CHAPTER 8: SYMMETRIES OF KNOTS

Knot diagrams can appear symmetrical, and for those that do not, the lack of symmetry is often an artifact of the diagram, and is not inherent in the knot itself. For instance, Figure 8.1 presents two diagrams for the knot 7_6. The first shows no apparent symmetry, while the second is quite symmetrical; a rotation of 180 degrees about a point in the plane leaves the diagram unchanged. As the example indicates, finding symmetrical diagrams for a knot can be a challenging task. On the other hand, powerful tools are available for proving that a knot does not have hidden symmetries.

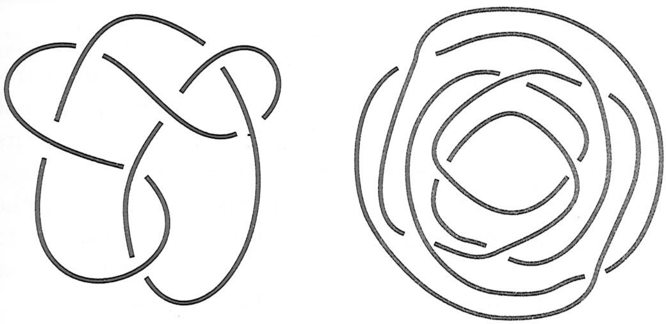

Figure 8.1

Section 1 expands on some of the basic types of symmetry discussed earlier. (For example, it was shown that the trefoil is distinct from its mirror image using the signature; the relationship between a knot and its mirror image will be discussed further.) The rest of the chapter is devoted to another type of symmetry, *periodicity*; roughly stated, a knot is called periodic if it has a diagram that is carried back to itself when rotated about the origin; Figure 8.1 shows that 7_6 is periodic, with period 2.

The two main results of the chapter are theorems of Murasugi and Edmonds. The first places algebraic restrictions on the Alexander polynomials of periodic knots. The second restricts their Seifert surfaces. Together these two theorems provide powerful means for studying the periods of knots. The examples in the final section will demonstrate the beautiful and subtle interplay between geometry and algebra.

1 Amphicheiral and Reversible Knots

Given an oriented knot, K, reversing the orientation creates a new oriented knot called its reverse, and denoted K^r. Changing all of its crossings yields an oriented knot denoted K^m. In Chapter 2, Exercise 5.6 asked you to prove that changing the crossings in a diagram for K yields a knot equivalent to the mirror image of K, corresponding to the reflection of its diagram through the y-axis of the knot diagram.

□ **DEFINITION.** *An oriented knot K is called reversible if K is oriented equivalent to K^r. It is called positive amphicheiral if it is oriented equivalent to K^m, and negative amphicheiral if it is oriented equivalent to K^{rm}.*

EXAMPLES

Figure 8.2 illustrates that a 180 degree rotation about the y-axis carries the knot 4_1 (the figure-8 knot) to itself, but reverses its orientation. Hence it is reversible. The reader should have no trouble showing that if all the crossings are changed, the resulting knot can be deformed to appear again as in the diagram. This shows that the figure-8 is amphicheiral, and, since it is reversible, it is both positive and negative amphicheiral. (See Exercise 1.1.)

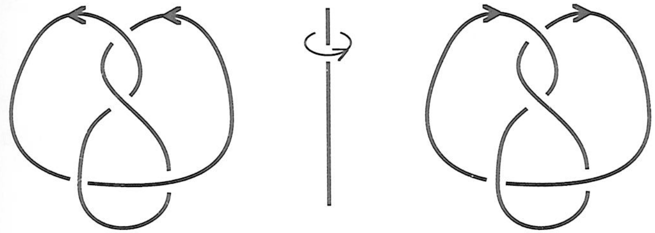

Figure 8.2

Figure 8.3 illustrates the (3,5,3)-pretzel knot. It too is reversible; rotate it 180 degrees about the vertical axis in the diagram. It is now known that the only reversible pretzel knots are those with two of the bands having an equal number of twists. A signature calculation shows that this pretzel knot is neither positive nor negative amphicheiral. (It follows from Exercise 1.8 of Chapter 6 that the signature of a knot and its mirror image are nega-

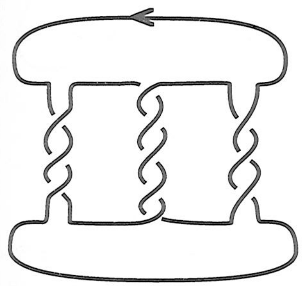

Figure 8.3

tives.) Finding knots that display one, but not both, forms of amphicheirality is at least as difficult as constructing nonreversible knots.

STRONG SYMMETRY

Although reversible and amphicheiral knots contain symmetries, the symmetry may be hidden. That is, it may be the case that the symmetry cannot be displayed in a diagram. In particular, some knots are reversible, but the reversal cannot be carried out in a simple manner as in the previous examples.

