group. That presentation is the same as the presentation of the knot group described in the previous section.

The connection with labeling can also be summarized. For each arc in the diagram of the knot there is an element in the fundamental group which is represented by a path that runs from the base point, p, directly to the arc, once around the arc, and then back to the basepoint. That element corresponds to the element in the knot group given by the variable label on the arc. It can be proved that relations between the elements in the fundamental group correspond to the relations in the knot group arising at the crossings.

A group is often studied by mapping it homomorphically onto a simpler group, say G, which is better understood. Given such a homomorphism of the fundamental group of a knot complement, composing it with the correspondence between the knot group and the fundamental group gives an assignment of an element in G to each arc in the diagram. That is, labelings of the diagram turn out to correspond to homomorphisms of the fundamental group of the knot complement. The consistency condition on the labeling corresponds to the map being a homomorphism. The generation condition corresponds to the map being surjective.

CHAPTER 6: GEOMETRY, ALGEBRA, AND THE ALEXANDER POLYNOMIAL

The discovery of connections between the various techniques of knot theory is one of the recurring themes in this subject. These relationships can be surprising, and have led to many new insights and developments. A recent example of this occurred with the discovery by V. Jones of a new polynomial invariant of knots . Although his approach was algebraic, the Jones polynomial was soon reinterpreted combinatorially. Almost immediately there blossomed an array of new combinatorial knot invariants which appear to be among the most useful tools available for problems relating to the classification of knots. An understanding of these new invariants from a noncombinatorial perspective is now a major problem in the subject, and one that will certainly lead to significant progress. Chapter 10 is devoted to a discussion of the Jones polynomial and its generalizations.

To demonstrate how various techniques can be related, this chapter presents geometric and algebraic approaches to the Alexander polynomial. The geometric approach introduces a new and powerful object, the *Seifert matrix*, and for this reason geometry will be the main focus here. The algebraic approach links the combinatorics to the geometry, and also demonstrates that the Alexander polynomial of a knot is determined by the knot group.

It is not surprising that bringing together the diverse methods developed so far involves difficult technical arguments. Even the definition of the Seifert matrix, given in Section 1, is fairly complicated. The benefit of this technical argument is seen in Section 2, where a simple algorithm for computing the Alexander matrix is given, and in Section 3, where new knot invariants are developed. Fox derivatives and their use in computing the Alexander polynomial from a presentation of the knot group are described in Section 4. This material may also appear quite technical; but again there are valuable insights gained from the approach.

1 The Seifert Matrix

If a surface is formed by adding bands to a disk, the cores of the bands along with arcs on the disk can be used to construct a family of oriented curves on the surface. This is illustrated in Figure 6.1. The choice of orientation of the curves is arbitrary. In the case where the surface is a Seifert surface for a knot, how these curves twist and link carries information about the knot. This linking and twisting information is captured by a matrix called the *Seifert matrix* of the knot.

In Exercise 2.5 of Chapter 3, linking numbers were defined. Exercise 2.6 of that chapter provided an alternative definition that is now summarized. Suppose that an oriented link of two components, K and J, has a regular projection. The *linking number* of K and J is defined to be the sum of the signs of the crossing points in the diagram

at which K crosses over J. The sign of a crossing is 1 if the crossing is right-handed, that is, if J crosses under K from the right to the left. The sign is -1 if the crossing is left-handed. The linking number is denoted $\ell k(K,J)$ and is symmetric: $\ell k(K,J) = \ell k(J,K)$.

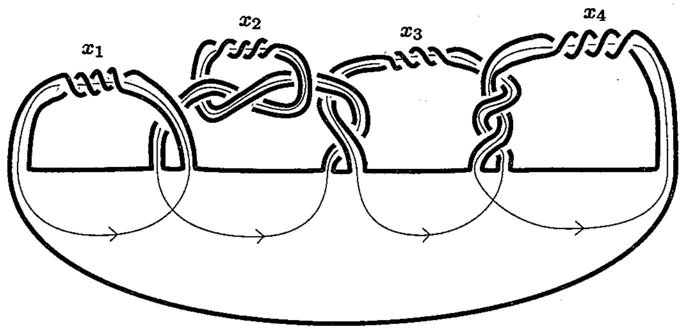


Figure 6.1

Given a knot K, fix a Seifert surface F for K. Since a Seifert surface is orientable, it is possible to distinguish one side of the surface as the "top" side. Formally this consists of picking a nonvanishing normal vector to the surface. Which direction is picked will not matter. With this done, given any simple oriented curve, x, on the Seifert surface, one can form the *positive push off* of x, denoted x^*, which runs parallel to x and lies just above the Seifert surface.

If the Seifert surface F is formed from a single disk by adding bands, it was shown in Figure 6.1 that there naturally arises a family of curves on F. If F is genus g there will be $2g$ curves, $x_1, x_2, \ldots, x_{2g}$. The associated *Seifert matrix* is the $2g \times 2g$ matrix V with (i,j)-entry $v_{i,j}$ given by $v_{i,j} = \ell k(x_i, x_j^*)$. The Seifert matrix clearly depends on the choices made in its definition, and by itself is not an invariant of the knot. However, in the next two sections it will be shown that the Seifert matrix can be used to define knot invariants, including the Alexander polynomial. The rest of this section is devoted to illustrating the computation of entries in a Seifert matrix.

Figure 6.2

EXAMPLE

Computing the entries of a Seifert matrix can be difficult, especially if the surface is very complicated. Let's consider the Seifert matrix for the Seifert surface and knot illustrated in Figure 6.1. The way the surface is oriented, the normal vector to the surface points toward the reader on the disk portion of the surface. Figure 6.2 illustrates the curves x_2 and x_3^*. Their linking number is 1, so that $v_{2,3} = 1$.

In Figure 6.3 the curves x_2 and x_2^* are drawn. The reader should redraw Figure 6.1 and check that the curve drawn as x_2^* actually lies above the Seifert surface. It is a delicate construction.

Using Figure 6.3, one computes $v_{2,2} = \ell k(x_2, x_2^*) = -5$. Continuing in this way (see Exercise 1.2) the final result is that the Seifert matrix

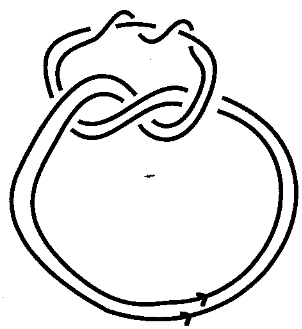


Figure 6.3

is given by

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

EXERCISES

1.1. In Figure 6.4 Seifert surfaces for the trefoil knot and its mirror image, the left-handed trefoil, are illustrated. Compute the Seifert matrix associated to each of these surfaces.

