Relativity, Mathematics, Statistics, & Lab Methods to know for the GRE

Special Relativity

Main Concepts and Equations

Main Concepts

Reference Frames

Velocity Addition

Lorentz Transformations

Relativistic collisions

Minkowski diagrams

Reference Frames

“Stationary” Frame:

Experiences time normally

Views objects approaching c as compressing

Fast moving objects appear to experience time more slowly

“Moving” Frame:

At speeds approaching c stationary objects move through time more slowly

Stationary objects appear to stretch and elongate

Within their reference frame moving objects experience time normally.

Velocity Addition

Classically: $\mathbf{v} = \boldsymbol{v_1} + \boldsymbol{v_2}$

Einstein Velocity addition: $\mathbf{v} = \frac{\boldsymbol{v_1} + \boldsymbol{v_2}}{\mathbf{1} + \left(\frac{\boldsymbol{v_1 v_2}}{\boldsymbol{c^2}}\right)}$

Example Problem:
A ship travelling at .8c launches a projectile at .6c relative to it, what is the speed of the projectile to a stationary observer?

Basic Relativistic Conversions

$$\boldsymbol{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Key relationships:
At .5c: $\gamma = 1.155$
At 3/5c:$\gamma = 1.25$
At 4/5c:$\gamma = 1.512$

Time Dilation
$\Delta t' = \gamma \Delta t$

Length Contraction
$\Delta x' = \frac{\Delta x}{\gamma}$

Relativistic Mass
$m = \gamma m_0$

Relativistic Momentum
p= $\gamma m_0 \vec{v}$

Relativistic Kinetic Energy
$E_k = (\gamma - 1) m_0 c^2$

Lorentz Transformations

Assuming a body moving with a velocity in the x direction:

$t' = \gamma(t - \frac{vx}{c^2})$	$t = \gamma(t' + \frac{vx'}{c^2})$
$x' = \gamma(x - vt)$	$x = \gamma(x' + vt')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$

The Lorentz transformations are derived from the following equation which is always true

$$\boxed{c^2t^2 - x^2 - y^2 - z^2 = c'^2t'^2 - x'^2 - y'^2 - z'^2}$$

Relativistic Collisions

Key Terms:

$E_{rest} = \gamma m_0 c_2$

p= $\gamma m_0 \vec{v}$

$E_m = \frac{p}{v}$

$E_{tot} = E_m + E_{rest}$

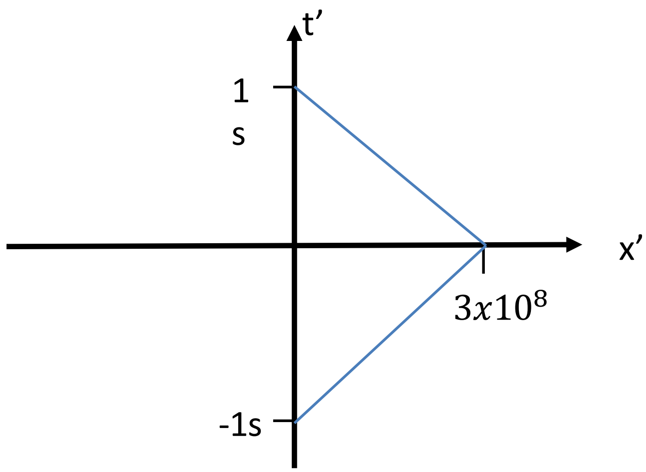
Example Question:
A particle of mass m is moving at 4/5c, collides inelastically with another particle of equal mass at rest.

Find its composite mass and velocity.

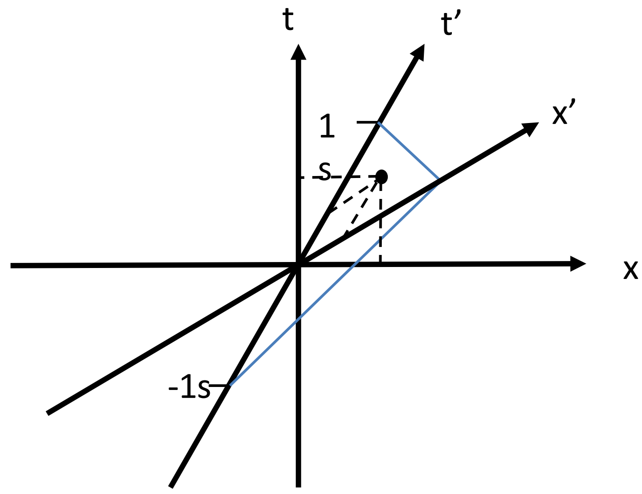
$$p_m = \frac{mv}{\sqrt{1-\left(\frac{4}{5}\right)^2}} = \frac{4}{3}m \quad E_m = \frac{p_m}{v} = \frac{\frac{4}{3}}{\frac{4}{5}}m = \frac{5}{3}m \quad E_{tot/c} = E_{rest} + E_m = \frac{8}{3}m$$

