the areas of all the "nearly polygons." If you imagine radii connecting each of the vertices of the "nearly polygons" to the center of the sphere, you are mentally dividing the volume of the sphere into many "nearly pyramids." Each of the "nearly polygons" is a base for a pyramid, and the radius, r, of the sphere is the height of the pyramid. So the volume, V, of the sphere is the sum of the volumes of all the pyramids. Now get ready for some algebra.



Step 1 Divide the surface of the sphere into 1000 "nearly polygons" with areas $B_1, B_2, B_3, \ldots, B_{1000}$. Then you can write the surface area, S, of the sphere as the sum of the 1000 B's:

$$S = B_1 + B_2 + B_3 + \ldots + B_{1000}$$

Step 2 The volume of the pyramid with base B_1 is $\frac{1}{3}(B_1)(r)$, so the total volume of the sphere, V, is the sum of the volumes of the 1000 pyramids:

$$V = \frac{1}{3}(B_1)(r) + \frac{1}{3}(B_2)(r) + \ldots + \frac{1}{3}(B_{1000})(r)$$

What common expression can you factor from each of the terms on the right side? Rewrite the last equation showing your factoring.

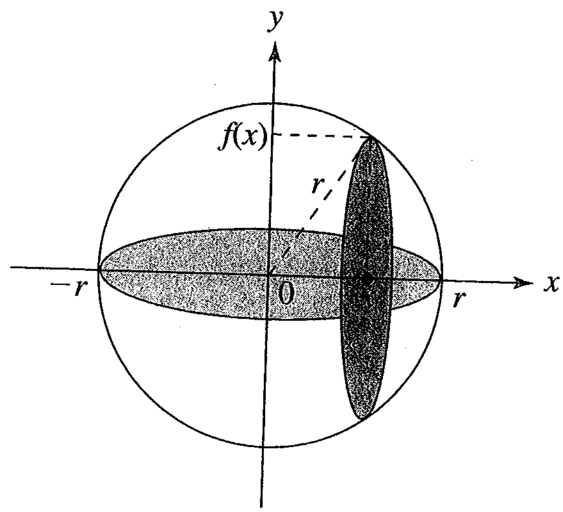
Step 3 But the volume of the sphere is $V = \frac{4}{3}\pi r^3$. Rewrite your equation from Step 2 by substituting $\frac{4}{3}\pi r^3$ for V and substituting for S the sum of the areas of all the "nearly polygons."

Step 4 Solve the equation from Step 3 for the surface area, S. You now have a formula for finding the surface area of a sphere in terms of its radius. State this as your next conjecture and add it to your conjecture list.

Sphere Surface Area Conjecture C-91

The surface area, S, of a sphere with radius r is given by the formula ?.

3. Volumes of Spheres Using Calculus

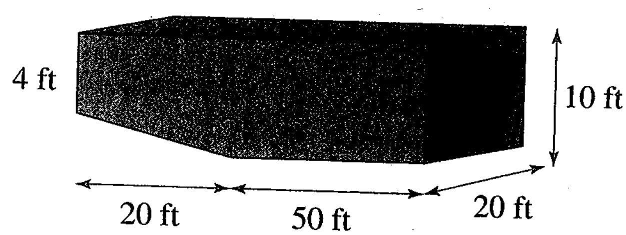


a. For each $-r \le x \le r$, describe the cross-section corresponding to x, then express its area $A(x)$ in terms of x and r.

b. Use Theorem 7.3.1 to determine the volume of the given sphere. ◆

EXERCISES 7.3

1. A swimming pool is in the shape of a right prism as in the following figure. How many cubic feet of water can this swimming pool hold?