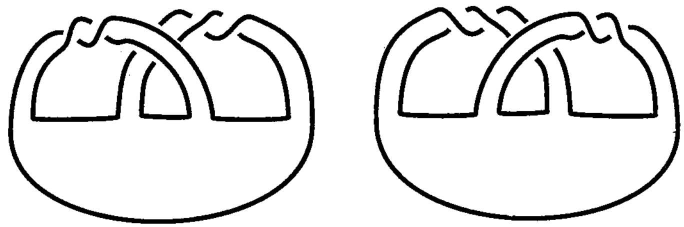


Figure 6.4

1.2. Complete the calculation of the Seifert matrix for the knot in Figure 6.1.

1.3. Figure 6.5 illustrates the Seifert surface of a knot, previously discussed in Exercise 2.2 of Chapter 3. (This particular example is the *3-twisted double* of the unknot.) Compute its Seifert matrix.

1.4. In Exercise 2.5 of Chapter 4 Seifert surfaces for the (p,q,r)-pretzel knot were constructed, for p, q, and r odd. Find the corresponding Seifert matrix.

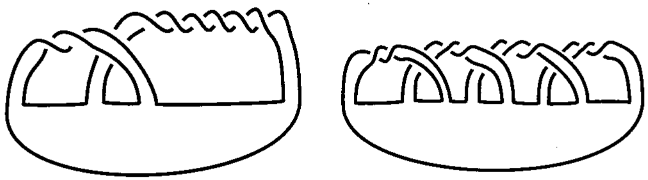


Figure 6.5 *Figure 6.6*

1.5. Figure 6.6 above shows a Seifert surface for the $(2,n)$-torus knot. (Only the (2,5)-torus knot is shown, but the pattern is clear.) Find the corresponding Seifert matrix.

1.6. What would be the effect of changing the orientation of the Seifert surface on the Seifert matrix?

1.7. Seifert surfaces for two knots can be used in order to form a Seifert surface for the connected sum of the knots. How are the corresponding Seifert matrices related?

1.8. In Exercise 1, the example of the trefoil and its mirror image can be generalized. What is the relation between the Seifert matrix of a knot, found using some given Seifert surface, and the Seifert matrix for its mirror image, found using the mirror image of the given Seifert surface?

2 Seifert Matrices and the Alexander Polynomial

The Alexander polynomial is easily computed using the Seifert matrix; recall, once again, that the polynomial is only defined up to multiples of $\pm t^i$. An immediate consequence will be a proof that the Alexander polynomial is symmetric. A proof of this based on the combinatorial definition of the Alexander polynomial is not at all evident.

□ **THEOREM 1.** *Let V be a Seifert matrix for a knot K, and V^t be its transpose. The Alexander polynomial is given by the determinant,* $\det(V - tV^t)$.

Later in this section it will be indicated why this determinant gives a well-defined knot invariant. The proof that it is the same as the combinatorially defined Alexander polynomial is a deeper result. The connection is via algebra: the complement of the knot can be decomposed using a Seifert surface and that decomposition leads to information about the structure of the knot group. In Section 4 a connection between the group of the knot and the Alexander polynomial will be presented. Carefully putting all these connections together yields the desired result.

One important corollary of Theorem 1 is the following.

□ **COROLLARY 2.** *The Alexander polynomial of a knot K satisfies $A_K(t) = t^{\pm i}A_K(t^{-1})$ for some integer i.*

PROOF
This is an immediate consequence of the fact that a matrix and its transpose have the same determinant: if a Seifert matrix V is used to compute the Alexander polynomial $A_K(t) = \det(V - tV^t) = \det((V - tV^t)^t) = \det(V^t - tV) = \det(tV - V^t) = \det(t(V - t^{-1}V^t)) = t^{2g}A_K(t^{-1})$. □

S-EQUIVALENCE OF SEIFERT MATRICES

The construction of the Seifert matrix of a knot depended on many choices. Two of these are especially critical.

Band moves: If a Seifert surface is presented as a disk with bands added, that surface can be deformed by sliding one of the points at which a band is attached over another band. The resulting surface is again a disk with bands added. However, the $2g$ curves formed from the cores of the new bands will not be the same as those formed from the cores of the original bands. The effect of this operation is to do a simultaneous row and column operation on the Seifert matrix; that is, for some i and j, a multiple of the i-th row is added to the j-th row, and then the same multiple of the i-th column is added to the j-th column. A sequence of these band slides changes the Seifert matrix from V to MVM^t where M is some invertible integer matrix.

Stabilization: Given a Seifert surface for a knot, it can be modified by adding two new bands, as illustrated in Figure 6.7 for the Seifert surface of the trefoil. One of the bands is untwisted and unknotted. The other can be twisted, or knotted, and can link the other bands.

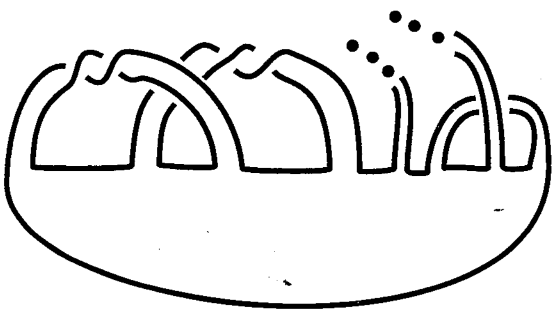


Figure 6.7

It is clear that the boundary of the new surface is the same knot as for the original Seifert surface. The effect

of this operation on the Seifert matrix is to add two new columns and rows, with entries as indicated.

$$\begin{pmatrix} & & & * & 0 \\ & & & * & 0 \\ & & & * & 0 \\ & & & * & 0 \\ * & * & * & * & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Two integer matrices are called S-equivalent if they differ by a sequence of operations of the two types described: right and left multiplication by an invertible integer matrix and its transpose, and addition or removal of a pair of rows or columns of the type shown above. These two matrix operations also include the changes that occur in a Seifert matrix if the bands are reordered, or reoriented.

A difficult geometric argument shows that for any two Seifert surfaces for a knot, there is a sequence of stabilizations that can be applied to each so that the resulting surfaces can be deformed into each other. A consequence is the following:

□ **THEOREM 3.** *Any two Seifert matrices for a knot are S-equivalent.*

□ **COROLLARY 4.** *If V_1 and V_2 are Seifert matrices associated to the same knot, then the polynomials* $\det(V_1 - tV_1^t)$ *and* $\det(V_2 - tV_2^t)$ *differ by a multiple of* $\pm t^k$.

PROOF

This is proved by checking the effect of the two basic operations of S-equivalence on the determinant. The first, multiplying by M and M^t has no effect on the determinant, since $\det(M) = 1$. The second has the effect of multiplying the determinant by t. □

EXAMPLE

In Section 1 the Seifert matrix of the knot illustrated in Figure 6.1 was presented. The Alexander Polynomial of that knot is given by the determinant of the matrix

$$\begin{pmatrix} 2-2t & 1 & 0 & 0 \\ -t & -5+5t & 1-t & 0 \\ 0 & 1-t & 2-2t & -1+2t \\ 0 & 0 & -2+t & -2+2t \end{pmatrix}$$

The determinant of this matrix is $64t^4 - 272t^3 + 417t^2 - 272t + 64$.

EXERCISES

2.1. Compute the Alexander polynomial of the trefoil knot using the Seifert matrices found in Exercise 1 of the previous section.

