

**Performing Numerical Integration without “advanced” tools.
(see accompanying Excel document online)**

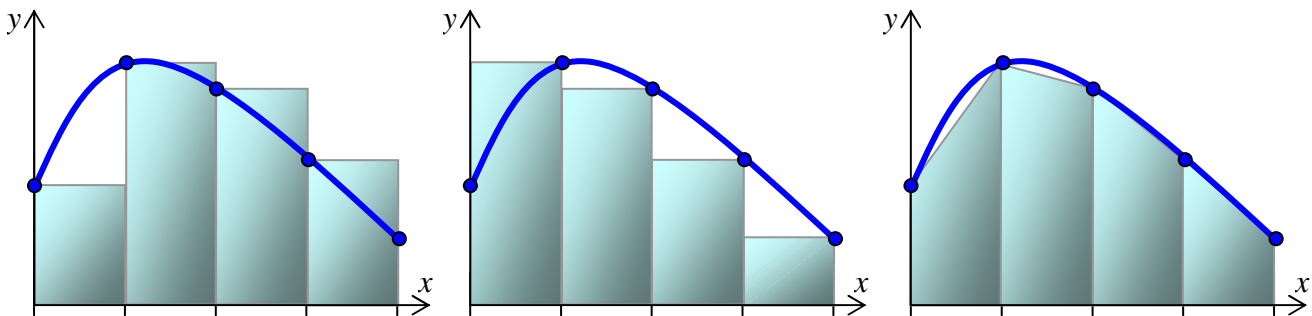
There are three commonly used ways of numerically integrating if you need a decent result fast: using right-side rectangles, or using left-side rectangles, or using trapezoids. Wait! Why are there two choices for rectangle? Each of the three shapes spans a region Δx across the bottom. So, there are two vertical sides for each of these choices. In the first method, the height of the rectangle is equal to the value of the function $y(x)$ on the left side of the span Δx , and the top of the rectangle extends to the right from there. In the second method, the height of the rectangle is equal to the value of $y(x)$ on the right side of Δx , and extends to the left of there. In the third and best method, you use the $y(x)$ values from both vertical edges.

By definition, the area of such a trapezoid is the average of the other two rectangles. So, for the three methods:

$$\begin{aligned} A_1 &= (\Delta x)(y_{\text{left}}) \\ A_2 &= (\Delta x)(y_{\text{right}}) \\ A_3 &= \frac{1}{2} (\Delta x)(y_{\text{left}} + y_{\text{right}}) \end{aligned}$$

In each of these three sketches, the number of “shapes” in use is 4, so I would sum the 4 areas to determine the total area under the curve. Notice that although I only have 4 shapes, I must evaluate $y(x)$ at 5 points (i.e., one more than the number of shapes). So, in the accompanying Excel document, I am using 100 shapes to test each method, so I need 101 rows of data (i.e., from 0 to 100 rather than from 1 to 100).

Generally, the first method under-predicts the true value when the function is increasing, but over-predicts it when the function is decreasing. The second method is the opposite of that. The third method gives the most error when the slope of the function is changing rapidly. In all cases, the results are better as Δx gets smaller. The downside of just making Δx ridiculously small is that the computational time (and space, if you’re doing it in Excel) becomes too large.

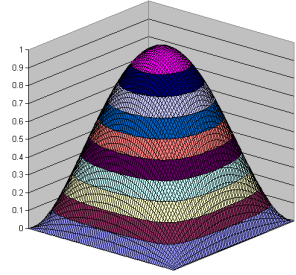


Consider once again the function $z = (1 - x^2)(1 - y^2)$.

When we found the volume of this hill before, we restricted the domain to:

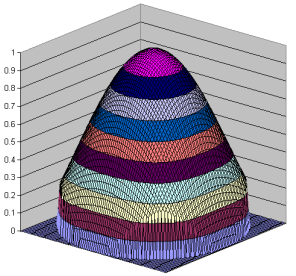
$-1 \leq x \leq +1$, and $-1 \leq y \leq +1$ (see first plot).

The resulting area was $16/9 \approx 1.777777$.



Now, I want to redo the volume of the same function z , but with a circular domain, rather than a square domain: $x^2 + y^2 \leq 1$ (second picture).

I expect the volume to be a little bit smaller (second picture).



The “floor plan” is also shown.

We need to describe this curve: $y(x) = \sqrt{1^2 - x^2}$.