2. **Classroom Connection 7.3.6: Revisiting Cylinders**

The following exploration is taken from pages 502–503, page 7-56, and page 7-57 in the sixth-grade textbook *Math Thematics, Book 1*. Answer the questions therein. ◆

GOAL

LEARN HOW TO...
- recognize a cylinder
- find the volume of a cylinder

AS YOU...
- explore the size and shape of a kiva

KEY TERM
- cylinder

Exploration

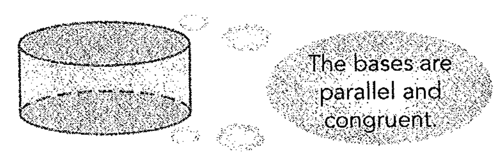
Volume of a Cylinder

SET UP *Work in a group of three. You will need:*
- *Labsheets 5B and 5C*
- *scissors*
- *tape*
- *rice*
- *ruler*

In the summer of 1891, Gustaf Nordenskiöld of Sweden and his team began to uncover the ruins at Mesa Verde. Part of their task was to remove the layers of dust and rubbish that had piled up over the centuries. After digging to a depth of $\frac{1}{2}$ m at one location, they began to see a kiva take shape.

12 How do you think Nordenskiöld could have estimated the amount of dust and rubbish in the kiva without removing it?

A kiva is shaped like a circular **cylinder**. A circular cylinder is a space figure that has two circular bases that are parallel and congruent.



13 **Use Labsheets 5B and 5C.** Cut out the nets for the open-topped *Prism A, Prism B and Cylinder.* Fold and tape each net.

14 How is the cylinder like a prism? How is it different?

15 Which has a larger volume, prism A or prism B? Explain.

16 Which do you think holds more, the cylinder or prism A? the cylinder or prism B? Explain your thinking in each case.

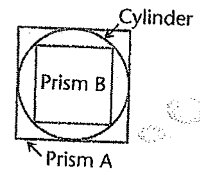
17 **a.** Fill prism B with rice and then pour the rice into the cylinder. Does the rice completely fill the cylinder, or is there too much or not enough rice?

b. Fill the cylinder with rice and then pour the rice into prism A. Does the rice completely fill prism A?

c. What can you conclude about the volume of the cylinder?

18 **a.** Place the cylinder inside the larger prism. Then place the smaller prism inside the cylinder.

b. For each of the prisms and the cylinder, find the area of a base and the height. Make a table to record your results.



top view

19 **Discussion** Add on to the table you completed in Question 18.

a. Find the volumes of prism A and prism B. Explain your method.

b. Use the same method you used in part (a) to find the volume of the cylinder.

c. Use your models and your results with rice to decide whether the volume you found for the cylinder is reasonable.

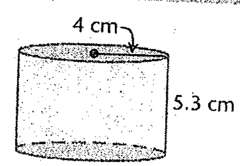
▶ **You can find the volume V of a cylinder with height h and a base with area B in the same way you find the volume of a prism.**

$$V = Bh, \text{ or } V = \pi r^2 h.$$

area of circular base

EXAMPLE

Find the volume of the cylinder shown to the nearest cubic centimeter. Use 3.14 for π.



SAMPLE RESPONSE

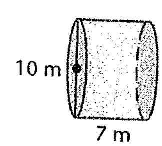
$V = \pi r^2 h$

$\approx 3.14 \cdot 4^2 \cdot 5.3 = 266.272$

The volume is about 266 cm^3.

Volume is measured in cubic units.

20 **✓ CHECKPOINT** Find the volume of the cylinder to the nearest cubic meter. Use 3.14 for π.



✓ QUESTION 20

...checks that you can find the volume of a cylinder.