2.2. Find the Alexander polynomial of the knot discussed in Exercise 1.3, using the Seifert matrix found there.

2.3. Check the calculation of the determinant that gives the Alexander polynomial of the knot in Figure 6.1.

2.4. Compute the Alexander polynomial of the (p,q,r)-pretzel knot, (p, q, and r odd) by using the Seifert matrix found in Exercise 1.4.

2.5. Use the result of Exercise 1.7 to show that the Alexander polynomial of the connected sum of knots is the product of their individual Alexander polynomials.

2.6. The Alexander polynomial of a knot can be normalized so that only positive powers of t appear and the constant term is nonzero. Show that the degree of the resulting polynomial is even. Hint: use the symmetry condition, along with the fact that $A_K(1)$ is odd. (If $A_K(1)$ is even, so is $A_K(-1)$ and the knot would have a mod 2 labeling. Now see Exercise 3.5, Chapter 3.)

2.7. Show that if the determinant of a $2g \times 2g$ Seifert matrix is nonzero, then the Alexander polynomial is degree $2g$ and has nonzero constant term.

3 The Signature of a Knot, and Other *S*-equivalence Invariants

In the last section it was seen that any two Seifert matrices for a knot are S-equivalent; that is, a pair of fairly simple operations will transform one to the other. Because of this many knot invariants can be defined using the Seifert matrix. This section discusses a few of them.

DETERMINANT

The determinant of the Seifert matrix can change under stabilization, and is not an invariant of the knot. However, if V is the Seifert matrix of a knot, then the determinant of $V + V^t$ is only changed by a sign if the matrix is stabilized. This is an easy exercise in determinants, and is given in the problems below. Multiplying by a matrix of determinant ± 1 can at most change the sign of the determinant as well. Hence, the absolute value of the determinant of $V + V^t$ is a well-defined knot invariant.

This is in fact the same as the determinant invariant defined in Chapter 3. The determinant of $V + V^t$ is the value of the Alexander polynomial evaluated at $t = -1$ up to a sign. The Seifert matrix approach leads to a simple calculation of the determinant.

THE SIGNATURE OF A KNOT

Given a symmetric ($A = A^t$) real matrix, there is a signature defined. One definition is constructive. By performing a sequence of simultaneous row and column operations the matrix can be diagonalized. The *signature* of the matrix is defined to be the number of positive entries minus the number of negative entries on the diagonal.

EXAMPLE

Consider the symmetric matrix A_1 below. Multiply the first row by $-1/4$ and add it to the second row. Now perform the same operation using the first column. The resulting matrix is listed as A_2.

$$A_1 = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & -10 & 2 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix}$$

$$\to \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -41/4 & 2 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix} = A_2$$

Using the second row and column the nondiagonal entries of the second row and column can be changed to 0. Finally, working with the third column and row reduces the matrix to diagonal form. The exercises ask you to check that the final result is

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -41/4 & 0 & 0 \\ 0 & 0 & 180/41 & 0 \\ 0 & 0 & 0 & -121/20 \end{pmatrix}$$

As there are 2 positive entries and 2 negative entries the signature is $2 - 2 = 0$.

A theorem of algebra, named for James J. Sylvester, states that if the symmetric matrix B is given by $B = MAM^t$, where M is invertible, then the signatures of A and B are equal.

For a Seifert matrix V of a knot K, the matrix $V + V^t$ is symmetric and its signature is called the *signature of* K, denoted $\sigma(K)$.

☐ **THEOREM 5.** *For a knot K, the value of $\sigma(K)$ does not depend on the choice of Seifert matrix, and is hence a well-defined knot invariant.*

PROOF

First, note that if Seifert matrices V and W are related by $W = MVM^t$, then $(W + W^t) = M(V + V^t)M^t$. Hence Sylvester's theorem implies that the signature of $(W + W^t)$ is the same as that of $(V + V^t)$. All that is left to check is that stabilization of V does not change the signature of $(V + V^t)$. Proving this is left to the exercises. ☐

EXAMPLE

A Seifert matrix V for the knot in Figure 6.1 was given in Section 1. For that V, $V + V^t$ is the matrix discussed in the previous example, and hence the signature of that knot is 0.

Using the Seifert matrix for the trefoil computed in Exercise 1.1 $V + V^t$ is given by

$$\begin{pmatrix} -2 & +1 \\ +1 & -2 \end{pmatrix}.$$

It has signature -2.

The same calculation for the left-handed trefoil gives a signature of 2. Hence, the right and left trefoils are inequivalent knots.

THE SIGNATURE FUNCTION

The signature of a knot can be generalized by using complex numbers. First recall that a complex matrix is called Hermitian if it equals its conjugate transpose. Any Hermitian matrix can be diagonalized performing a sequence of row and column operations. The only change from the diagonalization of real matrices is that if a row is multiplied by a complex number, then, when the corresponding column operation is performed, the column is multiplied by the conjugate of that number. Once diagonalized, the matrix has real entries, (as it equals its conjugate transpose) and the signature of the matrix is given by the number of positive entries minus the number of negative entries. Again, a theorem of linear algebra states that if a Hermitian matrix A is replaced by MAM^* where M is an invertible complex matrix and M^* is its conjugate transpose, the signature is unchanged.

Let V be the Seifert matrix for a knot K and let ω be a complex number of modulus 1. Consider the Hermitian matrix $(1-\omega)V+(1-\omega^{-1})V^t$. The signature of this matrix is called the *ω-signature* of K. Checking that S-equivalent Seifert matrices have the same ω-signature is straightforward; only stabilization remains to be checked. If one thinks of modulus 1 complex numbers as lying on the unit circle in the complex plane, this signature defines a function on the unit circle called the *signature function* of the knot.

Even for 2×2 Seifert matrices, the signature function can be difficult to compute. (See Exercise 3.8.) However, it can sometimes be used to distinguish knots where other methods fail. It also has many theoretical applications.

EXERCISES

3.1. Complete the diagonalization and signature calculations presented in this section.

3.2. Compute the signature of the $(3,5,-7)$-pretzel knot .

3.3. Compute the determinant of the (p,q,r)-pretzel knot.

3.4. For a Seifert matrix V, $\det(V+V^t)\neq 0$. (Why?) Conclude that the signature of a knot is always even.

3.5. Prove that stabilization does not change the signature of a matrix.

3.6. Use Exercise 1.7 to show that the signature of a connected sum of knots is the sum of their signatures.

3.7. Prove that the matrix $(1-\omega)V+(1-\omega^{-1})V^t$ has nonzero determinant for ω of modulus 1 unless ω is a root of the Alexander polynomial. Conclude that the signature function is constant on the circle, except for a finite number of jump discontinuities.

3.8. Compute the signature function for the trefoil and the figure-8 knot.

3.9. Compute the signature of the $(2,n)$-torus knot using Exercise 1.5.

4 Knot Groups and the Alexander Polynomial

In Chapter 5 it was shown how to construct a presentation of a group, starting with a knot diagram. The presentation consists of a set of n variables, and $n-1$ words in the variables (and their inverses.) In this section an algorithm will be presented that computes the Alexander polynomial of the knot starting with a group presentation of the form arising from the construction given in Chapter 5. The algorithm was discovered by Fox. It is, in fact, possible to compute the polynomial using any presentation of the group, but to do this the algorithm has to be generalized.