It’s still symmetric, so I’m still going to use the following:

$$V_{\text{total}} = 4V_{\text{quadrant 1}}$$

The integral for total volume is therefore:

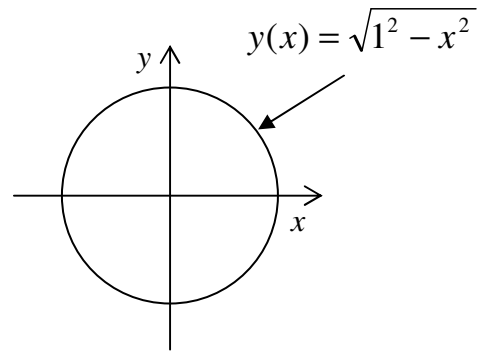
$$V_{\text{total}} = 4 \int dz dy dx = 4 \int z dx dy$$

$$V_{\text{total}} = 4 \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} (1-x^2)(1-y^2) dy dx$$

$$V_{\text{total}} = 4 \int_{x=0}^{x=1} (1-x^2) \left(y - \frac{y^3}{3} \right) \Big|_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$V_{\text{total}} = 4 \int_{x=0}^{x=1} \left((1-x^2)\sqrt{1-x^2} - \frac{(1-x^2)^3}{3} \right) dx$$

$$V_{\text{total}} = \frac{13\pi}{24} \approx 1.7017$$



→ It is a little smaller than the square domain version, but only by about 4%.

Choice of Coordinate Systems

Sometimes, it makes a huge difference whether you do an integral in Cartesian or some other coordinate system. If an object has a symmetry that matches a coordinate system, it's almost always best to take advantage of that. The following integral is simple in cylindrical (or polar) coordinates, but virtually impossible in Cartesian coordinates:

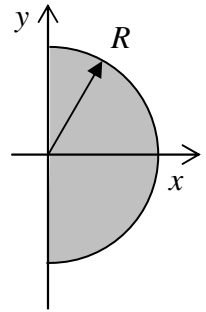
Area of a half-circle:

$$A = \int dA$$

$$A = \int_{\theta=-\frac{\pi}{2}}^{\theta=+\frac{\pi}{2}} \int_{r=0}^{r=R} r dr d\theta$$

$$A = \int_{\theta=-\frac{\pi}{2}}^{\theta=+\frac{\pi}{2}} \frac{R^2}{2} d\theta = \frac{R^2 \theta}{2} \Big|_{\theta=-\frac{\pi}{2}}^{\theta=+\frac{\pi}{2}} = \frac{R^2}{2} \left(\frac{\pi}{2} - \frac{-\pi}{2} \right)$$

$$A = \frac{\pi R^2}{2}$$



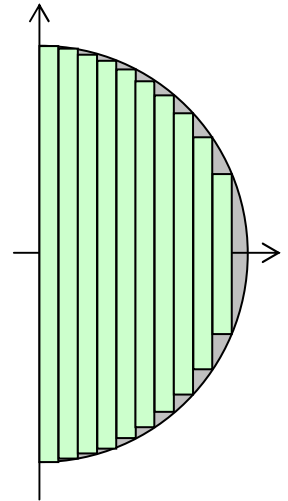
Every high school kid knows that! Let's (try to) do it in Cartesian:

$$A = \int dA$$

$$A = \int_{x=0}^{x=R} \int_{y=y_{\min}=-\sqrt{R^2-x^2}}^{y=y_{\max}=\sqrt{R^2-x^2}} dy dx$$

$$A = \int_{x=0}^{x=R} y \Big|_{y_{\min}}^{y_{\max}} dx$$

$$A = \int_{x=0}^{x=R} \left[2\sqrt{R^2-x^2} \right] dx = ????$$



At this point, your calculus teacher would say “and now you have to define a new variable θ using a trig substitution, and use it to replace x : $x = R\cos\theta$ ”. Which is just a fancy way of saying “and now, we convert everything into cylindrical coordinates and start over.”

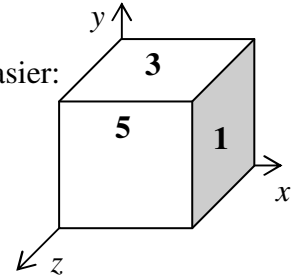
We already see the answer (above), so there's no point trying to do it again.

Divergence Theorem:

This technique is common in electricity & magnetism problems.

Main idea: there are two ways to perform the same integral, so pick whichever is easier:

$$\int (\nabla \cdot \vec{v}) dV_{(\text{Volume})} = \int \vec{v} \cdot d\vec{A}_{(\text{Surface Area})}$$



Consider a cubic volume of space, ranging from (0, 0, 0) to (1, 1, 1) as sketched.

A vector field exists within and near this region: $\vec{v} = 3x\hat{i} + 2\hat{j}$.

Determine the two integrals above.

A. The first is a volume integral. We have to start with finding the divergence $\nabla \cdot \vec{v}$:

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 3 + 0 + 0 = 3$$

$$\text{So, the integral is: } \int (\nabla \cdot \vec{v}) dV = \int (3) dV = 3V = \boxed{3.}$$

That was pretty easy. For this problem (but not all problems), the next part is harder.