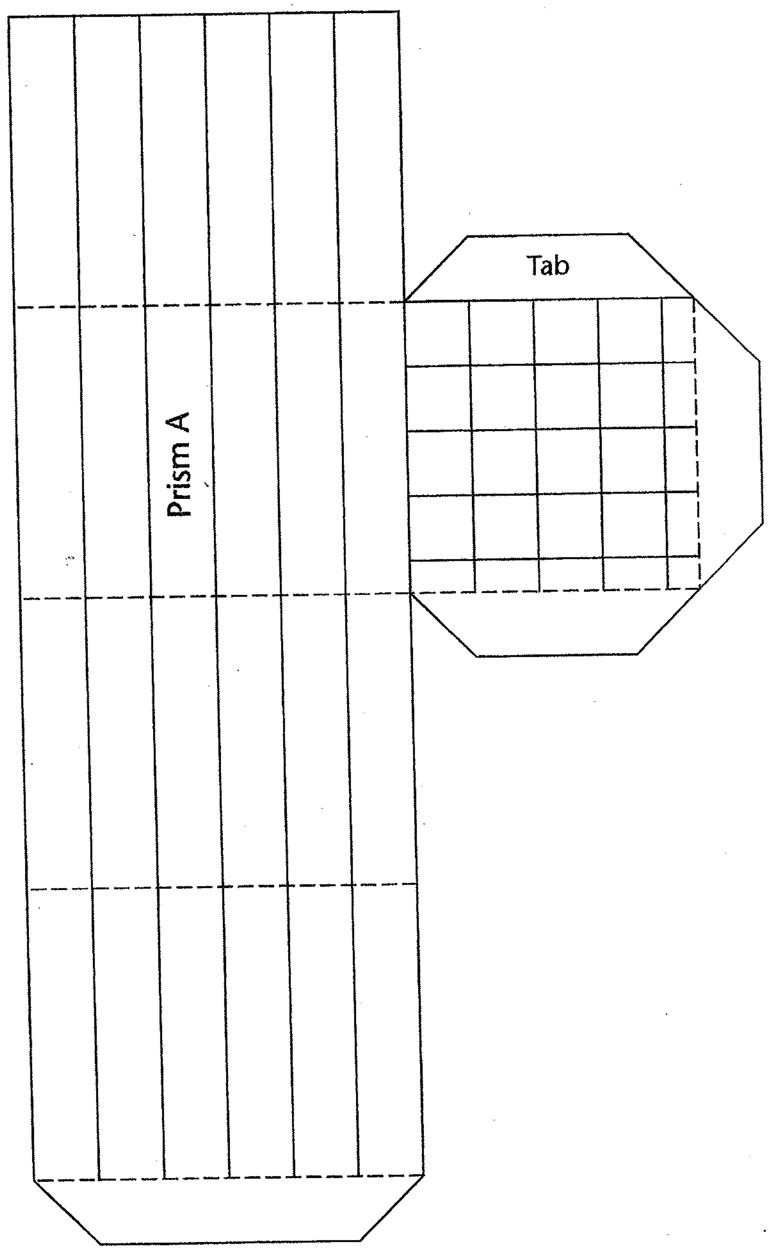
21 Gustaf Nordenskiöld reported that one of the kivas he uncovered had walls 2 m high with a diameter of 4.3 m. If this kiva was completely full of dust and rubbish, about how much material did Nordenskiöld have to remove?

HOMEWORK EXERCISES ▶ See Exs. 14–22 on p. 506.

Name Date

MODULE 7 LABSHEET 5B

Prism A (Use with Questions 13–19 on page 502–503.)