That the knot polynomial is determined by the group of the knot has certain theoretical implications. For instance, as mentioned in Section 2, the link between the combinatorial and geometric definition of the Alexander polynomial is provided by this algebra. On the practical side, Fox's algorithm provides one more means of computing the Alexander polynomial.

FOX DERIVATIVES

There is a procedure for defining the formal partial derivatives of monomials in noncommuting variables. In the present case these monomials will be the defining words of the group of a knot. The definition of the derivative begins with two basic rules, which in turn determine the derivative in general. Fox proved that these rules yield a well-defined operation on the set of words. Note that the derivative of a word will no longer be a single word, but rather a formal sum of words.

1. $(\partial/\partial x_i)(x_i) = 1$, $(\partial/\partial x_i)(x_j) = 0$, $(\partial/\partial x)(1) = 0$.
2. $(\partial/\partial x_i)(w \cdot z) = (\partial/\partial x_i)(w) + w \cdot (\partial/\partial x_i)(z)$, where w and z are words in variables $\{x_j,\ x_j^{-1}\}$.

One immediate consequence is that

$$\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}.$$

This follows from the calculations $(\partial/\partial x_i)(x_i \cdot x_i^{-1}) = (\partial/\partial x_i)(1) = 0$, and, using rule 2, $(\partial/\partial x_i)(x_i \cdot x_i^{-1}) = 1 + x_i(\partial/\partial x_i)(x_i^{-1})$.

EXAMPLE

The partial derivatives of the equation $xyxy^{-1}x^{-1}y^{-1}$ are computed in the following manner. Write the word as $(x) \cdot (yxy^{-1}x^{-1}y^{-1})$ and apply rule 2. To differentiate the second term, write it as $(y)(xy^{-1}x^{-1}y^{-1})$ and use rule 2 again. Proceed in this way, factoring out one term at a time. The final result is that the derivative with respect to x is $1 + xy - xyxy^{-1}x^{-1}$. The derivative with respect to y is $x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$. In the exercises you are called on to fill in the details of this calculation, and to compute some more complicated examples.

As a hint of things to come, note the following about this example. The equation $xyxy^{-1}x^{-1}y^{-1}$ is the defining equation for the group of the trefoil knot. If in the derivative, $1 + xy - xyxy^{-1}x^{-1}$, the variables are both replaced with t, then the polynomial $1 - t + t^2$ results. This is the Alexander polynomial of the trefoil. (Also, if the substitution is made in $x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$, the polynomial $-t^2 + t - 1$ results, which is the same as the first modulo a multiple of $\pm t^i$.)

USING THE FOX CALCULUS TO COMPUTE THE ALEXANDER POLYNOMIAL

Here is a new algorithm for computing the Alexander polynomial of a knot. Take any presentation of the group of the knot found by the procedure outlined in Chapter 5. The presentation will have one more generator than relation. Now form the Jacobian matrix consisting of all the partial derivatives of the equations, and eliminate any one column of the matrix. Substitute t for all the variables that appear. Finally, take the determinant of the matrix that results. This determinant is the Alexander polynomial.

In Chapter 5 it was shown that the group of the knot illustrated in Figure 5.8 was generated by x, y, and z, subject to the relations:

$$r_1 = yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}xy^{-1}x^{-1} = 1,$$
$$r_2 = z^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1} = 1.$$

(Recall that any one of the 3 relations is a consequence of the other 2.) If, in the Jacobian, the column corresponding to $\partial/\partial y$ is eliminated, the resulting matrix is 2×2. As an example, the (1,2) entry is $\partial/\partial z(r_1) = yx^{-1} - yx^{-1}zxy^{-1}xyx^{-1}z^{-1}$. Substituting t for each variable yields $-1+t$. If the other derivatives are computed and t substituted, the resulting matrix is

$$A(t) = \begin{pmatrix} -t^2+4-2 & -t+1 \\ -t+2 & 1-3t^{-1}+t^{-2} \end{pmatrix}$$

Taking the determinant yields an Alexander polynomial $-2t^2+10t-15+10t^{-1}-2t^{-2}$.

WHY THIS WORKS

The proof that this procedure actually produces the Alexander polynomial is fairly long and technical. The basic ideas are easily explained.

To begin, there is the following central observation. One presentation of the knot group is obtained with no algebraic manipulations. For each arc there is a generator and for each crossing there is a relationship. For instance, at a right-hand crossing there is the relation $x_i x_j x_i^{-1} x_k^{-1} = 1$. If the Jacobian matrix for this set of relationships is computed and then t is substituted for all the variables, the resulting matrix is just the matrix used in the combinatorial definition of the polynomial given in Chapter 3. The algebraic manipulations that reduce the number of variables in the presentation correspond to operations on the Jacobian matrix. A careful calculation shows that none of these changes affect the final determinant.

EXERCISES

4.1. The knot 5_1 has knot group

$$\langle x,\, y \mid xyxyxy^{-1}x^{-1}y^{-1}x^{-1}y^{-1}\rangle.$$

Compute its Alexander polynomial.

4.2. Find two generator presentations of the groups of the knots 6_2, 6_3, 7_1, and 7_5. In each case use the presentation to compute the Alexander polynomial.

4.3. Fill in the details of the calculation of the matrix $A(t)$ in this section.

4.4. If a knot diagram has n crossings, there is an n generator presentation of the knot group. Show that if this presentation is used to compute the Alexander polynomial, the result is the same as in the combinatorial calculation in Chapter 3.

CHAPTER 7: NUMERICAL INVARIANTS

A few methods for associating integers to knots have already appeared in the text. The genus is an important example. Others include the signature, the determinant, and the mod p rank. In this chapter many more will be described. Some of these will seem to be very natural quantities to study. Others, such as the degree of the Alexander polynomial, may at first seem artificial; it is the relationship between these invariants and the more natural ones that is particularly interesting and useful.

It will be clear in this chapter that with the introduction of each new invariant a host of questions arises concerning its relationship with other invariants. Some of these questions will be discussed, others will be presented in the exercises. A few open questions will appear along the way.

1 Summary of Numerical Invariants

Several knot invariants have been defined so far. These are reviewed in this section. In the next sections many new invariants will be described.

GENUS

Every knot forms the boundary of an oriented surface called a Seifert surface of the knot. The genus of a knot, $g(K)$, is the minimal genus that occurs among all Seifert surfaces. Only the unknot has genus 0; the pretzel knots form an infinite family of genus 1 knots. The proof of the prime decomposition theorem was based on the result that genus is additive under connected sum.

Another similar notion of genus is based on nonorientable surfaces. This concept plays a secondary role to orientable genus, and will not be pursued.

