Basic vector analysis

Cartesian

$\vec{r} = r$ "$vector$"
$\hat{r} = r$ "unit vector" or the direction of $\vec{r}$
r= magnitude of $\vec{r}$
$\vec{r} = (X\hat{\imath} + Y\hat{\jmath} + Z\hat{k})$
$\overrightarrow{r'} = (X'\hat{\imath} + Y'\hat{\jmath} + Z'\hat{k})$
$\vec{\mathcal{R}} = ((X - X')\hat{\imath} + (Y - Y')\hat{\jmath} + (Z - Z')\hat{k})$

Always Start with your r and r' definitions in Cartesian because cylindrical and spherical coordinates do not add and subtract easily.

Cylindrical

$\vec{r} = (s, \varphi, z)$
s= $\sqrt{x^2 + y^2}$
$\varphi = tan^{-1}(\frac{y}{x})$
$z = z$

Use cylindrical coordinates in cases where there is a line, ring, disk, or cylinder of charge

Sphrerical

$\vec{r} = (r, \Theta, \varphi)$
$r = \sqrt{x^2 + y^2 + z^2}$
$\Theta = tan^{-1}\sqrt{x^2 + y^2}/z$
$\varphi = tan^{-1}(\frac{y}{x})$

Use spherical in cases where there is a point charge, shell of charge, or sphere of charge.

Symmetry

In many problems certain axis and parameters can be eliminated via symmetric arguments

Symmetry involves a rotation, translation, or reflection about an axis resulting in the shape looking identical.

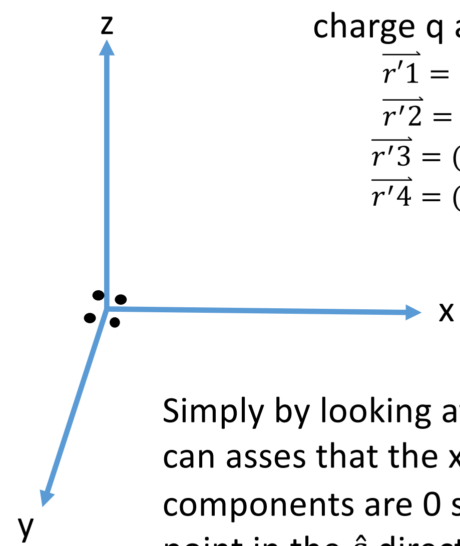
Symmetry can also result in components cancelling which can make a problem easier to deal with

Basic Electric field symmetries assuming constant charge density:

Sphere: variable in r
Infinite line: variable in s
Finite Cylinder: dependant on s and z
Infinite plane: constant

Ex: Find the direction of an E field of 4 points of charge q at positions

$$\overrightarrow{r'1} = (a\hat{\imath} + b\hat{\jmath} + 0\hat{k})$$
$$\overrightarrow{r'2} = (a\hat{\imath} - b\hat{\jmath} + 0\hat{k})$$
$$\overrightarrow{r'3} = (-a\hat{\imath} - b\hat{\jmath} + 0\hat{k})$$
$$\overrightarrow{r'4} = (-a\hat{\imath} + b\hat{\jmath} + 0\hat{k})$$



Simply by looking at the problem we can asses that the x and y components are 0 so the E field must point in the $\hat{s}$ direction

Brute Force $\vec{E}$ Field Equation

$\vec{r} = r\hat{r}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \rho \frac{\vec{\mathcal{R}}}{\mathcal{R}^3}\, d\boldsymbol{\tau}'$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \rho \frac{\hat{\mathcal{R}}}{\mathcal{R}^2}\, d\boldsymbol{\tau}'$$ 3D shape

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \sigma \frac{\hat{\mathcal{R}}}{\mathcal{R}^2}\, da'$$ 2D shape

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \lambda \frac{\hat{\mathcal{R}}}{\mathcal{R}^2}\, dl'$$ 1D shape

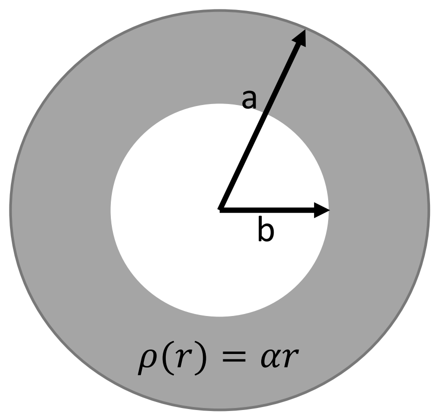
Gauss's Law Integral form $$\oint E(r) \cdot dA = \frac{1}{\varepsilon_0}\int \rho\, d\tau$$