B. The second is an area integral. The hard part is that there are 6 square areas that bound the volume, so I need to do an integral for each of the 6 faces, and then add the results together. I've numbered 3 of the faces on the sketch; face 2 is opposite face 1, etc. Each of the faces needs its own $d\vec{A}$:

$$\begin{aligned} d\vec{A}_1 &= (+\hat{i}) dydz & d\vec{A}_2 &= (-\hat{i}) dydz \\ d\vec{A}_3 &= (+\hat{j}) dx dz & d\vec{A}_4 &= (-\hat{j}) dx dz \\ d\vec{A}_5 &= (+\hat{k}) dx dy & d\vec{A}_6 &= (-\hat{k}) dx dy \end{aligned}$$

The six integrals are:

$$\int \vec{v} d\vec{A}_1 = \int_0^1 \int_0^1 (3x) dy dz = 3x = 3. \quad \text{Note : since this is face1, } x=1 \text{ on this face (see picture).}$$

$$\int \vec{v} d\vec{A}_2 = - \int_0^1 \int_0^1 (3x) dy dz = -3x = 0. \quad \text{Note : since this is face2, } x=0 \text{ on this face.}$$

$$\int \vec{v} d\vec{A}_3 = \int_0^1 \int_0^1 (2) dx dz = +2. \quad \int \vec{v} d\vec{A}_4 = - \int_0^1 \int_0^1 (2) dx dz = -2.$$

$$\int \vec{v} d\vec{A}_5 = \int_0^1 \int_0^1 (0) dx dy = 0. \quad \int \vec{v} d\vec{A}_6 = - \int_0^1 \int_0^1 (0) dx dy = 0.$$

$$\text{The total result is } 3 + 0 + 2 - 2 + 0 + 0 = \boxed{3.}$$

Fortunately, this matches the result found using the other method!

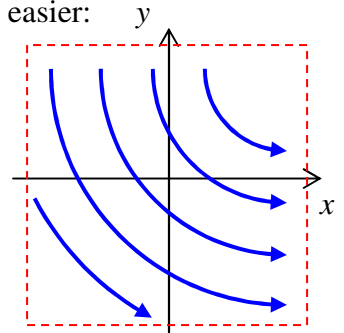
Stokes' Theorem:

This technique is also common in electricity & magnetism problems.

Main idea: there are two ways to perform the same integral, so pick whichever is easier:

$$\oint \vec{v} \cdot d\vec{s}_{(\text{boundary of a area})} = \int (\vec{\nabla} \times \vec{v}) \cdot d\vec{A}_{(\text{Area})}$$

You might also recall a circulation (Γ) problem involving an airplane from assignment #2 that required this kind of integral.



Example with velocity field:

The boundaries of the domain are: $-5 \leq x \leq +5$, and $-5 \leq y \leq +5$

The velocity field is $\vec{v} = (5 - y)\hat{i} + (x - 5)\hat{j}$

Looking at the second integral first, we need a curl: $\vec{\nabla} \times \vec{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{k} = 1 - (-1)\hat{k} = 2\hat{k}$

For the square area shown, $d\vec{A} = dydx(+\hat{k})$.

Therefore, the second integral is: $\int (2\hat{k})d\vec{A} = \int_{-5}^5 \int_{-5}^5 2dydx = 2xy \Big|_{x,y=-5}^{x,y=+5} = 2(10)(10) = \boxed{200}$

Let's try the other method. From the picture, the boundary has 4 edges, so I'll need to create four separate integrals and add the results together. These $d\vec{s}$ integrals require a direction, which is specified by the right hand rule. Since \hat{k} is "up" (out of the page), the right hand rule tells us that we have to create our boundary by traveling the 4 edges clockwise.



$$\begin{aligned} d\vec{s}_{\text{bottom}} &= (+\hat{i})dx & d\vec{s}_{\text{top}} &= (-\hat{i})dx \\ d\vec{s}_{\text{right}} &= (+\hat{j})dy & d\vec{s}_{\text{left}} &= (+\hat{j})dy \end{aligned}$$

The four integrals are then:

$$\int \vec{v}d\vec{s}_{\text{bottom}} = \int (5 - y)dx \rightarrow \text{but, on the bottom, } y = -5, \text{ so } \int (5 - (-5))dx = \int_{-5}^5 10dx = +100$$

$$\int \vec{v}d\vec{s}_{\text{right}} = \int (x - 5)dy \rightarrow \text{but, on the right, } x = +5, \text{ so } \int_{-5}^5 0dy = 0$$

$$\int \vec{v}d\vec{s}_{\text{top}} = \int -(5 - y)dx \rightarrow \text{but, on the top, } y = +5, \text{ so } \int_{-5}^5 0dx = 0$$

$$\int \vec{v}d\vec{s}_{\text{left}} = \int -(x - 5)dx \rightarrow \text{but, on the left, } x = -5, \text{ so } \int_{-5}^5 +10dy = +100$$

The total result is $+100 + 0 + 0 + 100 = \boxed{+200}$

It seemed that the "curl" method was easier this time. In practice, it is common to use the second method when the curl is infinity at some point in the flow field. In those cases, you can't even do the curl method at all, but the other method usually works fine.