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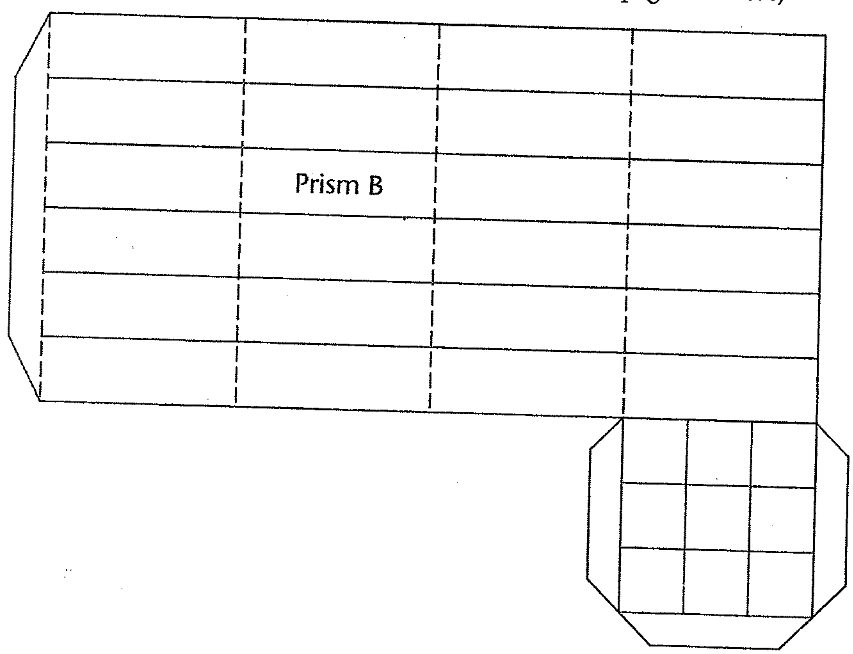
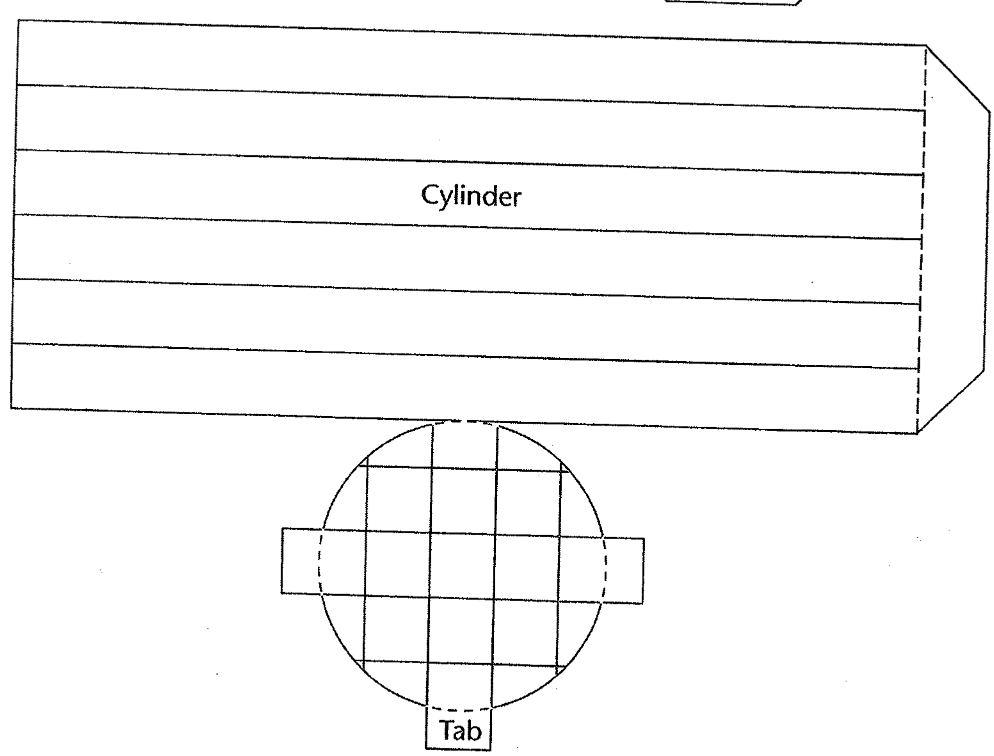
Name ______________________ Date ____________

MODULE 7 LABSHEET **5C**

Prism B and Cylinder (Use with Question 13–19 on page 502–503.)





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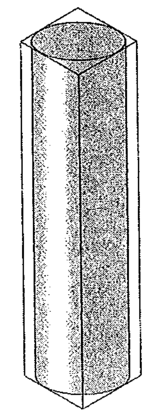
3. **Classroom Connection 7.3.7: A Couple of Problems on Cylinders**

The following extensions are from page 40 in the seventh-grade textbook *Connected Mathematics, Filling and Wrapping*. Answer the questions in Problems 12 and 13. ◈

Extensions

12. A cylindrical can is packed securely in a box as shown at right. The height of the box is 10 cm, and the sides of its square base measure 2 cm.