Standard E Field Solutions

Point Charge $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

Infinite Plane $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

Cylinder/Line $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$



Spherical surface (Ex):

$$\vec{E} = \frac{\alpha(a^4 - b^4)}{4\varepsilon_0 r^2}\hat{r}$$

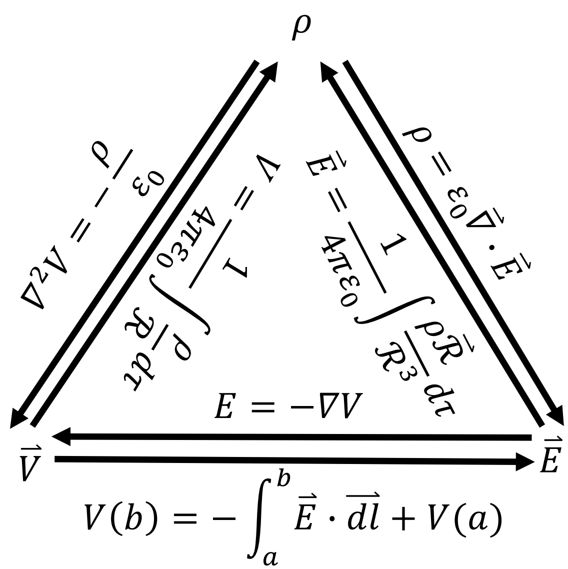
Electric Potentials

$E = -\nabla V$

$$V(b) = -\int_a^b \vec{E} \cdot \vec{dl} + V(a)$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r)}{\mathcal{R}} d^3$$



Conductors

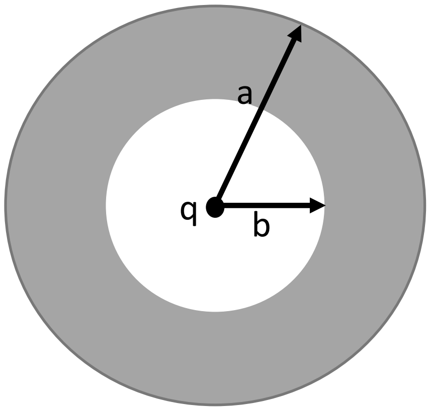
In a conductor E field is 0.

Outside a conductor the E field mimics a point charge and is perpendicular to the surface.

Net charge density is 0.

Any charge density is confined to the surface.

$$Q_{enc} = 0$$
$$V_{outside} = \frac{q}{4\pi\varepsilon_0 r}\hat{r}$$
$$V_{inside} = \frac{q}{4\pi\varepsilon_0 b}\hat{r}$$



Work and Energy

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i)$$

$$W = \frac{1}{2}\int \rho(r)V(r)d^3r$$

$$U_E = \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

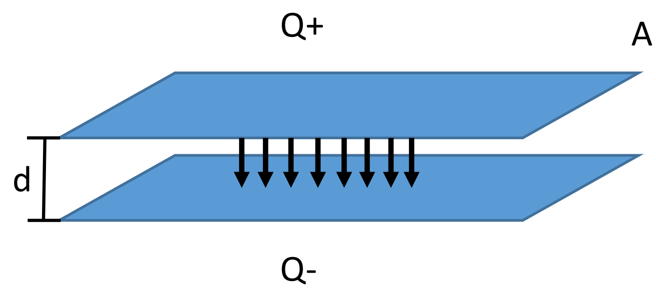
$$W = U_2 - U_1$$

Capacitors

$$Q = CV$$

$$Q = V \times \frac{A\epsilon_0}{d}$$

$$\mathrm{U} = \frac{\epsilon_0}{2}\left(\frac{Q^2}{A^2{\epsilon_0}^2}\right)(Ad) = \frac{1}{2}\frac{Q^2 d}{A\epsilon_0} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$