$$v_c = \frac{p_c}{E_c} = \frac{\frac{4}{3}m}{\frac{8}{3}m} = \frac{c}{2} \qquad m_c{}^2 = E_c{}^2 - p_c{}^2 \qquad m_c = \frac{4}{\sqrt{3}}m$$

Minkowski Diagrams



Two spaceships moving at .5 c are $3x10^8$m apart at t=-1, the first one shines a light at a mirror on the back of the second.

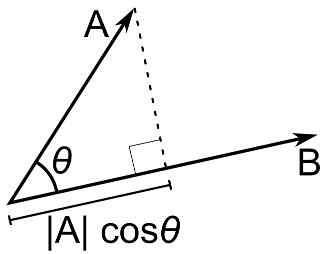


Mathematics

Dot & Cross Product Identities

Dot Product

$$\vec{A}\cdot\vec{B} = |A||B|\cos(\theta)$$



$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

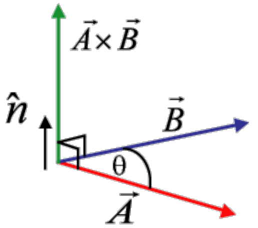
Yields a scalar quantity

Cross product

$$\vec{A}\times\vec{B} = |A||B|\sin(\theta)$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= (a_2 b_3 - a_3 b_2)\,\mathbf{i} + (a_3 b_1 - a_1 b_3)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k}$$



Yields a vector

Matrices & Determinants

- Det$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Gradient Theorem

$$\int_a^b (\vec{\nabla} f) \bullet d\vec{\ell} = f(b) - f(a)$$

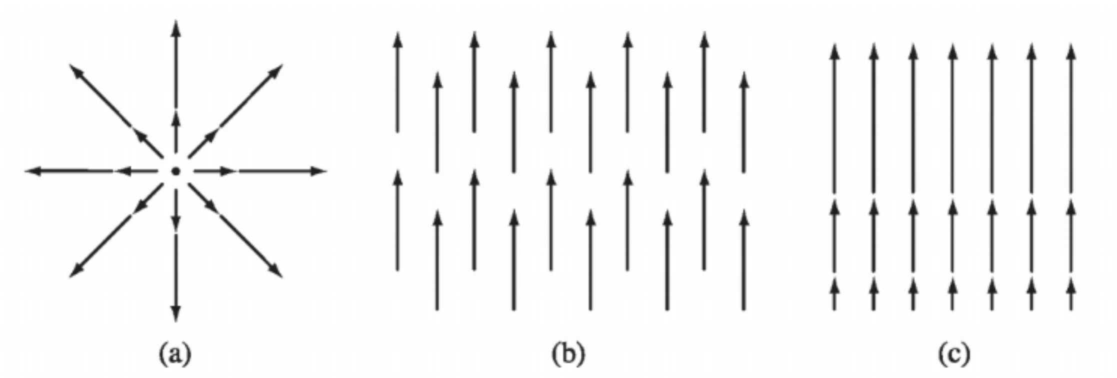
Interpretation: if you were to measure how high a building was by measuring one floor and counting the total floors and adding them, or by simply taking a measurement of the altitude at the top subtracted from that at the bottom, you expect to get the same answer.

