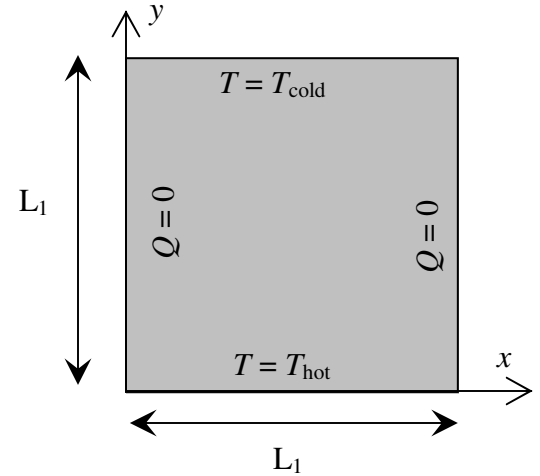


PDE: More Heat Equation with Derivative Boundary Conditions

Let's do another heat equation problem similar to the previous one. For this one, I'll use a *square* plate ($N = 1$), but I'm going to use different boundary conditions.

In the previous problem, the bottom was kept hot, and the other three edges were cold. In that case, energy moved as heat from the hot edge, into the plate, and then out of the plate into the cold stuff.

This time, the top boundary is still cold, and the bottom boundary is still hot. However, in this problem, the side edges are *insulated*. That means that there is no heat can flow through these edges. This requires us to remember a little bit of physics: since the heat flow Q is proportional to temperature gradient, then this boundary condition is the same thing as saying that $\frac{\partial T}{\partial x} = 0$ along both of these edges. Heat can move vertically along these edges (i.e., in the y direction, from the bottom to the top of the plate), but it can't cross out of the plate in the x direction. Additionally, if we had insulated the top edge, then we would have $\frac{\partial T}{\partial y} = 0$ there.



As before, we will non-dimensionalize this whole problem before starting it:

$$\frac{\partial^2 \Theta}{\partial x^{*2}} + \frac{\partial^2 \Theta}{\partial y^{*2}} = 0$$

$$\frac{\partial \Theta}{\partial x^*}(x^* = 0) = 0$$

$$\frac{\partial \Theta}{\partial x^*}(x^* = 1) = 0$$

$$\Theta(y^* = 0) = 1$$

$$\Theta(y^* = N = 1) = 0$$

Much of the early work is the same as before, so we'll steal from ourselves:

$$X = A \cos(kx) + B \sin(kx) \qquad Y = Ce^{ky} + De^{-ky}$$

$$\Theta = (A \cos(kx) + B \sin(kx))(Ce^{ky} + De^{-ky})$$

This time, we'll need the derivatives of Θ , too: $\frac{\partial \Theta}{\partial x} = (kA \sin(kx) - kB \cos(kx))Y(y)$

$$\text{Using BC \#1: } \frac{\partial \Theta}{\partial x}(x=0) = 0 \rightarrow \begin{aligned} 0 &= (kA \sin(k \cdot 0) - kB \cos(k \cdot 0))Y(y) \\ 0 &= kB \end{aligned}$$

Well, either k or B could be zero. If we choose $k = 0$, we end up with nothing at all. So, I guess this is telling us that $B = 0$.

$$\text{Using BC \#2: } \frac{\partial \Theta}{\partial x}(x=1) = 0 \rightarrow \begin{aligned} 0 &= kA \sin(k \cdot 1)Y(y) \\ k &= n\pi \end{aligned}$$

Well, that's a lot like what we had before. So far, then, we have:

$$\Theta = \sum_{n=0}^{\infty} A_n \cos(n\pi x) (C e^{ky} + D e^{-ky})$$

Note that this infinite series starts at *zero*, since we have cosine terms, but in the problem from the previous class, we only started at 1, since we had only sine terms.

I want to steal some more from ourselves as we evaluate the y -boundary conditions. After some effort the other day, we found that the Y part of the solution came down to a hyperbolic sine function; we have the same boundaries here today, so we'll just borrow that answer. So, our result so far is:

$$\Theta = \sum_{n=0}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-y))$$

Now, here's a hard part. I notice that today, n starts counting at zero, as just mentioned. However, when $n = 0$, then $k = 0$, too. Recall that when we started our Separation of Variables, we were solving two ODE's:

$$\begin{aligned} \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} &= -k^2 \\ \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} &= +k^2 \end{aligned}$$

We proceeded to solve them, resulting in the Θ above. But along the way, without noticing it, we implicitly assumed that $k \neq 0$. However, if $k = 0$, then these two equations become somewhat different:

$$\begin{aligned} \frac{\partial^2 X(x)}{\partial x^2} &= 0 \\ \frac{\partial^2 Y(y)}{\partial y^2} &= 0 \end{aligned}$$

The solution to these equations is NOT sine and cosine (or even exponentials), as we assumed in our answer. Instead, when $k = 0$, the answers to these two ODE's are simply:

$$X_{k=0} = C_1x + C_2, \text{ and } Y = C_3y + C_4.$$

In other words, the total solution is actually this, where I've moved the $n = 0$ part out of the summation and included it as its own thing:

$$\Theta = \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-y)) + (C_1x + C_2)(C_3y + C_4)$$

So THIS summation only starts at 1.

