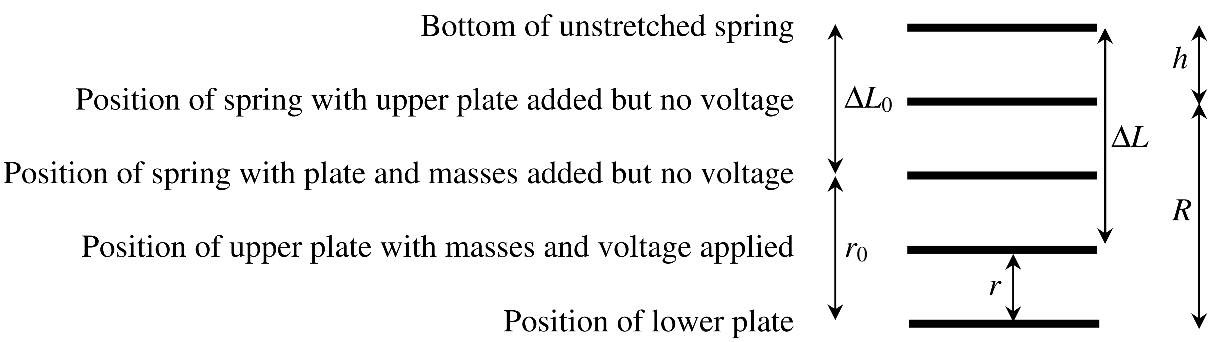
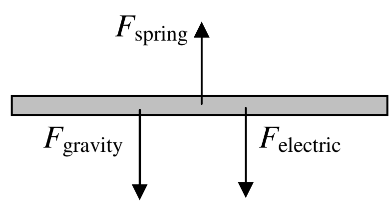
Free Body Diagram for Permittivity Experiment

First, we need a sketch of the possible positions of the spring. The upper line shows the position of the bottom of the spring when no mass is attached to it. When the plate M with any masses m is attached to it, the spring extends by a distance ΔL_0. When a voltage is then also applied, the spring extends to a new total distance ΔL. "r" is the distance between the two plates when a voltage is applied, and r_0 is this separation when there is no voltage, as shown in this sketch:



Note that $\Delta L + r = \Delta L_0 + r_0$, so $\Delta L = \Delta L_0 + r_0 - r$.

From a free body diagram of the upper plate:



At equilibrium without voltage, $\Sigma F = 0$, so:

$$F_{\text{spring}} = F_{\text{gravity}} + F_{\text{electric}}$$

$$k\Delta L_0 = (m+M)\,g + 0 .$$ We'll substitute this in shortly to remove $k\Delta L_0$.

Even when voltage is added, we still have equilibrium, so $\Sigma F = 0$, and:

$$k\Delta L = (m+M)\,g + \frac{\varepsilon_0 A V^2}{2r^2}$$

$$k\left(\Delta L_0 + r_0 - r\right) = k\Delta L_0 + \frac{\varepsilon_0 A V^2}{2r^2}$$

$$k\left(r_0 - r\right) = \frac{\varepsilon_0 A V^2}{2r^2}$$

$$\boxed{V^2 = \frac{2r^2 k\left(r_0 - r\right)}{\varepsilon_0 A} = \left(\frac{2k}{\varepsilon_0 A}\right)\left(r^2 r_0 - r^3\right)}$$

What is r when V reaches its maximum value? Since V is maximum when V^2 is maximum, we can set $\frac{dV^2}{dr} = 0$ instead of solving the more difficult $\frac{dV}{dr} = 0$.

$$\frac{dV^2}{dr} = 0 = \left(\frac{2k}{\varepsilon_0 A}\right)\left(2r\ r_0 - 3r^2\right) \qquad \rightarrow r = \frac{2}{3} r_0 \text{ when } V \text{ is maximum.}$$

Plugging this into our V^2 equation,

$$V_{max}{}^2 = \frac{2\left(\frac{2}{3} r_0\right)^2 k\left(r_0 - \frac{2}{3} r_0\right)}{\varepsilon_0 A} = \frac{8 r_0^3 k}{27 \varepsilon_0 A}$$

Solving for r_0:

$$r_0^3 = \frac{27 \varepsilon_0 A}{8k} V_{max}{}^2 \qquad \boxed{r_0 = \frac{3}{2}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3} V_{max}{}^{2/3}}$$

This answer doesn't seem to have the mass m in it anywhere (or even the mass of the upper plate, M), but really, r_0 is a different number every time we use a different amount of mass m. Let's go back to the original sketch, and try to find r_0 in terms of R, the separation of the plates when there are **no** masses m sitting on the plate, with no voltage applied. On the original sketch, h is the extension of the spring in this case, caused only by the mass M of the upper plate. From the sketch, we note that

$$R + h = \Delta L_0 + r_0, \text{ so } h = \Delta L_0 + r_0 - R.$$

Therefore, from the free body diagram when $m = 0$:

$$kh = Mg$$

When both m and M are present, we already looked at the free body diagram to discover that:

$$k\Delta L_0 = (m + M)\, g = mg + Mg = kh + mg$$
$$k\Delta L_0 = k\left(\Delta L_0 + r_0 - R\right) + mg$$
$$mg = k(R - r_0)$$
$$\boxed{r_0 = R - \frac{mg}{k}}$$

This tells us how r_0 changes with m.

Plugging this into the above, and solving for *m* results in:

$$R-\frac{mg}{k}=\frac{3}{2}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3} V_{\max}{}^{2/3}$$

$$\boxed{m=-\frac{3k}{2g}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3} V_{\max}{}^{2/3}+\frac{Rk}{g}}$$

Since *m* and *V* are our measured quantities, any plot we make has to have functions of these measurements plotted on each axis.

We'd like to get a plot that looks like a straight line whenever possible. This seems possible when we choose:

$$y=m$$
$$x=V_{\max}^{2/3}$$

This obviously gives us $slope=-\frac{3k}{2g}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3}$

and $intercept=\frac{Rk}{g}$.

Since ε_0 appears in the slope of this plot, we'll use the slope of the plot of *m* vs $V_{\max}^{2/3}$ to obtain a value for ε_0 (assuming that we already know *k* and *g*). That's Ok, since we're getting a value for *k* from a separate experiment. Similarly, if *k* and *g* are known, you could use the intercept to determine *R*. Of course, you could also find *R* by direct measurement with a ruler, but the accuracy would be pretty low. The truth is, we don't care what *R* is anyway… it's not a fundamental quantity; it just reflects how you chose to set up your spring the first time.