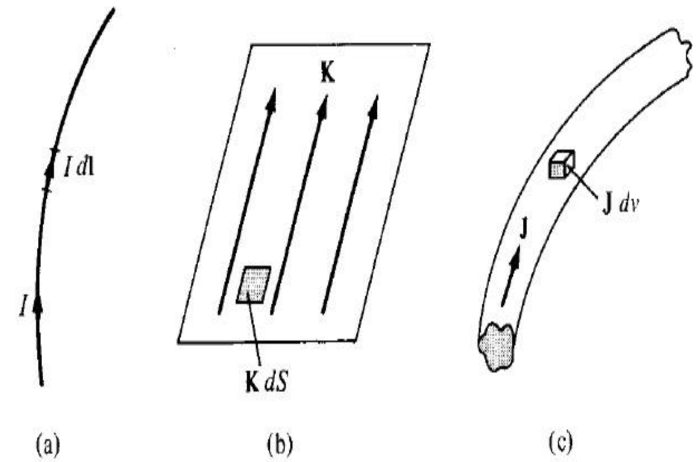
Magnetostatics

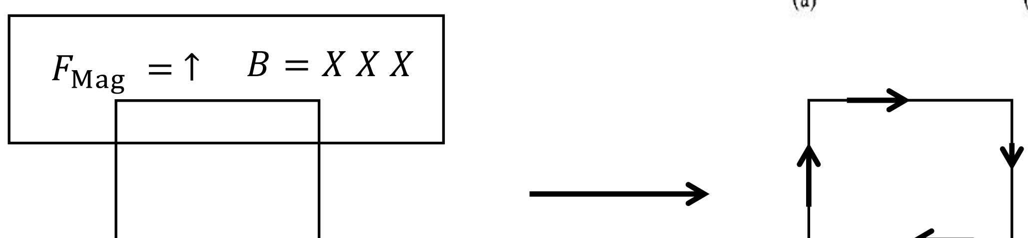
$$\boldsymbol{F}_{\mathbf{Mag}} = q(\boldsymbol{v} \; x \; \boldsymbol{B}) \quad \text{(Point charge)}$$

$$\boldsymbol{F}_{\mathbf{Mag}} = \int (\boldsymbol{v} \; x \; \boldsymbol{B})\lambda \; \mathrm{d}l = \int (\boldsymbol{I} \; x \; \boldsymbol{B}) \; \mathrm{d}l$$

$$\boldsymbol{F}_{\mathbf{Mag}} = \int (\boldsymbol{v} \; x \; \boldsymbol{B})\sigma \; \mathrm{d}a = \int (\boldsymbol{K} \; x \; \boldsymbol{B}) \; \mathrm{d}a$$

$$\boldsymbol{F}_{\mathbf{Mag}} = \int (\boldsymbol{v} \; x \; \boldsymbol{B})\rho \; \mathrm{d}\tau = \int (\boldsymbol{J} \; x \; \boldsymbol{B}) \; \mathrm{d}\tau$$

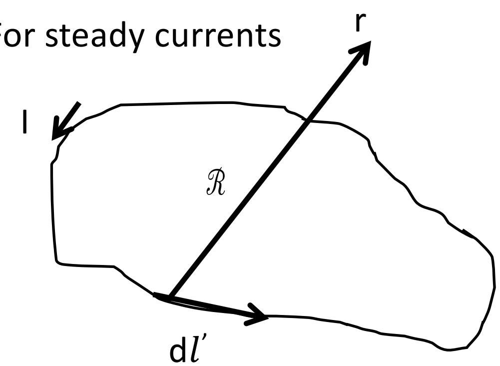




Steady Currents and Biot-Savart Law

$$\nabla.\boldsymbol{J} = -\frac{\partial \rho}{\partial t}$$ Continuity equation

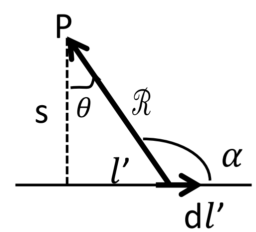
$$\nabla.\boldsymbol{J} = 0$$ For steady currents



$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{I}\ x\ \widehat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}\ \mathrm{d}l' = \frac{\mu_0}{4\pi} I \int \frac{d\boldsymbol{l}' x\ \widehat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}$$

Example

Find the magnetic field a distance s from a long straight wire carrying a steady current I



$$\boxed{\left| d\boldsymbol{l}' x\ \widehat{\boldsymbol{\mathcal{R}}} \right| = \mathrm{d}l'\ \mathrm{Sin}\alpha = \mathrm{d}l'\ \mathrm{Cos}\theta}$$

$$\boxed{l' = s\ \tan\theta\ \rightarrow\ \mathrm{d}l' = \frac{s}{\mathrm{Cos}^2\theta}\mathrm{d\theta}}$$

$$\boxed{s = \mathcal{R}\ \mathrm{Cos}\theta \rightarrow \frac{1}{\mathcal{R}^2} = \frac{\mathrm{Cos}^2\theta}{s^2}}$$