Green's Divergence Theorem / Gauss' Law

The divergence measures how much the function spreads out over time

$$\int (\vec{\nabla} \cdot \vec{A})\, d\tau = \oint \vec{A} \cdot d\vec{a}$$

$$\int source\ points\ inside\ volume = \oint flow\ out\ of\ surface$$



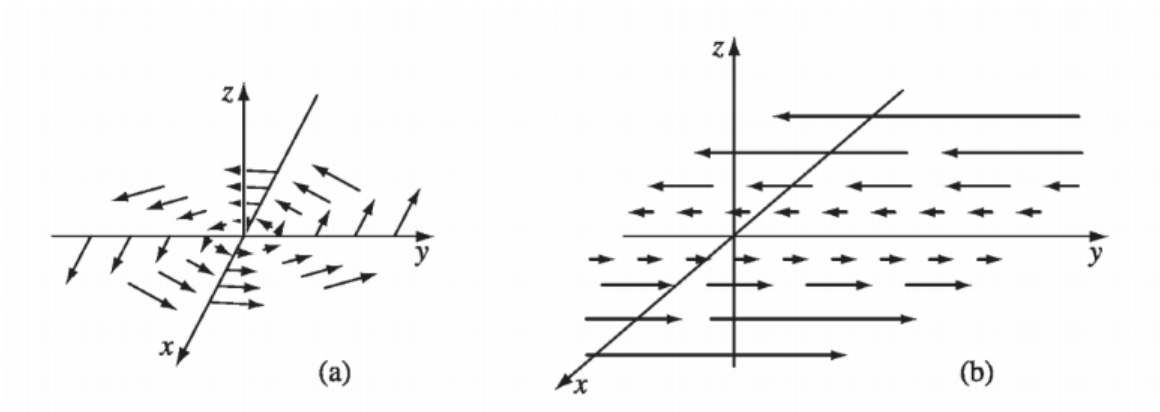
(a) & (c) show positive divergence

Stoke’s Theorem of Curl (Flux)

Curl over a region or surface is equal to the value of the function at the boundary

Curl measures how much rotation there is in the medium

$$\int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{\ell}$$



Both images have nonzero curl

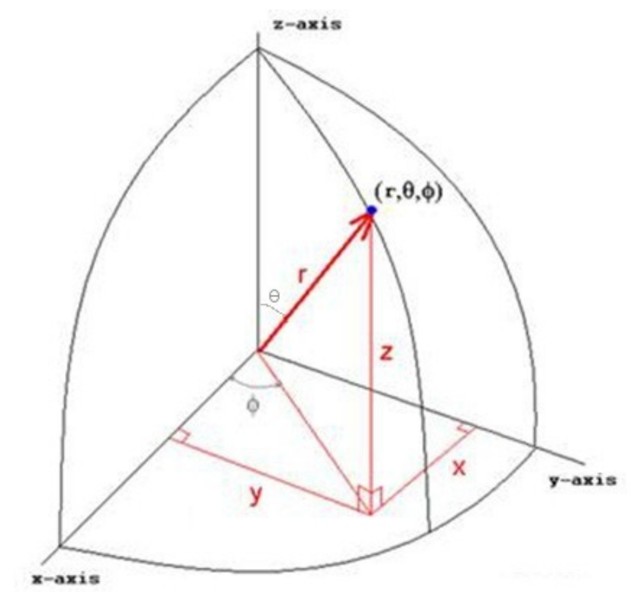
Laplacian- divergence of the gradient

- $\boldsymbol{\nabla}^2 T = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) = \frac{\partial^2 T}{\partial^2 x^2} + \frac{\partial^2 T}{\partial^2 y^2} + \frac{\partial^2 T}{\partial^2 z^2}$
- $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = 0$
- $\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \vec{V}\right) = 0$
- $\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \vec{V}\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \vec{V}\right) - \boldsymbol{\nabla}^2 \vec{V}$

Coordinate Systems: Spherical

- Cartesian to Spherical
  - $r = \sqrt{x^2 + y^2 + z^2}$
  - $\varphi = \tan^{-1} \frac{y}{x}$
  - $\theta = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}}$
- Spherical to Cartesian
  - $x = r \sin\theta \cos\varphi$
  - $y = r \sin\theta \sin\varphi$
  - $z = r \cos\theta$

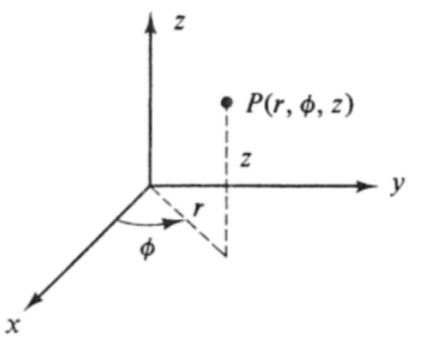
- Integration:
  - $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\varphi$