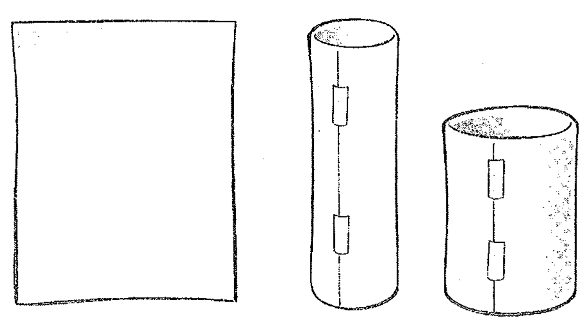
a. Find the radius and height of the can.

b. What is the volume of the empty space between the can and the box?

c. Find the ratio of the volume of the can to the volume of the box.

d. Make up a similar example with a different size can and box. What is the ratio of the volume of the can to the volume of the box for your example? How does the ratio compare to the ratio you got in part c?

13. Start with two identical sheets of paper. Tape the long sides of one sheet together to form a cylinder. Form a cylinder from the second sheet by taping the short sides together. Imagine that each cylinder has a top and a bottom.



a. Which cylinder has greater volume? Explain your reasoning.

b. Which cylinder has greater surface area? Explain your reasoning.

In Exercises 4–14, sketch the graphs $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$. Then find the volume of the solid obtained by revolving about the x-axis the region of the plane that is bounded by the two graphs. You may use your calculator only to check your final answers.

4. $f(x) = 3, g(x) = x$ and $a = 0, b = 3$
5. $f(x) = x^2, g(x) = 0$ and $a = 0, b = \sqrt[3]{2}$
6. $f(x) = x, g(x) = 2x$ and $a = 0, b = 4$
7. $f(x) = -x^3, g(x) = 0$ and $a = 1, b = 3$
8. $f(x) = \sqrt{x}, g(x) = x$ and $a = 0, b = 1$
9. $f(x) = 1/\sqrt{x^3}, g(x) = 0$ and $a = 1, b = 4$
10. $f(x) = \sqrt{4 - x^2}, g(x) = 0$ and $a = -2, b = 2$
11. $f(x) = 2\sqrt{1 - \frac{x^2}{9}}, g(x) = 0$ and $a = -3, b = 3$
12. $f(x) = \sqrt{e^x}, g(x) = x^2$ and $a = 0, b = 1$
13. $f(x) = x\sqrt{x^3 + 1}, g(x) = 0$ and $a = 0, b = 1$
14. $f(x) = |x|, g(x) = 1$ and $a = -1, b = 1$
15. Find the volume of the solid obtained by revolving about the x-axis the region inside the triangle of vertices $A(0,0)$, $B(0,2)$, and $C(1,1)$.
16. Find the volume of the torus obtained by revolving about the x-axis the disc of radius 1 centered at point $(0,2)$.
17. Find the volume of the region in the space obtained by revolving about the x-axis the region of the plane that is bounded by the graphs $y = |x|$ and $y = 1$ for $-2 \leq x \leq 2$.
18. Find the volume of the region in the space obtained by revolving about the x-axis the region of the plane that is bounded by the graphs $y = \sqrt{x}$ and $y = x$ for $0 \leq x \leq 2$.
19. **I. The Disk Method**

   a. Sketch the graph of a nonnegative continuous function $f(x)$ defined on the interval $[a, b]$.
   b. Sketch the solid obtained by revolving about the x-axis the region of the plane that is between the x-axis and the graph $y = f(x)$.
   c. Give a few examples of solids that can be obtained as previously described.
   d. Using Theorem 7.3.1, find an explicit formula for the volume of this type of solids.

   II. The Washer Method

   a. On the same coordinate system, sketch the graph of two continuous functions $f(x)$ and $g(x)$ defined on a given interval $[a, b]$ satisfying $0 \leq g(x) \leq f(x)$ for each $a \leq x \leq b$.