Because we are adding this change to the total solution, it makes us wish we had seen this problem coming before we applied the boundary conditions. In any case, for the side boundary conditions, we can just fix up this new part of the solution and plug it right in there.

Using BC #1: $\frac{\partial \Theta_{k=n=0}}{\partial x}(x=0) = 0 \rightarrow 0 = (C_1)(C_3y + C_4)$, resulting in $C_1 = 0$.

$$\Theta_{k=0} = C_2(C_3y + C_4)$$

Here, I'll combine $C_2 \cdot C_3$ into a new constant, and the same with $C_2 \cdot C_4$:

$$\Theta_{k=0} = C_5y + C_6$$

This brings our total result so far to: $\Theta = (C_5y + C_6) + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-y))$

Top edge: BC#3: $\Theta(y=1) = 0 \rightarrow 0 = (C_5 \cdot 1 + C_6) + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-1))$
 $0 = (C_5 + C_6) + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \cdot 0$

So we learn that $C_5 = -C_6$

Our equation so far: $\Theta = C_5(y-1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-y))$

Finally, the last BC: Bottom edge: BC #4: $\Theta(y=0) = 1$

$$1 = C_5(0-1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-0))$$

$$1 + C_5 = \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi)$$

We've seen this kind of thing before. Today, I'll invent a new parameter (I seem to love doing that...) called "a":

$$a_n = A_n \sinh(n\pi)$$

$$1 + C_5 = \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

We need to solve this for a_n . This is a Fourier series problem again:

$$a_n = 2 \int_0^1 (1 + C_5) \cdot \cos(n\pi x) dx = (\text{MathCAD}) = 0 \text{ for all } n!!$$

Well, that's unexpected good news. This boundary condition reduces to $1 + C_5 = 0$
 $C_5 = -1$

And the total solution reduces to: $\Theta = 1 - y$

Well, that's nuts. That was way too much work for such a simple answer. This answer says that the temperature is not really a function of x after all, and that the table gets cooler as you go from the bottom to the top.

$$\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} = 0 + 0 = 0 \text{ check!}$$

I notice that: $\frac{\partial \Theta}{\partial x} = 0$ everywhere; check!

$$\Theta(y=0) = 1 \text{ check!}$$

$$\Theta(y=1) = 0 \text{ check!}$$

Non-Uniform Temperature along Bottom Edge

Because this solution proved too boring, let's do a different, but similar problem. For this problem, the lower edge temperature is $\Theta_{\text{bottom}} = x$, (as opposed to equals 1). Here, we need to go back to just before we applied BC#4:

$$\Theta = C_5 (y-1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-y))$$

$$x = C_5 (0-1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi(1-0))$$

$$x + C_5 = \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_n = 2 \int_0^1 (x + C_5) \cdot \cos(n\pi x) dx \rightarrow a_n = 0 \text{ for even } n$$

$$a_n = \frac{-4}{n^2 \pi^2} \text{ for odd } n$$

This suggests that we want to count $n = 1, 3, 5, 7$. In fact, we are counting $m = 1, 2, 3, 4$.

So, to isolate the odd values, we use the same old trick as always: $a_m = \frac{-4}{(2m-1)^2 \pi^2}$.

$$\text{Our solution so far: } \Theta = C_5 (y-1) + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2 \pi^2} \frac{\cos((2n-1)\pi x) \sinh((2n-1)\pi(1-y))}{\sinh((2n-1)\pi)}$$

I guess there's only one more thing to discover, and that is C_5 .

Using BC#1, I notice that $\Theta(x=0, y=0) = 0$:

$$0 = C_5 (0-1) + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2 \pi^2} \frac{\cos((2n-1)\pi 0) \sinh((2n-1)\pi(1-0))}{\sinh((2n-1)\pi)}$$

$$0 = -C_5 + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2 \pi^2} \frac{1 \sinh((2n-1)\pi)}{\sinh((2n-1)\pi)}$$

$$0 = -C_5 + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2 \pi^2}$$

$$0 = -C_5 - \frac{1}{2} \quad (\text{infinite sum done in MathCAD}).$$

$$C_5 = -\frac{1}{2}$$

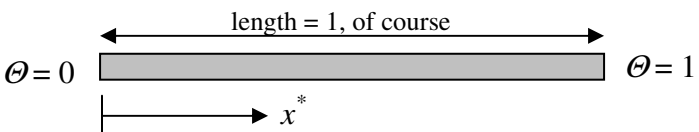
$$\boxed{\Theta = \frac{1-y}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2 \pi^2} \frac{\cos((2n-1)\pi x) \sinh((2n-1)\pi(1-y))}{\sinh((2n-1)\pi)}}$$

This is plotted in MathCAD. Notice the lack of slope along the side edges, and the linear profile along the bottom edge, in agreement with out boundary conditions.