$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{\mathrm{Cos}^2\theta}{s^2}\right) \left(\frac{s}{\mathrm{Cos}^2\theta}\right) \mathrm{Cos\theta\ d}\theta \ = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \mathrm{Cos}\theta\ \mathrm{d}\theta = \frac{\mu_0 I}{4\pi s} (\mathrm{Sin}\theta_2 - \mathrm{Sin}\theta_1)$$

$$\boxed{B_{\text{infinite\_wire}} = \frac{\mu_0 I}{2\pi s}}$$

Ampere's Law

$$\nabla\ x\ \boldsymbol{B} = \mu_0 \boldsymbol{J} \qquad \int (\nabla x \boldsymbol{B}).\,\mathrm{d}\boldsymbol{a} = \oint \boldsymbol{B}.\,\mathrm{d}\boldsymbol{l} = \mu_0 \int \boldsymbol{J}.\,\mathrm{d}\boldsymbol{a}$$

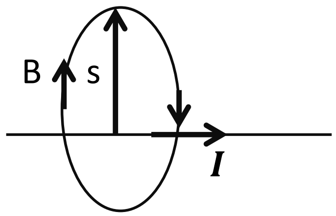
$$\int \boldsymbol{J}.\,\mathrm{d}\boldsymbol{a} = \text{total current passing through the surface}$$

$$\boxed{\oint B.\,\mathrm{d}l = \mu_0 I_{\mathrm{enc}}}$$

Example

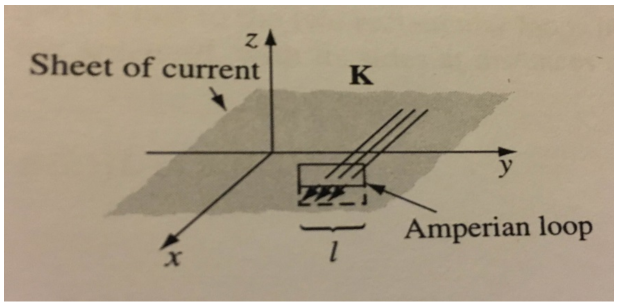
Find the magnetic field a distance s from a long straight wire carrying a steady current I

$$\oint B.\,\mathrm{d}l = B \int \mathrm{d}l = B2\pi s = \mu_0 I_{\mathrm{enc}} = \mu_0 I \quad ==> \quad \boxed{B = \frac{\mu_0 I}{2\pi s}}$$



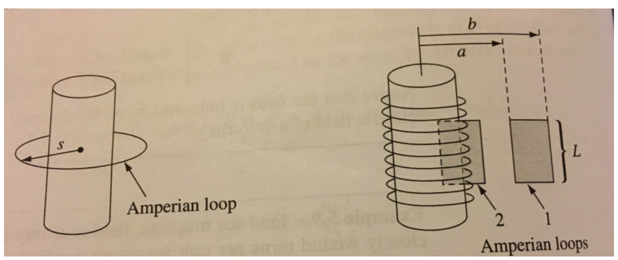
B-Field for different shapes

Magnetic field of an infinite uniform surface current $\mathbf{K} = k\hat{x}$



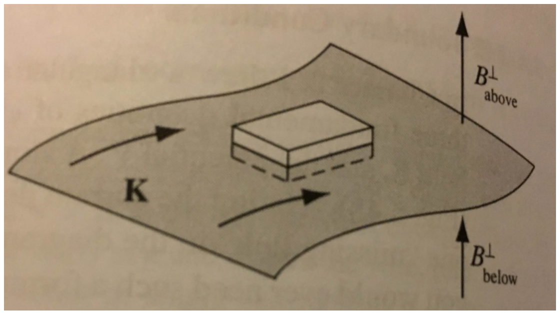
$$B = \begin{cases} \left(\frac{\mu_0}{2}\right) k\hat{y} & \text{for } z < 0 \\ -\left(\frac{\mu_0}{2}\right) k\hat{y} & \text{for } z > 0 \end{cases}$$