□ **DEFINITION.** *A knot is called strongly reversible if it is equivalent to a knot that is carried to its reverse by a 180 degree rotation about the y-axis.*

If the standard diagram for the (3,5,3)-pretzel knot is rotated by 180 degrees about the y-axis, then the representative for the knot is clearly fixed. On the other hand, the connected sum of the left- and right-handed trefoils (see Figure 8.4) is not invariant under that rotation; show it is strongly reversible nonetheless.

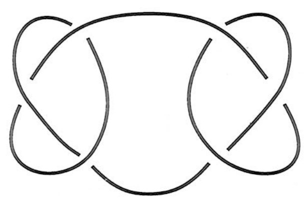

Figure 8.4

It was once conjectured that a reversible knot is necessarily strongly reversible. This is now known to be false. The double of a knot is always reversible, as the reversal can be carried out inside a torus, as illustrated in Figure

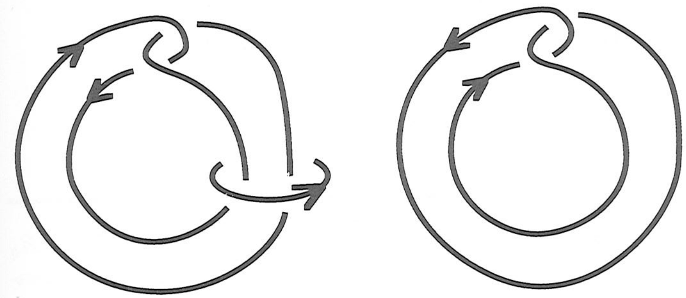

Figure 8.5

8.5. However, Whitten proved that for a double of a knot to be strongly reversible, the original knot itself has to be reversible. The proofs depend on difficult geometric constructions.

There are also similar notions of strong amphicheirality. A knot K is called *strongly positive amphicheiral* if there is a self-map T of 3-space with T^2 = identity, such that $T(K) = K^m$. Similarly K is called *strongly negative amphicheiral* if there is such a T with $T(K) = K^{rm}$. As our only example, the connected sum $K \# K^{rm}$ is strongly negative amphicheiral. Such a connected sum is illustrated in Figure 8.4. Let T be rotation by 180 degrees about the y-axis. The effect of T is the same as changing all the crossing in the diagram. As with reversibility, examples exist demonstrating the distinction between the various notions of amphicheirality.

EXERCISES

1.1. Prove that for reversible knots, being positive amphicheiral is equivalent to being negative amphicheiral.

1.2. (a) Verify that 6_3 is amphicheiral.

(b) Show that 6_3 is reversible.

1.3. Verify that the second knot in Figure 8.1 is 7_6.

2 Periodic Knots

For any integer $q \geq 2$, let R_q denote the linear transformation of R^3 consisting of a rotation about the z-axis of $360/q$ degrees. For any knot K, the diagrams for K and $R_q(K)$ differ by a rotation of $360/q$ degrees about the origin.

□ **DEFINITION.** *A knot K is called periodic with period q if K has a diagram which misses the origin and which is carried to itself by a rotation of* $360/q$ *degrees about the origin.*

Figure 8.6

The diagram in Appendix 1 for the trefoil, 3_1, displays its 3-fold symmetry; the trefoil is periodic of period 3. Similarly, the diagrams of 5_1 and 7_1 show that they have periods 5 and 7, respectively. Figure 8.6 is another diagram of 5_1, showing that it is also a period 2 knot. The first diagram in the chapter, Figure 8.1, displayed 7_6 as a period 2 knot, although no symmetry at all is evident in the figure in Appendix 1. The reader should scan through the appendix and identify the clearly periodic diagrams.

The quotient knot and linking numbers

Given a periodic diagram for a knot, there is a simple procedure for constructing a simpler knot, called the *quotient knot*.