Coordinate Systems: Cylindrical

- Cartesian to Cylindrical
  - $s = \sqrt{x^2 + y^2}$
  - $\varphi = \tan^{-1} \frac{y}{x}$
  - z=z
- Cylindrical to Cartesian
  - $x = s \cos \varphi$
  - $y = s \sin \varphi$
  - z=z

- Integration:
  - $d\tau = s\, ds\, d\varphi\, dz$



Trig Identities

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\sin\theta = \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos\theta\cos\phi = \tfrac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)]$$

$$\sin\theta\sin\phi = \tfrac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)]$$

$$\sin\theta\cos\phi = \tfrac{1}{2}[\sin(\theta + \phi) + \sin(\theta - \phi)]$$

Fourier series

- Fourier series are the expansion of a periodic function into sin and cos terms.
- The sin and cos terms are orthogonal, so they do not contribute at the same terms.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, 3, \ldots$$

Binomial distribution

- For situations with discrete binary outcomes (Y/N, probability p of Y)
- The probability of k successes over n trials is

$$P(K = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

mean $\mu = np$

standard deviation $\sigma = \sqrt{np(1-p)}$

Gaussian distribution

- A continuous normalized probability distribution about a mean, shaped like a bell. The normalization constant is in front.

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Poisson distribution

- A discrete probability distribution of events happening at a known rate and independent probability of one another. (ex. radioactive decay)
- If λ is the expected number of events in a given time period, the probability of k events is, with mean λ and standard deviation $\sqrt{\lambda}$.

$$P(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Taylor series

- The approximation of a function f(x) at x = a is represented by

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

Taylor series:

$$\frac{1}{1-x} = \frac{1}{1-a} + \frac{x-a}{(1-a)^2} + \frac{(x-a)^2}{(1-a)^3} + \ldots$$

$$\cos x = \cos a - \sin a\,(x-a) - \tfrac{1}{2}\cos a\,(x-a)^2 + \tfrac{1}{6}\sin a\,(x-a)^3 + \ldots$$

$$e^x = e^a\left[1 + (x-a) + \tfrac{1}{2}(x-a)^2 + \tfrac{1}{6}(x-a)^3 + \ldots\right]$$

$$\ln x = \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \ldots$$

$$\sin x = \sin a + \cos a\,(x-a) - \tfrac{1}{2}\sin a\,(x-a)^2 - \tfrac{1}{6}\cos a\,(x-a)^3 + \ldots$$

$$\tan x = \tan a + \sec^2 a\,(x-a) + \sec^2 a \tan a\,(x-a)^2 + \sec^2 a\left(\sec^2 a - \tfrac{2}{3}\right)(x-a)^3 + \ldots.$$

Lab Methods

Lab Methods

- Dimensional Analysis→ GRE asked for the slope of a plot in terms of fundamental constant, just look at dimensions of the ratio y/x.
- Log Plots: Never show zero on the axes, A straight line on log-log plot corresponds to $y=ax^b$, A straight line on a log-linear plot corresponds to an exponential growth law, $y=C.\ 10^{bx}$

Statistics

- Estimate of spread: Sample Variance

$$\sigma_s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- Uncertainty : uncertainty of 10% means

$$\sigma / \bar{x} = .1$$

- Precise measurement: small variance
- Accurate measurement: Closer to the true value

Error Propagation

$$f = aA \qquad \sigma_f = a\sigma_A$$

$$f = A + B \qquad \sigma_f = \sqrt{\sigma_A^2 + \sigma_B^2}$$

$$f = AB \qquad \sigma_f = f\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$$

$$f = {}^{A}\!/_{B} \quad \sigma_f = f\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$$

Poisson Processes

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

- For large N $\sigma = \sqrt{N}$
- How rare the process is, if λ is small, likely to observe no events $P(0) = e^{-\lambda}$

Electronics

- Capacitor $$Z = \frac{1}{i\,\omega\,C}$$
- Inductor $$Z = i\,w\,L$$
- Resistor $$Z = R$$

- Series $$Z_{tot} = Z_1 + Z_2 + \cdots Z_n$$
- Parallel $$Z_{tot}^{-1} = Z_1^{-1} + Z_2^{-1} + \cdots Z_n^{-1}$$