MOD p RANK

Finding mod p labelings of a knot diagram can be reduced to solving a system of linear equations mod p. The dimension of that solution space is called the mod p rank of the knot. In Exercise 4.6 of Chapter 3 it was shown that, if K has mod p rank n, then the number of mod p labelings is $p(p^n - 1)$. It follows that mod p rank is additive under connected sum. (See Exercise 1.1.)

DETERMINANT, DET(K)

The determinant was first defined combinatorially. However, the simplest definition is based on Seifert matrices. If V is a Seifert matrix for a knot K, then the determinant of K, $\det(K)$, is the absolute value of the determinant of $V + V^t$. Thus, the determinant of the connected sum of knots is the product of their determinants (see Chapter 6).

SIGNATURE, $\sigma(K)$

The Seifert matrix also provides a means of defining the signature of a knot. If V is a Seifert matrix for K, then $\sigma(K)$ is the signature of $V + V^t$. Signature is additive under connected sum. See Exercise 3.6, Chapter 6, for a proof. (ω-signatures can also be defined using V.)

As shown earlier, the right- and left-handed trefoils have signature -2 and 2, respectively. Hence, the connected sum of the two trefoils, called the *square knot*, has signature 0. Connected sums of square knots provide an infinite family of knots with signature 0.

DEGREE OF THE ALEXANDER POLYNOMIAL

Although not yet discussed, this invariant derives easily from the polynomial itself. By multiplying by the appropriate power of t, the Alexander polynomial of a knot can be normalized to have no negative powers of t, and so that the constant term is nonzero. The degree of this polynomial is called the degree of the Alexander polynomial.

The Alexander polynomial of a connected sum of knots is the product of their individual polynomials (see Chapter 6). Hence, the degree of the Alexander polynomial adds under connected sum. An infinite family of knots, all with Alexander polynomial 1 can be constructed from the connected sums of copies of a single nontrivial polynomial 1 knot. Families containing only prime knots also exist.

EXERCISE

1.1. If a knot K has mod p rank n, then the number of mod p labelings is $p(p^n - 1)$. Use this to show that the number of labelings including ones with all labels the same is given by p^{n+1}. Use this to prove that mod p rank adds under connected sum.

2 New Invariants

The two invariants defined in this section are the most natural in the study of knots. Surprisingly, although they are so simple to define their calculation turns out to be especially difficult, and the most natural questions concerning them are unanswered.

CROSSING INDEX, $C(K)$

Each regular projection of a knot has a finite number of double points. Different projections of a knot can have different numbers of double points, since Reidemeister moves 1 and 2 change the number of double points. The least possible number of double points in a projection of a knot is called the crossing index of the knot.

For example, the unknot has crossing index 0. It is fairly easy to see that if a knot has a projection with one or two crossings it is unknotted. Hence there are no knots of crossing index 1 or 2. The trefoil has crossing index 3.

Although there are clearly only a finite number of knots with a given crossing index, listing them all is difficult. The chart of prime knots in the appendix is arranged by crossing index. The number of knots of a given crossing index seems to grow very rapidly, but little is known in detail about this number.

At the present time it is conjectured, but unproven, that the crossing index adds under connected sum. (This has been proved for knots with alternating projections; a knot diagram is alternating if, travelling around the knot, overpasses and underpasses are met alternately. This result for alternating knots is discussed again in Chapter 10.) As a measure of the present state of ignorance, we cannot rule out the possibility that the connected sum of two knots can have crossing number less than either factor!

UNKNOTTING NUMBER, $U(K)$

Given a knot diagram, it is always possible to find a set of crossings such that if each is switched from right- to

left-handed or vice versa, the knot becomes unknotted. One way to discover one set of such switches is to draw a new knot diagram starting with the projection of the knot. Trace the knot *projection* starting at a point p. Each crossing point will be met twice in the tracing, and when it is met for the second time, have that strand go under the first. This is best understood via an example; the result of this construction for a particular knot is illustrated in Figure 7.1. The proof that the algorithm produces an unknot is left to the exercises.

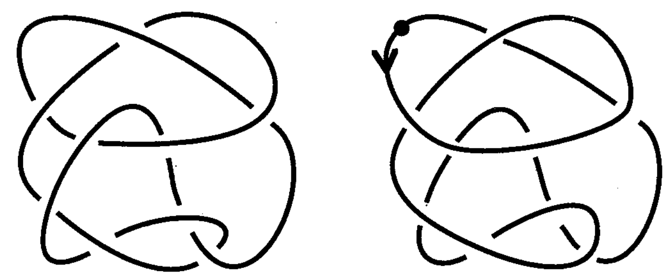


Figure 7.1

For a given knot diagram several different choices of crossing change can lead to the unknot, and the number of crossing changes that are required might depend on the choice of diagram. The minimal number of crossing changes that is required, ranging over all possible diagrams, is called the unknotting number of the knot.

Given that the definition is taken over all possible diagrams, the unknotting number seems difficult to compute, and in general it is. However, only the unknot has unknotting number 0. The n-twisted doubled knots considered in Exercise 2.2 of Chapter 3 (see also Exercise 1.3 of Chapter

6) provide an infinite family of unknotting number 1 knots. They are distinguished by their Alexander polynomials.

How the unknotting number behaves under connected sums is a mystery. It is easily proved that the unknotting number of the connected sum of knots is at most the sum of their unknotting numbers, and the conjecture is that unknotting number is additive. Scharlemann has proved that the connected sum of two unknotting number one knots is always of unknotting number two.

A fascinating example concerning the unknotting number was discovered by S. Bleiler. Figure 7.2 presents two diagrams of the same knot, the second with more crossing than the first. No two crossing changes in the first diagram produces an unknot, but changing the indicated crossings in the second diagram does unknot it.

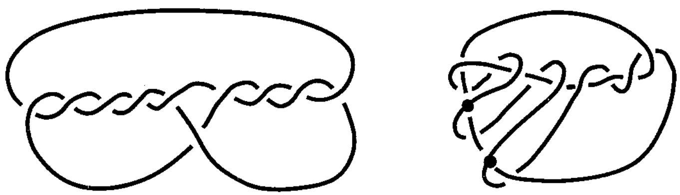


Figure 7.2

Bleiler proved that to demonstrate that the knot has unknotting number 2 the crossing number of the diagram used cannot have the minimal number of crossings for the knot. (Figure 7.2 presents only one minimal crossing diagram of the knot; there conceivably could be more.) The next section includes the needed techniques to prove that this knot has unknotting number ≥ 2.

EXERCISES

2.1. Draw all knot diagrams having 2 crossings.

2.2. Prove that there are only a finite number of n-crossing knots for each integer n.

2.3. Prove that the procedure outlined in the text actually produces an unknotted curve.

2.4. Check that making the indicated crossing changes in Bleiler's example (Figure 7.2) produces the unknot. Show that no two crossing changes in the first diagram gives the unknot.

3 Braids and Bridges

Although somewhat less intuitive then the crossing index and the unknotting number, both of the invariants described in this section have a long history in the study of knots. The study of braids is particularly fascinating in that it introduces group theory into the study of knots in a completely new way.