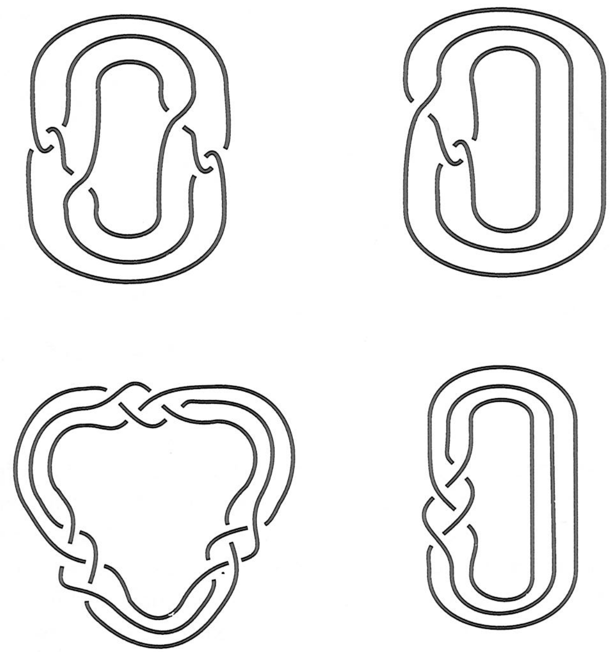

Figure 8.7

In Figure 8.7, two periodic knots and their quotients are drawn. Knots of period 2 and 3 are drawn on the left. Their respective quotients are drawn on the right. Note that for the first the quotient is itself unknotted, and for the second the quotient is the Figure-8 knot.

The construction just given can be reversed: given a knot diagram that misses the origin and an integer $q \geq 2$, one can construct a knot, or link, having the original knot as a quotient. Figure 8.8 illustrates a case in which this so-called *covering link* has more than one component. Deciding whether or not the covering link is a knot calls for the introduction of linking numbers into the study.

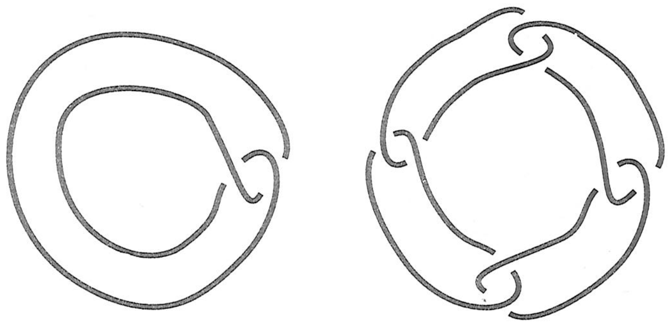

Figure 8.8

Given a diagram for a knot which misses the origin, choose an orientation. Also, pick a ray from the origin such that none of the points of intersection of the ray and knot are tangential. (For a polygonal knot, choose the ray so that it misses all the vertices of the knot.) The *linking number* of the diagram with the z-axis, to be denoted λ, is computed as the absolute value of the intersection number of the knot with the ray. The intersection number is the number of intersection points at which the knot crosses the ray in the clockwise direction minus the number of counterclockwise intersections. For the knot diagram in Figure 8.1, $\lambda = 5$. For the knots in Figure 8.7, the linking numbers are $\lambda = 1$ and $\lambda = 3$. For the knot in Figure 8.8, $\lambda = 0$.

If a knot diagram is periodic, it is easily seen that the linking number of the knot with the z-axis is the same as the linking number of the quotient with the z-axis. (See Exercise 2.6.) Conversely, if a periodic diagram for a knot arises from the covering construction, the linking numbers are the same. It remains to determine when the covering link is a knot.

□ **THEOREM 1.** *If a knot diagram for K misses the origin, the corresponding q-fold covering link L has a single component if the linking number is relatively prime to q. More generally, the number of components in L is the greatest common divisor of the linking number λ and q.*

PROOF

Observe that neither changes in crossings nor deformations that do not cross the origin affect the linking number or the number of components in the cover. Such deformations determine periodic deformations of the covering link (these are called *lifts* of the deformation on the quotient), and crossing changes clearly have no effect on the algorithm that computes the linking number.

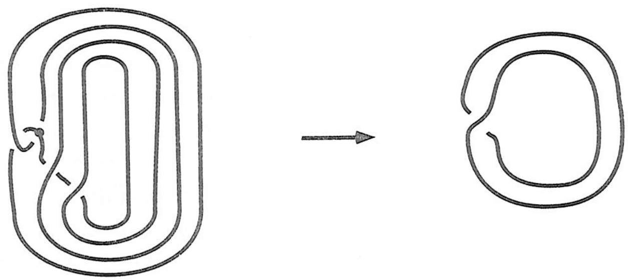

Figure 8.9

Now, by an appropriate sequence of crossing changes and deformations, the knot diagram can be transformed into one that runs monotonically around the axis. Crossing changes are used to eliminate any clasps that occur. This is illustrated in Figure 8.9; after changing the indicated crossing, a deformation (that does not cross the origin) results in a knot diagram that runs clockwise about the origin. Denote the new knot by K'.

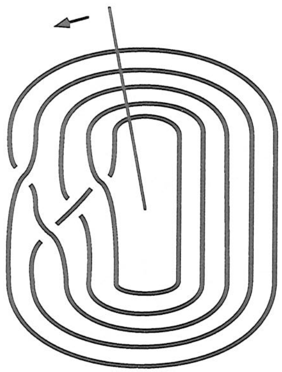

Figure 8.10

Pick a ray from the origin meeting the knot in λ points, and label the points with integers from 1 to λ. Given any point of intersection on the ray, a new point is determined by travelling once around the origin along K'. Hence, a permutation ρ in S_λ is determined. For the knot illustrated in Figure 8.10, $\rho = (13452)$.