Magnetic field of a very long solenoid with $\mathrm{K} = n\, I$

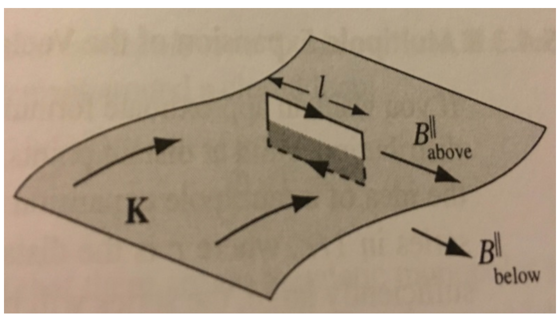


$$B = \begin{cases} \mu_0 n \mathrm{I}\, \hat{z} & \text{inside the solenoid} \\ 0 & \text{outside the coil} \end{cases}$$

Boundary Conditions



$$B^{\text{perp}}{}_{\text{above}} = B^{\text{perp}}{}_{\text{below}}$$



$$B^{||}{}_{\text{above}} - B^{||}{}_{\text{below}} = \mu_0 K$$

Other stuff to know

$\boldsymbol{\nabla} . \boldsymbol{B} = 0$ $\quad W_{\mathrm{mag}} = 0$ $\quad \boldsymbol{m} = I\boldsymbol{a}$ (Magnetic dipole moment)

Here $\boldsymbol{a}$ is the vector area of the loop

Maxwell's Equations

Equation	Name
$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{E}} = \frac{\boldsymbol{\rho}}{\boldsymbol{\epsilon_0}}$	Gauss' Law
$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = -\frac{\boldsymbol{d\vec{B}}}{\boldsymbol{dt}}$	Faraday's Law
$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}} = \mathbf{0}$	No-Name Law/ Gauss' Law for B-fields
$\boldsymbol{\nabla} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} + \mu_0 \boldsymbol{\epsilon_0} \frac{\boldsymbol{d\vec{E}}}{\boldsymbol{dt}}$	Ampere's Law

$\boldsymbol{\rho}$ — volume charge density (C/m^3)

J -- current density (A/m^2)

Faraday's Law

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

According to Stokes' Theorm ....

$$\int (\nabla \times \vec{E}) \cdot ds = \int \vec{E} \cdot dl = \frac{-d\phi_B}{dt}$$

$$\phi_B = \oint B \cdot \mathrm{d}S$$

$$\boxed{\varepsilon = -\frac{d\phi_B}{dt}}$$

Lenz's Law (emf)

***Induced currents always oppose changes in magnetic field*

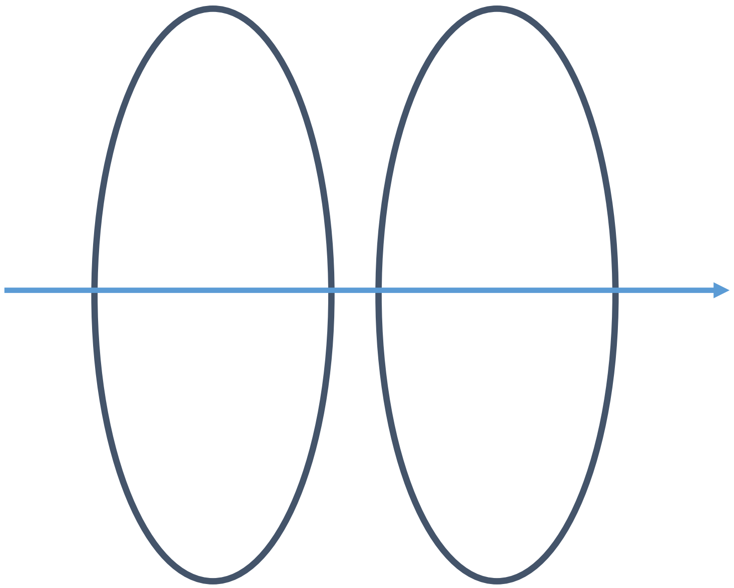
Inductors: Loop of Wire

A loop of current produces current in any nearby loops.