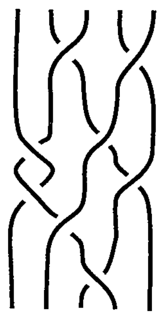


Figure 7.3

BRAIDS

An n-stranded braid consists of n disjoint arcs running vertically in 3 space. The set of starting points for the arcs must lie immediately above the set of endpoints. Figure 7.3 illustrates a 5-braid. A formal definition need not be given, and could be supplied by the reader.

A braid can be turned into a link by attaching arcs to the top and bottom, as illustrated in Figure 7.4. Braids are of interest in the study of knots and links because of a theorem that states that every knot and link arises from a braid in this way. The proof is constructive, as follows.

Draw the knot polygonally, and orient it. Also pick a point in the projection plane which does not lie on the knot. This point will be called the *braid axis*. The goal of the construction is to arrange for every segment of the polygon to run clockwise with respect to the chosen point. If some segment runs counter clockwise, it can be divided up into several smaller segments, each of which can be pulled across the axis. This is illustrated in Figure 7.5. Exercise 2 asks that you apply this algorithm to several knots to draw them as closed braids.

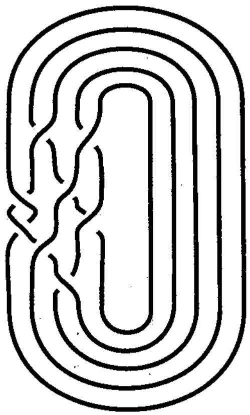


Figure 7.4

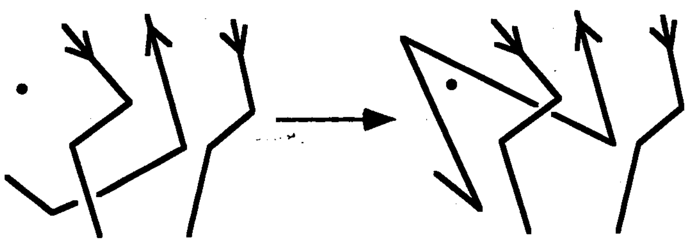


Figure 7.5

Different braids can close to form the same knot; the *braid index* of a knot, denoted $\mathrm{brd}(K)$, is defined to be the minimum number of strands that are required in a braid description of a knot. Braid index is subadditive under connected sum; that is, $\mathrm{brd}(K\#J) \leq \mathrm{brd}(K) + \mathrm{brd}(J)$. To see this, note that given braid descriptions of two knots, there is a simple way to construct a braid description of their connected sum. This is illustrated in Figure 7.6.

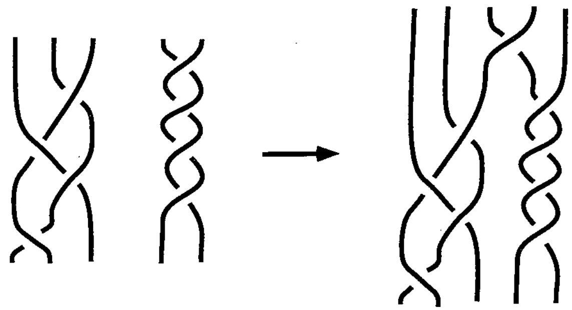


Figure 7.6

Artin introduced braids into the study of knots. What is most fascinating about braids is that there is a natural way to form groups using them. Given two n-stranded braids, placing one on top of the other produces a new braid. This operation induces a group operation on the set of equivalence classes of n-stranded braids, where two braids are equivalent if one can be deformed into the other fixing all endpoints. In the exercises you are asked to derive a few properties of this group, called the *braid group*.

One important theorem in the study of braids deserves notice. As was mentioned, two distinct braids can produce the same knot or link when closed up. For instance, stabilization, as indicated in Figure 7.7, does not affect the resulting link. Also, if a given braid is multiplied on the right and left by a second braid and its inverse (in the braid group) the resulting links are the same. This operation is called conjugation in the braid group. A theorem

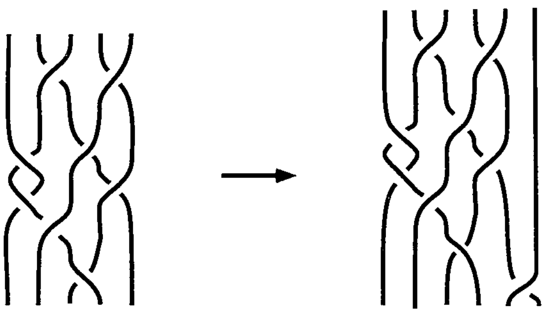


Figure 7.7

of Markov states that if two braids give the same knot or link, then each can be repeatedly stabilized and conjugated so that the same braid results. This theorem, along with a knowledge of the structure of the braid group, was crucial for Jones' discovery of new polynomial invariants of knots. More on that later.

BRIDGE INDEX, brg(K)

Any projection of a knot can be perturbed so that there are a finite number of relative maxima. Figure 7.8 illustrates a knot with the maxima and minima marked. You can prove that the number of minima equals the number of maxima. Different diagrams of a knot can certainly have a different number of maxima. The minimum number of such maxima (taken over all possible projections) is called the *bridge index* of the knot, denoted brg(K).

It should be clear that only the unknot has bridge index 1. Hence the bridge index of the trefoil is two, as can be seen in its standard projection.

The first 3-bridge knot in the table of prime knots is 8_5. A theorem proved by Schubert states that the bridge index behaves nicely under the connected sum operation.

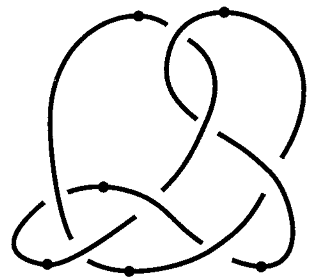


Figure 7.8

□ **THEOREM 1.** *For knots K and J,* $\text{brg}(K\#J) = \text{brg}(K) + \text{brg}(J) - 1$.

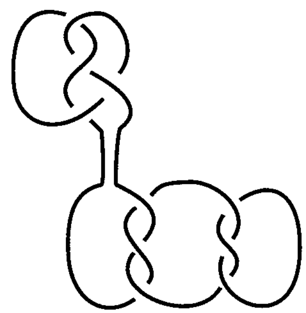


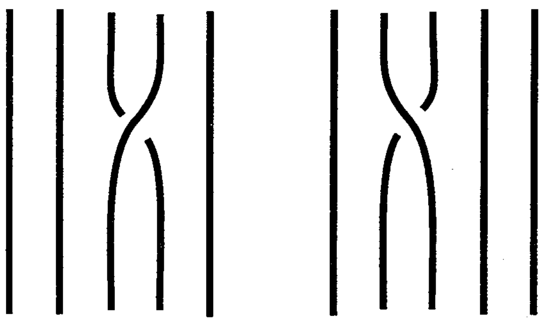
Figure 7.9

The proof is quite difficult. One step is demonstrated easily in a diagram; the bridge index satisfies the inequality $\text{brg}(K\#J) \leq \text{brg}(K) + \text{brg}(J) - 1$. Figure 7.9 illustrates the connected sum of a 2-bridge knot and a 3-bridge knot drawn so that it has 4 bridges.