Next observe that as K' is connected, the corresponding permutation is a λ-cycle. In general, K' would have 1 component for each cycle in a decomposition of ρ as a product of disjoint cycles, including 1-cycles.

The cover of K', say L', similarly corresponds to a permutation, ρ', and it is easily seen from the construction that $\rho' = \rho^q$. Now if q is relatively prime to λ then the q-th power of a λ-cycle is again a λ-cycle. More generally, the q-th power of a λ-cycle is the product of d disjoint λ/d cycles, where d is the greatest common divisor of q and λ. Proving this is one more exercise concerning the symmetric group. □

Note that different periodic diagrams of a given knot can have different linking numbers. The trefoil has a periodic diagram of period 3 and linking number 2. It also has a periodic diagram of period 2 and linking number 3, as is shown in Figure 8.11. (A consequence of results of the next section imply that, for a given knot, any two diagrams of the same period also have the same linking number.)

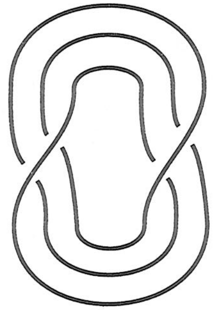

Figure 8.11

EXERCISES

2.1. Figure 8.1 shows that 7_6 can be described as the closure of the square of a 5-strand braid. Show that the same is true for 6_3. The resulting periodic diagram of 6_3 will have 8 crossings.

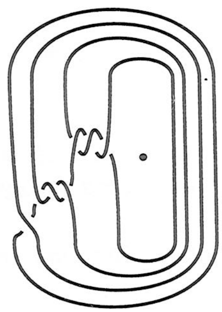

Figure 8.12

2.2. Find 2 crossing changes that convert the knot illustrated in Figure 8.12 into a braid about the origin.

2.3. The braid that results from the crossing changes in Exercise 2 determines a cyclic permutation. Find it.

2.4. Does the statement of Theorem 1 hold when the linking number is 0? Recall that the greatest common divisor of 0 and q is q.

2.5. In the definition of period, it was required that the knot diagram misses the origin. Why is this relevant only in the case of period 2?

2.6. Show that the linking number of a periodic knot with the z-axis is the same as the linking number for the quotient knot.

3 The Murasugi Conditions

Murasugi gave simple but powerful criteria for testing a knot for possible periods; these criteria were based on the Alexander polynomial. He discovered that if a knot has a periodic diagram, then the Alexander polynomial of the knot and its quotient are closely related.

Suppose that a knot K has period $q = p^r$, with p prime. Let J denote the quotient knot of a period q diagram of K, and let λ be the linking number of J with the axis.

□ **THEOREM 2.** (*Murasugi Conditions*) (1) *The Alexander polynomial of J, $A_J(t)$, divides the Alexander polynomial of K, $A_K(t)$.*

(2) *The following* mod p *congruence holds for some integer i:*

$$A_K(t) = \pm t^i (A_J(t))^q (1 + t + t^2 \cdots + t^{\lambda-1})^{q-1} \pmod{p}.$$

PROOF

The proof of these congruences consists of a lengthy and clever argument in matrix manipulation. Although the details cannot be presented, the idea is fairly simple.

To compute the Alexander polynomial one begins with a labeling of the knot diagram. If the diagram is periodic the labeling can also be chosen to be periodic. For example, if in the quotient knot an arc is labeled x_i, in the covering knot the various lifts of that arc can be labeled $x_i^1, x_i^2, \ldots, x_i^q$. Hence, the corresponding Alexander matrix decomposes into blocks corresponding to the sets $\{x_i^1\}$, $\{x_i^2\}, \ldots, \{x_i^q\}$. The individual blocks are closely related to the Alexander matrix of the quotient knot. It is perhaps not surprising that the determinant of the large matrix is related to the q-th power of the determinant of the quotient knot. The details of the proof consist of a careful study of the relationship. □

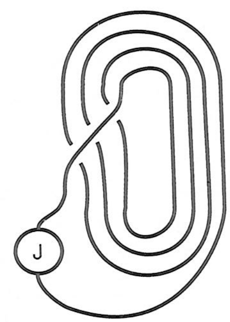

Figure 8.13

One comment about the second condition offers a little insight. The simplest construction of a period q, linking number λ, knot with quotient J is given by lifting the diagram in Figure 8.13. The covering knot consists of a (q,λ)-torus knot with q copies of J added on. Condition 2 states that any period q knot with the same quotient and linking number has the same polynomial as this basic example, modulo p. Essentially, changes in the diagram of the quotient only change the polynomial of the covering by multiples of p.