Problem: We want a forward difference expression for the **first** derivative that is accurate to $\mathcal{O}(h^2)$. In other words, we want to do algebra to find $f'(x)$ in terms of h, f_i, f_{i+1}, f_{i+2}, etc.., but which may not include any higher derivatives like f'' or f''' in it, even though such terms might appear in the original Taylor series.

Starting Point: $$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \ldots \qquad [1]$$

Similarly, $$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{iv}(x) + \ldots \qquad [2]$$

Solving [1] for f': $$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(x) - \frac{h^3}{24}f^{iv}(x) + \ldots \qquad [3]$$

Let's find a way to get rid of f''. Algebraically performing EQ [2] – 2 × EQ [1] results in:

$$f(x+2h) - 2f(x+h) =$$

$$\left[f(x) - 2f(x)\right] + \left[2hf'(x) - 2hf'(x)\right] + \left[\frac{(2h)^2}{2!}f''(x) - 2\frac{h^2}{2!}f''(x)\right] + \left[\frac{(2h)^3}{3!}f'''(x) - 2\frac{h^3}{3!}f'''(x)\right] + \ldots$$

f' isn't supposed to depend on f''. So, let's solve for f'' in terms of other stuff. First, we simplify the above:

$$f(x+2h) - 2f(x+h) = \left[-f(x)\right] + \left[0\right] + \left[h^2 f''(x)\right] + \left[\mathcal{O}(h)^3\right] = -f(x) + h^2 f''(x) + \mathcal{O}(h^3)$$

We accidentally eliminated f''', too, which is good luck for us! Rearranging this for f'' results in:

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} + \mathcal{O}(h^1) \qquad [4]$$

Next, we substitute our EQ[4] into EQ[3] so that f'' finally disappears:

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2}\left[\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}\right] + \mathcal{O}(h^2) \qquad [5]$$

Expanding and distributing:

$$f'(x) = \frac{f(x+h) - f(x) + \left[-\frac{1}{2}f(x+2h) + f(x+h) - \frac{1}{2}f(x)\right]}{h} + \mathcal{O}(h^2) + \mathcal{O}(h^2)$$

One more rearrangement:

$$f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + \mathcal{O}(h^2)$$

Rewriting with subscript notation: $$\boxed{f_i' = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h} + \mathcal{O}(h^2)}$$