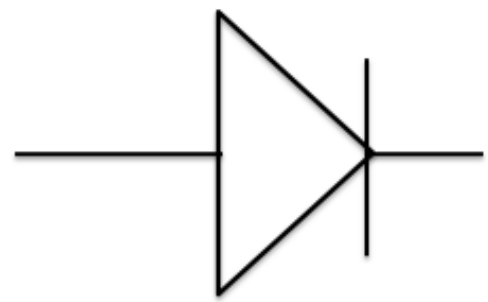
One Inductor and Capacitor Circuit

$$Z_{LC} = \frac{1}{i\omega C} + i\omega L = \frac{i(1 - \omega^2 LC)}{\omega C}$$

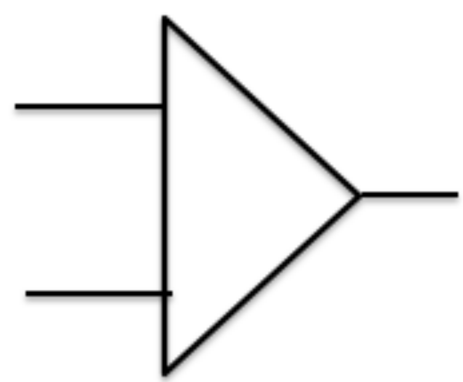
- The numerator vanishes when $\omega = \frac{1}{\sqrt{LC}}$, the resonant frequency

Advance Circuit Elements

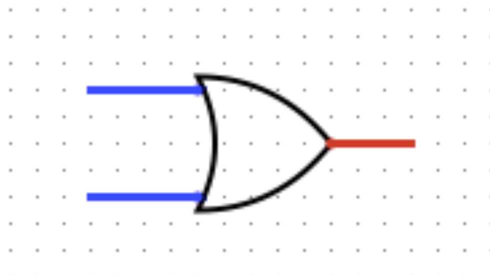
- Diode: Doesn't follow Ohm's law, current only flows one direction

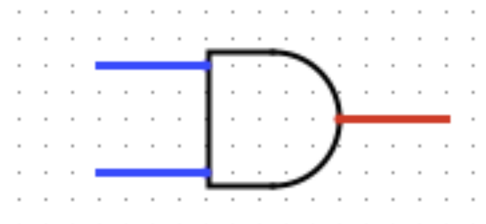


- Op-amp: Two inputs one output, output proportional to the difference of input voltages
-



Logic Gates





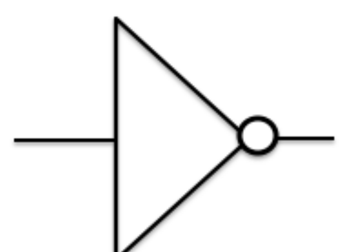
- Or gate (A+B)
- True either A or B is true
- False when both A &B are false

And gate(A.B)

True if both A & B are true

False otherwise

- The inputs can be inverted using a NOT gate



- Inverting output of AND and OR gates give NAND and NOR gates
- All basic logic gates can be constructed from the combination of these two gates
- De Morgan's Laws in Boolean Algebra:

$$\overline{A \cdot B} = \bar{A} + \bar{B}$$

$$\overline{A + B} = \bar{A} \cdot \bar{B}$$

Interactions of Charged Particles with Matter

- Range: Nuclei are stopped faster than the electrons, energy loss/length is much higher for nuclei than for electrons
- Path Shape: Nuclei tend to travel is straight line
- Energy Loss: Nuclei lose energy exclusively due to collisions, rather than by emitting radiation.

Photon Interactions

- Photoabsorption(photoelectric): Photon completely absorbed by an atom, and an electron is emitted.
- Compton Scattering: Photon scatters off an atomic electron, the wider the scattering angle, the more energy it loses to the electron.
- Compton Wavelength $\lambda = \frac{h}{mc}$

Lasers

- The process of photons from decaying excited states being absorbed by other excited states, called stimulated emission, starts a chain reaction, the product of which is an exponentially large number of photons, all with exactly the same frequency and phase, this is laser light.

Types of Lasers

- Solid-state lasers: (Nd-YAG)
- Collisional Gas Lasers: (He-Ne)
- Molecular Gas Lasers: (CO_2)

Interferometer