$$\boldsymbol{\phi_{\mathrm{B}} = LI}$$

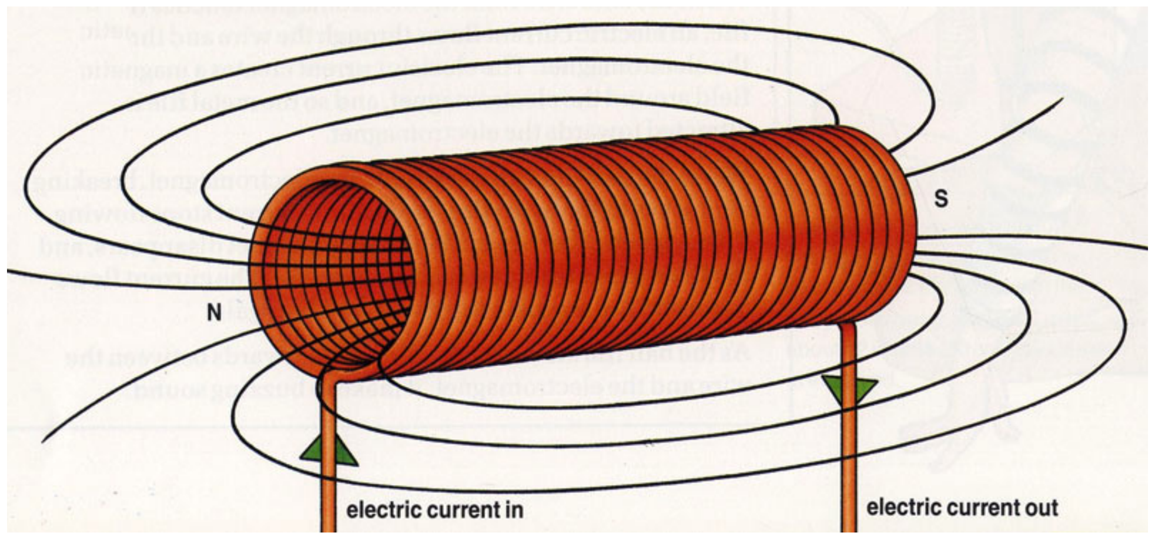
For a magnetic field *produced by wires themselves*

$$\boldsymbol{\xi = -L\frac{\mathrm{d}I}{\mathrm{d}t}}$$



Inductors: Solenoid

$$L = \frac{\mu_0 N^2 A}{l}$$



***N*-** number of turns

***A*-** cross-sectional area

***l*-** length

Electric Dipoles

Dipole moment

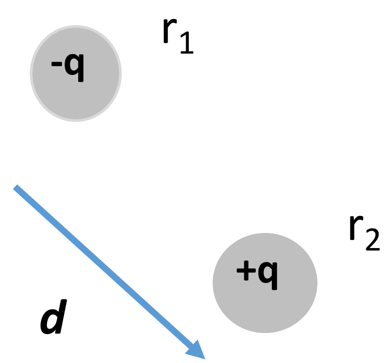
$$\vec{\boldsymbol{p}} = \boldsymbol{q r_1} - \mathbf{q}\boldsymbol{r_2} = \boldsymbol{q d}$$

Dipole moment- collection of charges

$$\vec{p} = \sum_i q_i d_i$$

Electric potential from dipole moment

$$\mathbf{V}(\vec{\mathbf{r}}) = \frac{\mathbf{1}}{\mathbf{4\pi}\boldsymbol{\varepsilon}_{\mathbf{0}}}\frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{\mathbf{r}^{\mathbf{2}}}$$



**Note V→$1/r^2$ and E→ $1/r^3$

Magnetic Dipoles

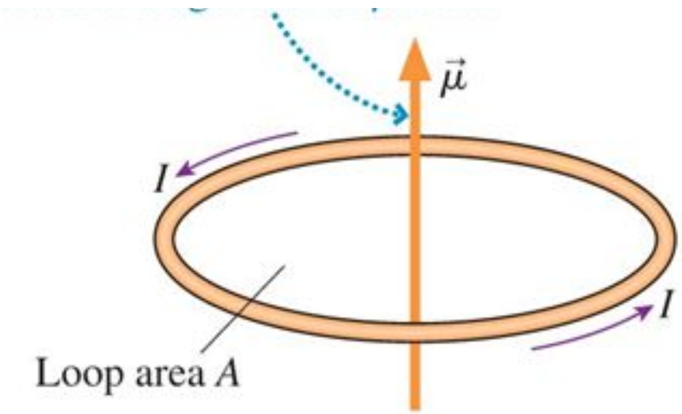
Remember:

--no magnetic monopoles

--cannot be decomposed like E dipoles

Dipole moment

$$\boldsymbol{\vec{m} = IA}$$



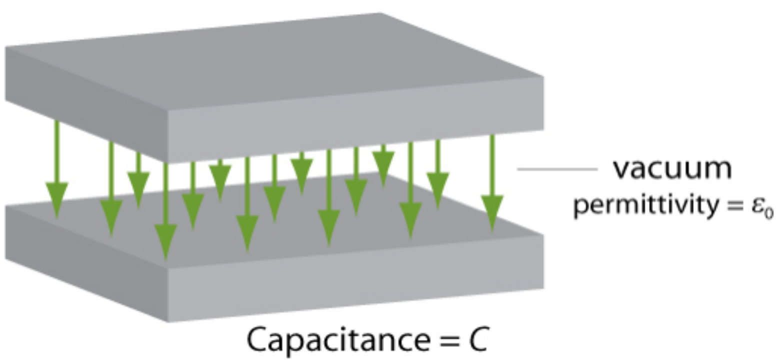
m- magnetic dipole moment (Am^2)
A- vector normal to surface (area)

Matter Effects - Dielectrics

Materials that can be polarized in an applied field and thus slightly cancel the electric field. Parameterized by making the substitution:

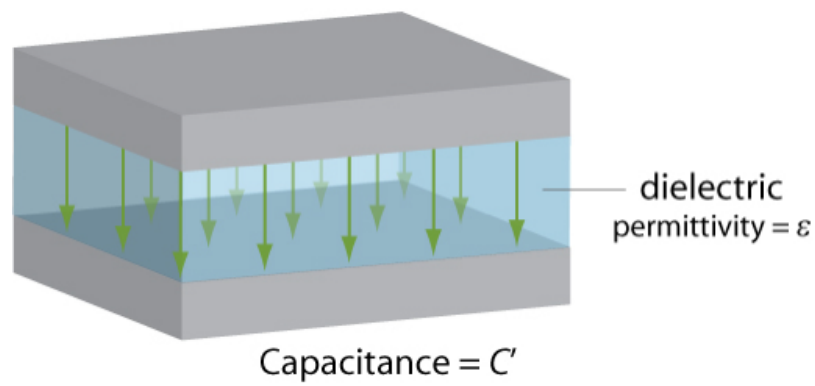
$$\varepsilon_0 \rightarrow \varepsilon = \kappa\varepsilon_0$$

κ is the dielectric constant



Most common: insulator placed between two plates, capacitance becomes

$$C = \frac{\varepsilon A}{d} = \kappa\frac{\varepsilon_0 A}{d}$$



Electromagnetic waves - Wave Equation

Assuming vacuum

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

And

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Electromagnetic waves - Wave Equation

Wave solutions have the explicit form

$$\tilde{\mathbf{E}}(\boldsymbol{r}) = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\hat{\boldsymbol{n}} \qquad \tilde{\mathbf{B}}(\boldsymbol{r}) = \frac{1}{c}\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}})$$

Where $\hat{\boldsymbol{k}}$ is the propagation vector describing the direction in which the wave travels and $\hat{\boldsymbol{n}}$ is the polarization vector of the electric field only.

Rule for notation: calculate everything in the complex formalism and then take the real part $\mathbf{E} = \mathrm{Re}(\tilde{\mathbf{E}})$

Fine for superpositions, since the real part of a sum is the sum of the real parts, but it's more complicated for products.

Electromagnetic waves - Poynting Vector

Flux of energy of the wave

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{2\mu_0}\mathrm{Re}(\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^*)$$

Intensity of the wave is often easier to deal with because of extremely high frequencies

$$I = \langle S \rangle = \frac{1}{2} c \varepsilon_0 E_0^2$$

Electromagnetic waves - Radiation

Larmor formula: An accelerating point charge radiates total power

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} = \frac{\mu_0 q^2 a^2}{6\pi c}$$

Only holds for small velocities.
Oscillating dipole with dipole moment

$$\mathbf{p}(\mathrm{t}) = p_0 \cos(\omega t)\hat{\mathbf{z}}$$

And intensity

$$\langle S \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{sin^2\theta}{r^2}$$

Falls off like $\frac{1}{r^2}$ and the $sin^2\theta$ means no radiation occurs along the dipole axis

Electromagnetic waves - Radiation

Integrate over a sphere of radius r gives the total power

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

Analogous formula for the magnetic dipole radiation

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

Where m_0 is the average magnetic dipole moment.