Free Body Diagram for Permittivity Experiment

First, we need a sketch of the possible positions of the spring. The upper line shows the position of the bottom of the spring when no mass is attached to it. When the plate M with any masses m is attached to it, the spring then extends by a distance ΔL_0. When a voltage is then also applied, the spring extends to a new total distance ΔL. "r" is the distance between the two plates when a voltage is applied, and r_0 is this separation when there is no voltage, as shown in this sketch:

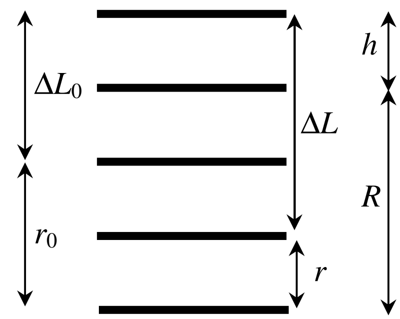
Bottom of unstretched spring

Position of spring with upper plate added but no voltage

Position of spring with plate and masses added but no voltage

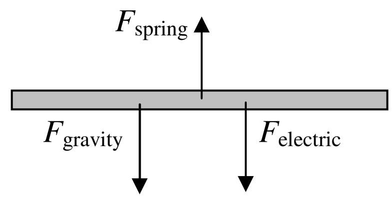
Position of upper plate with masses and voltage applied

Position of lower plate



Note that $\Delta L + r = \Delta L_0 + r_0$, so $\Delta L = \Delta L_0 + r_0 - r$.

From a free body diagram of the upper plate:



At equilibrium without voltage, $\Sigma F = 0$, so:

$$F_{\text{spring}} = F_{\text{gravity}} + F_{\text{electric}}$$

$$k\Delta L_0 = (m+M)\,g + 0 \,.$$ We'll substitute this in shortly to remove $k\Delta L_0$.

Even when voltage is added, we still have equilibrium, so $\Sigma F = 0$, and:

$$k\Delta L = (m+M)\,g + \frac{\varepsilon_0 A V^2}{2r^2}$$

$$k\left(\Delta L_0 + r_0 - r\right) = k\Delta L_0 + \frac{\varepsilon_0 A V^2}{2r^2}$$

$$k\left(r_0 - r\right) = \frac{\varepsilon_0 A V^2}{2r^2}$$

$$\boxed{V^2 = \frac{2r^2 k\left(r_0 - r\right)}{\varepsilon_0 A} = \left(\frac{2k}{\varepsilon_0 A}\right)\left(r^2 r_0 - r^3\right)}$$

What is r when V reaches its maximum value? Since V is maximum when V^2 is maximum, we can set $\frac{dV^2}{dr} = 0$ instead of solving the more difficult $\frac{dV}{dr} = 0$.

$$\frac{dV^2}{dr} = 0 = \left(\frac{2k}{\varepsilon_0 A}\right)\left(2r\ r_0 - 3r^2\right) \qquad \rightarrow\ r = \frac{2}{3}r_0 \text{ when } V \text{ is maximum.}$$

Plugging this into our V^2 equation,

$$V_{max}{}^2 = \frac{2\left(\frac{2}{3}r_0\right)^2 k\left(r_0 - \frac{2}{3}r_0\right)}{\varepsilon_0 A} = \frac{8r_0^3 k}{27\varepsilon_0 A}$$

Solving for r_0:

$$r_0^3 = \frac{27\varepsilon_0 A}{8k}V_{max}{}^2 \qquad \boxed{r_0 = \frac{3}{2}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3} V_{max}{}^{2/3}}$$

This answer doesn't seem to have the mass m in it anywhere (or even the mass of the upper plate, M), but really, r_0 is a different number every time we use a different amount of mass m. Let's go back to the original sketch, and try to find r_0 in terms of R, the separation of the plates when there are **no** masses m sitting on the plate, with no voltage applied. On the original sketch, h is the extension of the spring in this case, caused only by the mass M of the upper plate. From the sketch, we note that

$R + h = \Delta L_0 + r_0$, so $h = \Delta L_0 + r_0 - R$.

Therefore, from the free body diagram when $m = 0$:

$$kh = Mg$$

When both m and M are present, we already looked at the free body diagram to discover that:

$$k\Delta L_0 = (m+M)\,g = mg + Mg = kh + mg$$
$$k\Delta L_0 = k\left(\Delta L_0 + r_0 - R\right) + mg$$
$$mg = k(R - r_0)$$
$$\boxed{r_0 = R - \frac{mg}{k}}$$

This tells us how r_0 changes with m.

Plugging this into the above and solving for m results in:

$$R - \frac{mg}{k} = \frac{3}{2}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3} V_{\max}{}^{2/3}$$

$$\boxed{m = -\frac{3k}{2g}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3} V_{\max}{}^{2/3} + \frac{Rk}{g}}$$

Since m and V are our measured quantities, any plot we make has to have functions of these measurements plotted on each axis.

We'd like to get a plot that looks like a straight line whenever possible. This seems possible when we choose:

$$y = m$$
$$x = V_{\max}^{2/3}$$

This obviously gives us $$slope = -\frac{3k}{2g}\left(\frac{\varepsilon_0 A}{k}\right)^{1/3}$$

and $$intercept = \frac{Rk}{g}.$$

Since ε_0 appears in the slope of this plot, we'll use the slope of the plot of m vs $V_{\max}^{2/3}$ to obtain a value for ε_0 (assuming that we already know k and g). That's ok, since we're getting a value for k from a separate experiment. Similarly, if k and g are known, you could use the intercept to determine R. Of course, you could also find R by direct measurement with a ruler, but the accuracy would be pretty low. The truth is, we don't care what R is anyway… it's not a fundamental quantity; it just reflects how you chose to set up your spring the first time.