Unsteady Heat Equation

The real heat equation is: $\nabla^2 T = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}$ where α is a material property related to the conductivity. We will non-dimensionalize this as before. The only new feature this time is that we need to invent a dimensionless time t^* : $t^* = \left(\frac{\alpha^2}{L^2}\right)t$. Making this substitution, and being too

lazy to write the “*” symbol after every x , y , and t , the heat equation becomes: $\nabla^2 \Theta = \frac{\partial \Theta}{\partial t}$. Since ∇^2 is really a function of all 3 spatial dimensions x , y , and z , this problem is way too hard for us. So to simplify it, we’ll look at an object that has only one spatial dimension: x . So, this is still a 2D problem, with independent variable x and t .

$$\frac{\partial^2 \Theta}{\partial x^2} - \frac{\partial \Theta}{\partial t} = 0$$


In this problem, I’ll need three “boundary” conditions: two relating to x , and only one about t . Here’s the problem we’ll solve today: the boundary conditions are that the left edge ($x = 0$) is held at $\Theta = 0$, and the right edge ($x = 1$) is held at $\Theta = 1$ (“hot”). The third “initial” condition is that $\Theta = 0$ everywhere along the bar when we start. In other words, it starts off all cold, but it heats up over time as heat leaks into the bar from the right hand edge.

We’ve learned some tricks about such problems from the last two examples. We will assume that the answer is comprised of two functions X and T that are multiplied together (Separation of Variables), and we will add on an extra bit like we did in the last problem for $n = 0$ to account for the steady state solution:

$$\Theta = X(x) \cdot T(t) + \Theta_{ss}$$

Adding the steady state solution is a lot like what we did with non-homogenous ODE problems, where we assumed that $y_{\text{complete}} = y_C + y_P$.

If a steady state temperature distribution is reached, then by definition, the temperature is done changing. Let’s deal with this first. So, as $t \rightarrow \infty$,

$$\frac{\partial \Theta_{ss}}{\partial t} = 0 \quad \text{as } t \rightarrow \infty. \text{ Therefore, from the heat equation,}$$

$$\frac{\partial^2 \Theta_{ss}}{\partial x^2} = 0 \quad \text{as } t \rightarrow \infty. \text{ Integrating twice results in:}$$

$$\Theta_{ss} = C_1 x + C_2$$

We can use the BC for the steady state solution: $\Theta_{ss}(x=0) = 0 \rightarrow C_2 = 0$.

$$\Theta_{ss}(x=1) = 1 \rightarrow C_1 = 1.$$

So, we know this part already: $\Theta_{ss} = x$

Now, let's do the Separation of Variables: $\Theta_c = X(x) \cdot T(t)$

$$\frac{\partial \Theta_c}{\partial x} = X'T \qquad \frac{\partial \Theta_c}{\partial t} = XT'$$

$$\frac{\partial^2 \Theta_c}{\partial x^2} = X''T$$

$$X''T - XT' = 0$$

The heat equation therefore becomes: $\frac{X''}{X} - \frac{T'}{T} = 0$

As before, this is only possible if each group is a constant:

$$\frac{X''}{X} = -k^2 \qquad -\frac{T'}{T} = +k^2$$

$$X'' + k^2 X = 0 \qquad T' + k^2 T = 0$$

From our chapter on ODE's, the answers to these two equations are:

$$X = A \cos(kx) + B \sin(kx) \qquad T = C e^{-k^2 t}$$

Since I'll be multiplying these expressions, the constant C is redundant, and we'll work it into A and B as we've done with all the other examples so far.

Now, we apply the boundary conditions:

BC #1: $\Theta(x=0) = 0 \rightarrow 0 = (A \cos(0) + B \sin(0)) e^{-k^2 t} + 0$ (this last bit is Θ_{ss})
 $\rightarrow \boxed{A = 0}$

$$1 = (B \sin(k)) e^{-k^2 t} + 1 \quad (\text{this last bit is } \Theta_{ss})$$

BC #2: $\Theta(x=1) = 1 \rightarrow 0 = (B \sin(k)) e^{-k^2 t}$ Since $B \neq 0$, it must be true that:
 $\boxed{k = n\pi}$ Well, we've seen that before!

Our answer so far: $\Theta = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2 \pi^2 t} + x$

Finally, we apply the initial condition: BC #1: $\Theta(t = 0) = 0 \rightarrow$

$$0 = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2 \pi^2 t} + x$$
$$-x = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

Note that n starts counting at 1, not 0, because this is a sine series.
This is a regular old Fourier series that we've gotten good at:

$$B_n = 2 \int_0^1 (-x) \sin(n\pi x) dx$$

MathCAD says: $B_n = \frac{2(-1)^n}{n\pi}$.

So, we're just about done: $\Theta = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin(n\pi x) e^{-n^2 \pi^2 t} + x$

Some plots are shown in MathCAD. Each separate time represents a time, and the vertical axis is temperature.