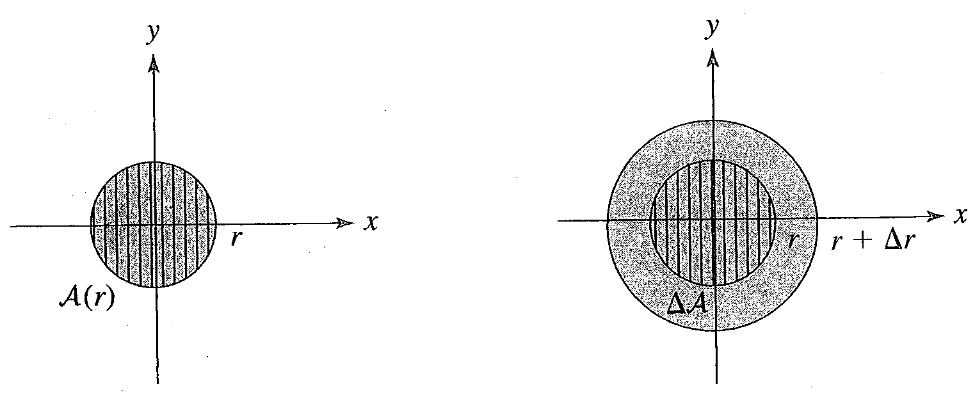
b. Sketch the solid obtained by revolving about the x-axis the plane region that lies between the two preceding graphs.
c. Give a few examples of solids that are obtained in this manner.
d. How does the volume $\mathcal{V}$ of the solid in IIb compare to the volumes of the solids that are obtained by revolving about the x-axis the region below the graph of f and above the x-axis and the region below the graph of g and above the x-axis?
e. Using Problem Id, find an explicit formula for the volume of the solid in Problem IIb.

20. The graph $y = x^2$, where $0 \le x \le h$, was rotated first about the x-axis to form Solid A, then about the y-axis to form Solid B.
a. Find the volume $V_A(h)$ of Solid A.
b. Find the volume $V_B(h)$ of Solid B.
c. For which values of h do the two solids have the same volume?
d. For which values of h is $V_A(h)$ larger than $V_B(h)$?
e. For which values of h is $V_B(h)$ larger than $V_A(h)$?

PROJECTS AND EXTENSIONS 7.3

I. Relations between Length, Area, and Volume Using Derivatives

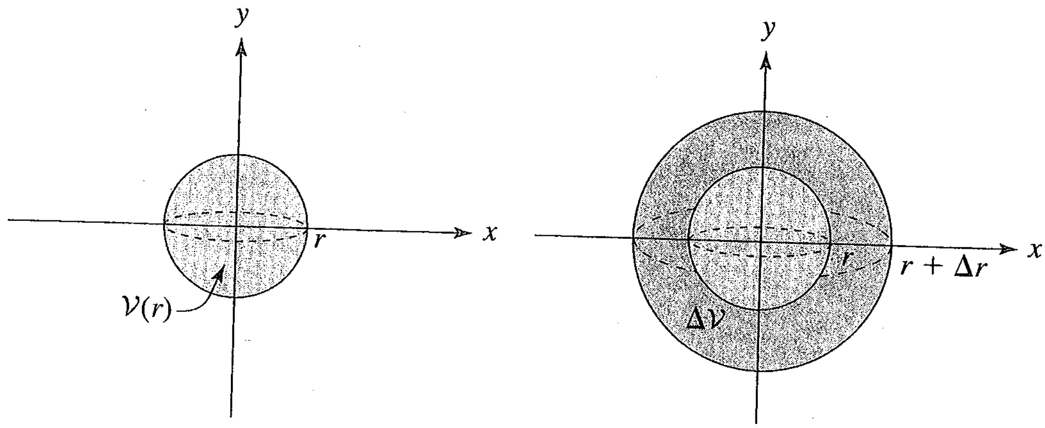
1. Let $\mathcal{C}(r)$ denote the circumference of a circle of radius r, and let $\mathcal{A}(r)$ denote the area of a disc of radius r. Recall the expressions of $\mathcal{C}(r)$ and $\mathcal{A}(r)$.
2. Using differential calculus, find a relation between the functions $\mathcal{C}(r)$ and $\mathcal{A}(r)$.
3. Does the preceding relation also hold between the perimeters and areas of squares?
4. Is this only a coincidence or are there reasons for the relation you have found between $\mathcal{C}(r)$ and $\mathcal{A}(r)$? To answer this question, follow the ideas in the proof of the Fundamental Theorem of Calculus given in Section 6.3.



