Definitions:

$\phi_1 = \int V_x dx + C_1(y)$

$\phi_2 = \int V_y dy + C_2(x)$

$\psi_1 = \int V_x dy + C_3(x)$

$\psi_2 = -\int V_y dx + C_4(y)$

ϕ_1 and ϕ_2 must agree.

ψ_1 and ψ_2 must agree.

$\nabla^2\phi = 0 \qquad \nabla^2\psi = 0$

ϕ exists if $\vec{\nabla}\times\vec{V} = 0$

A flow is incompressible if:
$\vec{\nabla}\cdot\vec{V} = 0$

Example: Compute curl of this velocity: $V_x = 6xy + 1.5y^2 - 1.5x^2 + 2$
$V_y = 3x^2 + 3xy - 3y^2 + 1$

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

$$\vec{\nabla}\times\vec{V} = 0\hat{i} + 0\hat{j} + \big((6x+3y) - (6x+3y)\big)\hat{k} = 0$$

→ We knew this had to be true, since ϕ existed and was already found in class!

Example: Given $V_x = 6xy + y^2$, what is V_y if the flow is incompressible?

$$\vec{\nabla}\cdot\vec{V} = 0 = (6y) + \frac{\partial V_y}{\partial y} \rightarrow V_y = \int -6y\,dy$$

$$\boxed{V_y = -6\frac{y^2}{2} + C(x) = -3y^2 + C(x)}$$

Example: Given $V_x = 5 - y$, and $V_y = x - 5$. Find curl, divergence, ϕ, and ψ.

$$\vec{\nabla}\times\vec{V} = 0\hat{i} + 0\hat{j} + \big(1 - (-1)\big)\hat{k} = -2\hat{k}\frac{1}{\mathrm{s}}$$ Not Zero!!

$$\vec{\nabla}\cdot\vec{V} = 0 + 0 = 0$$ So this flow is incompressible.

$$\psi = 5y - \frac{y^2}{2} + C_1(x) \text{ and } \psi = -\frac{x^2}{2} + 5x + C_2(y)$$

$$\therefore C_1(x) = -\frac{x^2}{2} + 5x \text{ and } C_2(y) = 5y - \frac{y^2}{2}$$

$$\boxed{\psi = 5y - \frac{y^2}{2} - \frac{x^2}{2} + 5x}$$ Can also add any constant to this answer.

$$\phi = 5x - xy + C_3(y) \text{ and } \phi = xy - 5y + C_4(x) ???$$

ϕ doesn't exist! ← (because curl is not zero)

"Solid Body rotation"… this is a rotating disk. Outer points move faster.

Example: Given: $\vec{V} = (x^2 - y^2)\hat{i} + (-2xy)\hat{j}$. Find ϕ and ψ.

$$\psi = x^2y - \frac{y^3}{3} + C_1(x) \text{ and } \psi = +2\frac{x^2}{2}y + C_2(y)$$

$$\therefore C_1(x) = 0 \text{ and } C_2(y) = -\frac{y^3}{3}$$

$$\boxed{\psi = x^2y - \frac{y^3}{3}}$$ Can also add any constant to this answer.

$$\phi = \frac{x^3}{3} - y^2x + C_1(y) \text{ and } \phi = -2x\frac{y^2}{2} + C_2(x) \quad \rightarrow C_1(y) = 0 \text{ and } C_2(x) = \frac{x^3}{3}$$

$$\boxed{\phi = \frac{x^3}{3} - y^2x}$$ Can also add any constant to this answer.