A simple corollary of the Schubert theorem is that 2-bridge knots are prime (See Exercise 3.3.) Even this is a difficult geometric exercise without the aid of Schubert's general result.

EXERCISES

3.1. The n stranded braid group is generated by the twists σ_i which put a half twist between the i-th and $(i+1)$-th strand, as indicated in Figure 7.10 below. Show that the two relations hold: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and $\sigma_j\sigma_i = \sigma_i\sigma_j$, $|i-j| > 1$. (In fact, these two sets of relations generate all the relations in the braid group.)



σ_3 σ_2^{-1}

Figure 7.10

3.2. Draw the knots 4_1 and 5_2 as closed braids.

3.3. How does Theorem 1 imply that 2-bridge knots are prime?

3.4. Any 2-bridge knot can be drawn with one strand straightened and not crossing any of the other strands, as illustrated in Figure 7.11 below. Describe a method for converting a 2-bridge diagram into this form. (With this observation the classification of 2-bridge knots can be stated. Any 2-bridge knot is determined by a sequence of integers, $[c_1, c_2, \ldots, c_n]$, where c_i is the number of right- or left-handed twists, depending on i odd or even.)

The knot illustrated to the left corresponds to [2,2,3]. To such a sequence one can form the continued fraction,

$$\frac{p}{q} = c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \cdots}}.$$

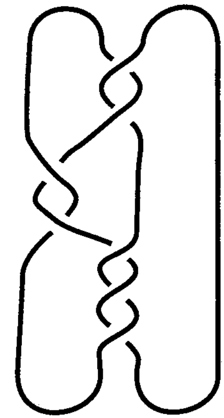


Figure 7.11

Now Schubert proved that two 2-bridge knots, with corresponding fractions p/q and p'/q', are equivalent if and only if $p = p'$ and $q - q'$ is divisible by p.)

3.5. Apply your algorithm from Exercise 3.4 above to illustrate the knots 7_3 and 8_2 in standard form. What are the associated fractions for each?

3.6. How does the continued fraction corresponding to a 2-bridge knot compare to that of its mirror image? Which two bridge knots are equivalent to their mirror images?

4 Relations between Numerical Invariants

Many of the numerical invariants studied so far are closely related. For instance, the combinatorial algorithm for computing Alexander polynomials immediately implies that the degree of the Alexander polynomial is less than the crossing number. Hence, the $(2,n)$-torus knot cannot be drawn with fewer than n crossings; the degree of its polynomial was discussed in Chapter 3, Section 5, and shown to be $n-1$. This section will focus on demonstrating a few of the less obvious connections.

The next section will deal with the independence of some of the invariants.

THE CROSSING NUMBER AND THE GENUS

Recall that Seifert's algorithm provides a means of building a Seifert surface for a knot from its diagram. In Exercise 3.4 of Chapter 4, it was show that the genus of the resulting Seifert surface is given by $2g = cr - s + 1$, where cr is the crossing number of the diagram and s is the number of Seifert circles. Unless K is unknotted, $s > 1$, so $2g \leq cr - 1$. For the trefoil knot, $2g = cr - 1$.

BRIDGE INDEX AND MOD p RANK

Any mod p labeling of an n-bridge knot is determined by the labels on the n top arcs, or bridges. Hence, there can be at most an n-dimensional space of labelings. Taking into account the 1-dimensional space of trivial labelings, one has that the mod p rank of a knot is at most the $\text{brg}(K) - 1$. As an application, the (3,3,3)-pretzel knot has mod 3 rank 2, and so cannot be drawn with 2 bridges. It is clearly a 3-bridge knot.

SIGNATURE AND THE UNKNOTTING NUMBER

Arguments concerning the unknotting number are much more difficult. The result here states that $2u(K) \geq |\sigma(K)|$. The proof depends on showing that changing a crossing in a knot changes the signature by at most 2.

Fix a knot diagram and a crossing in the diagram. If Seifert's algorithm is applied to the diagram the resulting Seifert surface is built from many disks and the given crossing corresponds to a band joining two of the disks. To find the Seifert matrix the surface must be deformed into a single disk with bands added. For the calculation this must be done in such a way that the given band corresponds to a single band on the final surface.

To see that this is possible, cut the Seifert surface across the band of interest. The remaining surface can be assumed to be connected. (Why?) Deform it into a single disk with bands added. The original Seifert surface can be recovered by reattaching the band that was cut to the disk. Order the bands so that this final band is the last in the ordering.

Changing the crossing of interest will have the effect of twisting the last band. This will in turn only affect the last diagonal entry of the Seifert matrix, V. Hence, the diagonalization of $V + V^t$ only changes in its last entry, and the signature can change by at most 2. The signature of Bleiler's example is 4, and this is how he proves it does not have unknotting number 1.

MOD p RANK AND UNKNOTTING NUMBER

In general the unknotting number is at least as large as the mod p rank, for all p. All that will be proved here is that unknotting number 1 knots have mod p rank ≤ 1. The reader should interpret the statement and argument in terms of colorings. (Colorings are often used in expository talks on knot theory to prove that the trefoil is not unknotted. The following argument translates into an easy proof of the much subtler fact that the square knot cannot be unknotted with a single crossing change, regardless of how it is drawn.)

Suppose that a knot K has unknotting number 1, and fix a diagram for K and the crossing which changes K into an unknot when reversed. If there is a nontrivial labeling of K for which both the over and undercrossings are labeled 0 a contradiction arises. The given labeling remains consistent when the crossing is changed, yielding a nontrivial labeling of the unknot.

If the knot has mod p rank > 1, then there are two linearly independent labelings, both of which are 0 on the overcrossing. Neither can be 0 on the undercrossing by the previous argument. However subtracting some multiple of one labeling from the other yields a labeling with the bottom label 0. (Recall that the multiple is taken mod p.) The new labeling is nontrivial by linear independence.

EXERCISES

4.1. Prove that for any knot K, the degree of the Alexander polynomial is at most twice the genus.

4.2. Prove that the $|\sigma(K)| \leq 2g(K)$.

4.3. (a) Prove that the bridge index of a knot is at most equal to the braid index.

(b) Find an example of a 2-bridge link that has braid index greater that 2. (Linking numbers should help here.) Find a similar example of a knot.

4.4. (a) Prove that for n even, an n-crossing knot has genus at most $(n-2)/2$.

(b) Prove that if K has crossing number n, with n odd, then either K is a $(2,n)$-torus knot, or K has genus at most $(n-3)/2$. (The torus knot has genus $(n-1)/2$.)

5 Independence of Numerical Invariants

While some numerical invariants are closely related, others are completely independent. In most cases, this is demonstrated by constructing families of examples. Some of the families of examples are constructed from a few basic examples and connected

sums. Others are much more complicated. Here a few will be surveyed, with the main focus on bridge index.