5. Let $\mathcal{S}(r)$ and $\mathcal{V}(r)$ be, respectively, the area and volume of a sphere of radius r. Recall the expressions of $\mathcal{S}(r)$ and $\mathcal{V}(r)$.
6. Using differential calculus, find a relation between the functions $\mathcal{S}(r)$ and $\mathcal{V}(r)$.

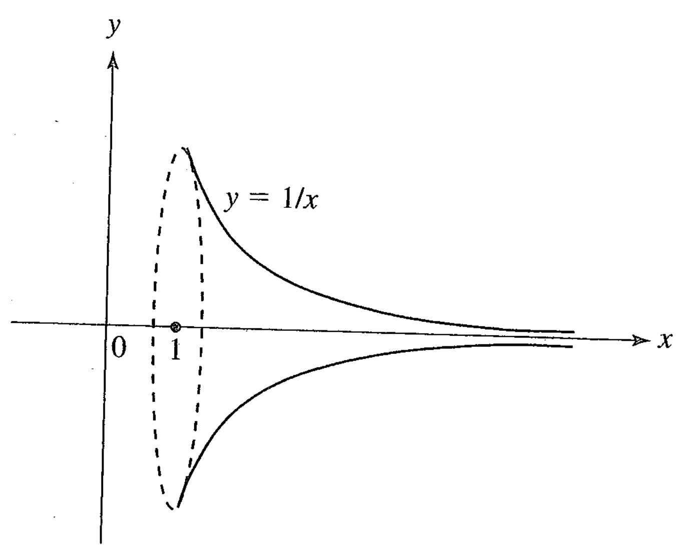
7. Consider a cylinder whose height equals its radius. Does the preceding relation also hold between the area and volume of the cylinder?

8. Is this simply a coincidence or are there reasons for the relation you have found between $\mathcal{S}(r)$ and $\mathcal{V}(r)$? To answer this question, follow the ideas in the proof of Theorem 7.3.1.



II. Gabriel's Trumpet

Consider the graph $y = 1/x$ for $x \geq 1$. The object obtained by revolving this graph about the x-axis is called *Gabriel's trumpet.*



Notice that the trumpet is infinitely long, since its shank (small end) stretches along the x-axis forever. Suppose you wanted to paint the outside of this trumpet. How much paint would be required? If, instead, you poured the paint into the trumpet, how much paint would be needed to fill the trumpet? The answers may surprise you! Begin by fixing $a > 1$ and considering the portion of the trumpet corresponding to $1 \leq x \leq a$. Denote by $\mathcal{S}(a)$ its area and by $\mathcal{V}(a)$ its volume.

1. Using calculus, evaluate by hand $\mathcal{V}(a)$.
2. What is the $\lim_{a \to \infty} \mathcal{V}(a)$?
3. How much paint can Gabriel's trumpet hold?
4. Using calculus, express $\mathcal{S}(a)$ as a definite integral. Do not evaluate this integral, but simplify the integrand as much as possible.
5. (T) Using your calculator, fill in the following table.

n	$\mathcal{S}(10^n)$
10	
10^2	
10^3	
10^4	
10^5	
10^6	
10^7	
10^8	
10^9	
10^{10}	

6. Make a conjecture about the $\lim_{a \to \infty} \mathcal{S}(a)$.
7. How much paint is needed to paint Gabriel's trumpet?
8. Can you resolve the paradox?

III. Displacement and Density

Research the concepts of displacement and density. Then, plan a lesson on how these quantities are used to find the volume of solids whose shapes are irregular.

CHAPTER 7 REVIEW

Roughly speaking, the length of a curve is the amount of string it takes to exactly cover the curve.

The length of the graph $y = f(x)$, where $a \leq x \leq b$, can be approximated by the lengths of polygonal paths approximating the curve. Let $n \geq 1$ be an integer, and denote by $x_i = a + i\frac{b-a}{n}$ for $0 \leq i \leq n$. The length $\mathcal{L}_n$ of the polygonal path obtained by joining the points whose x-coordinates are $x_0, x_1, ..., x_n$ is given by the formula

$$\mathcal{L}_n = \sum_{i=0}^{n-1} \sqrt{\frac{(b-a)^2}{n^2} + \left[f(x_{i+1}) - f(x_i)\right]^2}.$$