BRIDGE INDEX AND THE DEGREE OF THE ALEXANDER POLYNOMIAL

There is no relationship between the degree of the Alexander polynomial and the bridge index of a knot. The $(2,n)$-torus knots provide examples of two bridge knots with arbitrarily high degree Alexander polynomial. On the other hand, by forming the connected sum of many polynomial 1 knots, a polynomial 1 knot with large bridge index is created.

INDEPENDENCE OF *mod p* RANKS

The trefoil knot has mod 3 rank 1 and mod 5 rank 0; the (2,5)-torus knot has mod 3 rank 0 and mod 5 rank 1. Hence, the connected sum of k trefoils and j 5-twist knots has mod 3 rank k and mod 5 rank j. It follows that in general there is no relationship between the mod 3 and mod 5 ranks.

Given any finite set of primes, similar examples can be constructed showing the independence of mod p ranks. Note that it is not possible to find a knot with specified mod p ranks for all primes. For a given knot only a finite number of the mod p ranks are positive. The determinant of a knot provides a bound on the number of primes p for which the mod p rank can be positive. Exercise 5.1 asks for a precise bound.

SIGNATURE AND BRIDGE INDEX

The $(2,n)$-torus knot knot has signature $n-1$, and is a two-bridge knot. (See Exercise 3.9, Chapter 6) Hence no bound on the signature can be based on the bridge index. On the other hand, the connected sum of square knots has 0 signature, but large bridge index, so no bound on the bridge index follows from the signature.

UNKNOTTING NUMBER AND THE BRIDGE INDEX

The $(2,n)$-torus knots give a family of 2-bridge knots with arbitrarily high unknotting number. (Consider the signature.) The process of doubling a knot, as illustrated in Figure 7.12, produces unknotting number 1 knots of large bridge index.

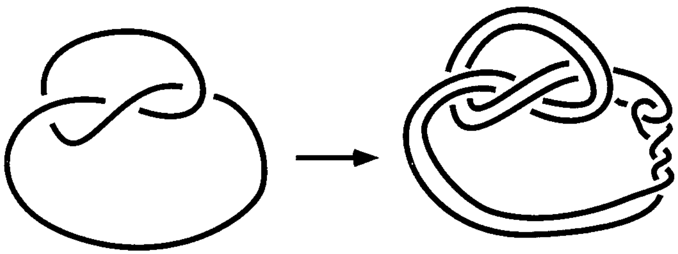


Figure 7.12

Schubert proved that if a knot is doubled the bridge index of the resulting knot is twice that of the original knot, except in one special case. (See Exercise 5.3.) It is clear that the bridge index of a doubled knot is at most twice that of the original knot, but showing that there is an equality is a lengthy and delicate geometric argument.

Without that delicate geometry, it is possible to prove that certain doubled knots have high bridge index, using the algebraic methods of Chapter 5, specifically labelings from the symmetric group, S_n. One part of the argument is based on the following theorem.

□ **THEOREM 2.** *If a knot K can be labeled with transpositions from S_n then* $\text{brg}(K) \geq n-1$.

PROOF

Given such a labeling of K, the set of labels generates S_n. However, the labels on the bridges determine all the other labels, as was seen in Chapter 5. Hence, the labels that occur on the bridges must generate S_n. According to Exercise 1.8 of Chapter 5, S_n cannot be generated by fewer than $n-1$ transpositions. The result follows. □

To apply this to the construction of examples, suppose that one starts with a knot diagram that has been consistently labeled with 3-cycles from S_n. (It is not required, or for that matter even possible, for the labels to generate S_n.) This labeling leads to a consistent labeling of some double of the knot using transpositions, as follows: On the bridges of the knot, if the original arc was labeled with the 3-cycle (a,b,c), label the two strands with (a,b) and (a,c). The consistency condition leads to a labeling of the rest of the doubled knot. Any problem with consistency at the bottom can be cured by adding twists.

It may not be immediately clear why a consistent labeling occurs in general. The following observations should clarify the situation. The two transpositions on a parallel pair of strands on a bridge were chosen so that their product is the 3-cycle with which the original strip was labeled. When the consistency condition is used to determine the rest of the labels, this property for adjacent pairs of labels is true everywhere. That is, the labels on any parallel pair of arcs have product equal to the 3-cycle that the original arc of the knot was labeled with. It is now easily checked that along the bottom strands, if the labels do not match up, twists can be added to the pair of strands so that they do match.

The discussion above shows how, given a knot which is consistently labeled with 3-cycles from S_n, it is possible to produce some double of the knot which can be consistently labeled with transpositions from S_n. These transpositions will generate S_n if the original set 3-cycle labels formed a *transitive* set. (A set of permutations is called transitive if for every positive integer $i \leq n$, some product of elements in the set maps 1 to i.) The proof of this algebraic condition is left to the reader as another exercise concerning the symmetric group. The construction is completed by noting that the connected sums of k (2,5)-torus knots can be consistently labeled with a transitive set of 3-cycles from S_{3+2k}. Hence, an explicit example is constructed by forming the connected sum of k (2,5)-torus knots, consistently labeled with a transitive set of 3-cycles from S_{3+2k}.

GENUS AND THE BRIDGE INDEX

The $(2,n)$-torus knots provide examples of 2-bridge knots of arbitrarily high genus. On the other hand, doubled knots have genus 1. Figure 6.5 illustrates a genus one surface bounded by a double of the unknot; the right-hand band on that surface can itself be knotted so that the resulting surface forms a genus 1 Seifert surface for an arbitrary doubled knot. It was just shown that doubled knots can have arbitrarily large bridge index.

EXERCISES

5.1. The number of primes for which a knot can have nontrivial mod p labelings is bounded by a function of the determinant. Find one such bound.

5.2. Why do doubled knots all have unknotting number 1?

5.3. Find the example of a double of a knot for which the bridge index is not twice the bridge index of the original knot.

5.4. Check the details of the construction of the labeling of a doubled knot with transpositions, given a 3-cycle labeling of the knot being doubled. In particular, check that consistency can be assured by adding the appropriate twists at the bottom.

5.5. Show that the connected sum of k (2,5)-torus knots can be labeled with 3-cycles from S_{3+2k} so that the set of labels form a transitive set.

5.6. Figure 7.13 illustrates a genus 3 Seifert surface. Show that its boundary has unknotting number 1. Show that its Alexander polynomial is of degree 6, and hence the knot is exactly genus 3. Generalize this example to find unknotting number 1 knots of arbitrarily large genus. It is more difficult, but possible, to show that there are genus 1 knots of high unknotting number.

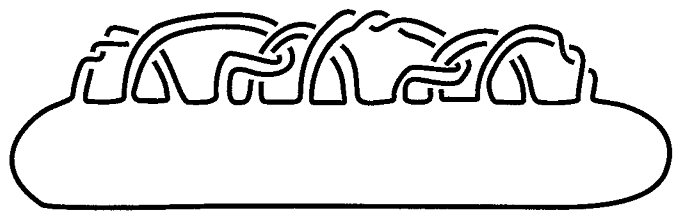


Figure 7.13